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Research Article

Group Action on the Set of Nonunits in Rings

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Let R be a ring, G be the group of all units of R, and $X = R - G \cup \{0\}$. In this paper, we investigate $|\{av(x) \mid x \in X\}| = |\{o(x) \mid x \in X\}|$ for a ring R, where $av(x)$ is the set of all vertices of the zero-divisor graph of R adjacent to x. We also investigate the question on zero-divisor graphs posed in the literature such that when the equality $o(1-e) = av(e)$ holds in a commutative regular ring R with identity. Here, e is a nonzero idempotent of R which is not the identity element of R.

1. Introduction

In 1988, Beck introduced zero-divisor graphs for commutative rings [1]. The modified definition of a zero-divisor graph was given by Anderson and Livingston [2]. Let R be a ring with identity $1 \neq 0$, $Z(R)$ be the set of zero-divisors, and $Z^*(R) = Z(R)\backslash\{0\}$ is the set of nonzero zero-divisors of R. We define a zero-divisor graph $\Gamma(R)$ with vertex set $Z^*(R)$ such that vertex x is adjacent with vertex y if and only if $xy = 0$. This graph has been studied extensively by several authors like [3, 4]. The notion has been developed for noncommutative rings in [5]. The articles [6, 7] provided similar notions for the commutative semigroups. The research idea of regular group action in rings was introduced by Han (see [8–11]).

Throughout, X will denote the set of all nonzero nonunit elements of R, G is the group of all units of R, and J is the Jacobson radical of R. We consider a group action of G on X given by $(g, x) \rightarrow gx$ from $G \times X \rightarrow X$, called the regular action. For each $x \in X$, we define the orbit of x as $o_l(x) = \{gx \mid \forall g \in G\}$ under the given group action and $av(x)$ is the set of all vertices of $\Gamma(R)$ which are adjacent to x, for all $x \in X$. In fact, $av_l(x) = \mathrm{ann}_l(x)\backslash\{0\}$. In [8], the authors showed that if R is a local ring, $J^n \neq 0 = J^{n+1}$ and X are a union of n orbits under the regular action on X by G.

The first draft of this work appeared in [12] where, in Section 3, we investigate some properties of regular action in noncommutative rings with X as a finite union of orbits under the regular action. In Section 4, we will investigate the question on zero-divisor graphs posed by Han in 2010 [13], that is, for any idempotent element e ($\neq 0, 1$) in a commutative regular ring R with identity, when does the equality $o(1-e) = av(e)$ hold?

2. Preliminary Notes

In this section, we recall some lemmas.

Lemma 1 (see [9]). *Let R be a ring with identity 1 such that $J \neq (0)$ and X is a union of n orbits under the regular action on X by G. If there exists $x \in J$ such that $x^n \neq 0$, then*

(i) $X = o(x) \cup o(x^2) \cup \cdots \cup o(x^n)$
(ii) R is a local ring
(iii) $J^{n+1} = (0)$, that is, J is a nilpotent ideal of R
(iv) J^n is one-dimensional vector space over R/J

Lemma 2 (see [9]). *If R is a ring such that X is a union of n orbits under the regular action on X by G, then there exists $x \in X$ with $x^{n+1} = 0 \neq x^n$ if and only if R is a local ring and X is the set of all right zero-divisors of R.*

A sufficient condition is given in [13], for a ring to be a division ring.

Lemma 3. *Let R be a regular ring with identity. If R has no nontrivial idempotents, then R is a division ring.*

In [14], authors proved the following lemma.

Lemma 4. *Let R be a commutative local ring with identity 1. There are nontrivial rings R_1 and R_2 such that $R \simeq R_1 \times R_2$ if and only if there exists a nontrivial idempotent $e \in R$. In this case, one can choose $R_1 = Re$ and $R_2 = R(1-e)$.*

In [15], authors proved the following lemma.

Lemma 5. *Let R be a local ring such that X is a union of n distinct orbits under the regular action of G on X. If $J^n \neq 0 = J^{n+1}$, then the set of all ideals of R is exactly*

$$\left\{\operatorname{ann}(x), \operatorname{ann}\left(x^2\right), \ldots, \operatorname{ann}(x^n), R, (0)\right\}. \tag{1}$$

3. Regular Action in Rings

In this section, we investigate $|\{av(x)\colon x \in X\}|$ in commutative rings with identity.

Lemma 6. *If there exists $x \in X$ such that $X = ann(x)$, then R is a local ring.*

Proof. Let $x \in X$ such that $X = \operatorname{ann}(x)$, implies X is a unique maximal ideal of R. Therefore, $J = X = \operatorname{ann}(x)$ and R are a local ring. □

Theorem 7. *Let I be ideal of R and $X + I = \{x + I \mid x + I$ is a nonunit$\}\backslash\{I\}$ be the subset of R/I. Then, $G/I = \{g + I \mid g \in G\} = G + I$ is the group of all units of R/I. Moreover, the action $(G + I) \times (X + I) \rightarrow X + I$ defined by $(g + I, x + I) \rightarrow gx + I$ is a regular action of $G + I$ on $X + I$.*

Proof. Let $g + I, g' + I \in G + I$. Then, $(g + I)(g' + I) = gg' + I \in G + I$, $1+I \in G + I$ and $g^{-1} + I \in G + I$ such that $(g + I)(g^{-1} + I) = gg^{-1} + I = 1+I$. So, $G + I$ is the group consisting of all units of R/I.

Let $x + I \in X + I$ and $g + I, g' + I \in G + I$, then $(g + I)(x + I) = gx + I \in X + I$. In addition, $(1 + I)(x + I) = 1x + I = x + I$ and $(g + I)((g' + I)(x + I)) = (g + I)(g'x + I) = g(g'x) + I) = (gg')x + I = (gg' + I)(x + I) = ((g + I)(g' + I))(x + I)$. Thus, $G + I$ acts on $X + I$. □

Theorem 8. *Let R be a commutative ring with identity and X is a union of n orbits under the regular action of G on X. If $I(\neq (0), R)$ is ideal of R, then there exists i_m, where $1 \leq i_m \leq n$, such that $I = o(x_{i_1}) \cup o(x_{i_2}) \cup \cdots o(x_{i_m})$ and $|I| \leq n$, where I is a nontrivial left ideal of R.*

Proof. Let $X = o(x_1) \cup o(x_2) \cup \cdots \cup o(x_n)$. Since $I \neq (0)$ is an ideal of R, then $\forall\, x \in I$. There exists i and $g \in G$ such that $x = gx_i$. So, $x_i \in I$ and $o(x_i) \subseteq I$. Thus, there exists i_m, where $1 \leq i_m \leq n$, such that $I = o(x_{i_1}) \cup o(x_{i_2}) \cup \cdots o(x_{i_m})$. Hence, $|I| \leq n$. □

Theorem 9. *Let R be a commutative ring with identity, $X = o(x_1) \cup o(x_2) \cup \cdots \cup o(x_n)$, I is a nontrivial ideal of R and $I \neq X$. Then, there exists an integer m such that $X + I$ is a union of m distinct orbits under the regular action of $G + I$ on $X + I$.*

Proof. Suppose $x_1, x_2, \ldots, x_k \in I$ and $I = o(x_1) \cup o(x_2) \cup \cdots \cup o(x_k)$. Now, we have $X + I = o(x_{i_1} + I) \cup o(x_{i_2} + I) \cup \cdots \cup o(x_{i_m} + I)$, $m \leq n - k$, and $x_{i_j} + I$ is nonunit, $1 \leq j \leq m$. On the other hand, for each $x + I \in X + I$, there exist $x_{i_j} \in X$, $(1 \leq i_j \leq n)$ and $g \in G$ such that $x = gx_{i_j}$. Hence,

$$x + I = gx_{i_j} + I = (g + I)\left(x_{i_j} + I\right) \in o\left(x_{i_j} + I\right). \tag{2}$$

Thus, $X + I \subseteq o(x_{i_1} + I) \cup o(x_{i_2} + I) \cup \cdots \cup o(x_{i_m} + I)$. However, $o_l(x_{i_j} + I) \subseteq X + I$, for each $1 \leq j \leq m$. Consequently, $X + I = o(x_{i_1} + I) \cup o(x_{i_2} + I) \cup \cdots \cup o(x_{i_m} + I)$. □

Theorem 10. *Let R be a commutative ring with identity, $J^n \neq (0) = J^{n+1}$, X is a union of n orbits under the regular action of G on X, then $|\{av(x) \mid x \in X\}| = |\{o(x) \mid x \in X\}| = n$.*

Proof. Let $X = o(x_1) \cup o(x_2) \cup \cdots \cup o(x_n)$. We know that $|\{av(x) \mid \forall x \in X\}| \leq n$. We show that $|\{av(x) \mid x \in X\}| = n$. The proof is done by induction on n. For $n = 2$, let $X = o(x_1) \cup o(x_2)$. So, $|\{av(x) \mid x \in X\}| \leq 2$ and $\operatorname{ann}(x_i) \neq 0$, where $i = 1, 2$. Suppose $av(x_1) = av(x_2)$. If $x \in \operatorname{ann}(x_1)$, then $xx_1 = 0$ and so there exists an element $g \in G$ such that $x = gx_1$ or $x = gx_2$. Therefore, $gx_1x_2 = 0$. This implies, $x_1x_2 = 0$ and $o(x_i) \subseteq \operatorname{ann}(x_1) = \operatorname{ann}(x_2)$, where $i = 1, 2$. Thus, $X = \operatorname{ann}(x_1)$ is the unique maximal ideal, which implies that R is a local ring and $\operatorname{ann}(x_1) \neq \operatorname{ann}(x_2)$ (by Lemma 5), which is a contradiction. Thus, $|\{av(x) \mid x \in X\}| = 2$.

Now, suppose the assertion holds for $n-1$ and let $x_i, x_j \in X$ such that $\operatorname{ann}(x_i) = \operatorname{ann}(x_j)$. Without loss of generality, let $\operatorname{ann}(x_1) = \operatorname{ann}(x_2)$. If $x_i \in \operatorname{ann}(x_1)$ for all i, $1 \leq i \leq n$, then R is a local ring, and by Lemma 5, $\operatorname{ann}(x_1) \neq \operatorname{ann}(x_2)$, which is a contradiction. Thus, there exists an i such that $\operatorname{ann}(x_1) = \operatorname{ann}(x_2)$ does not contain x_i, where $1 \leq i \leq n$. Assume that $\operatorname{ann}(x_1)$ does not contain x_j, and $I = \operatorname{ann}(x_j)$. By Lemma 3, there exist an integer m such that $X + I$ is a union of m orbits under the regular action of $G + I$ on $X + I$. By induction hypothesis, $|\{av(x + I) \mid x + I \in X + I\}| = m$. Moreover, $\operatorname{ann}(x_j)$ does not contain x_1 and x_2. We consider following two cases. □

Case 11. $o(x_1 + I) \neq o(x_2 + I)$. In this case, $\operatorname{ann}(x_1 + I) \neq \operatorname{ann}(x_2 + I)$. Then, there exists an element $s + I \in \operatorname{ann}(x_1 + I)$ and $s + I \notin \operatorname{ann}_l(x_2 + I)$. Therefore, $(s + I)(x_1 + I) = I$ but $(s + I)(x_2 + I) \neq I$. So $sx_1 \in I$ and $sx_2 \notin I$. On the one hand, $sx_1x_j = 0$ implies $x_j \in \operatorname{ann}_r(sx_1) = \operatorname{ann}(sx_1)$.

Therefore, $x_j s x_1 = 0$ and $x_j s \in \mathrm{ann}(x_1)$. On the other hand, $sx_2x_j \neq 0$; hence, $x_j \notin \mathrm{ann}_l(sx_2) = \mathrm{ann}(sx_2)$. Also, $x_j s x_2 \neq 0$. Thus, $\mathrm{ann}(x_2) = \mathrm{ann}(x_1)$ does not contain $x_j s$. Consequently, $\mathrm{ann}(x_1) \neq \mathrm{ann}(x_2)$, which is a contradiction.

Case 12. We first let $o(x_1 + I) = o(x_i + I)$, for every $x_i + I \in R/I$. Then, there exists $g_i \in G$ such that $x_i + I = g_i x_1 + I$. We show that R is a local ring, and $M = \{rx_1 + i \mid r \in R, i \in I\}$ is the unique maximal left ideal of R. Let N be an ideal of R such that $N \nsubseteq M$ and $x \in N$ be an arbitrary element. Since $x + I \in R/I$, there exists an orbit $o(x_i + I) \subseteq X + I$ such that $x + I \in o(x_i + I)$. This yields the existence of an element $g' \in G$ such that $x + I = g' x_i + I = g' g x_1 + I$. Therefore, $x = g' g x_1 + i$ and $x \in M$. So, $N \subseteq M$. Therefore, R is a local ring and $\mathrm{ann}(x_1) \neq \mathrm{ann}(x_2)$ (by Lemma 5).

Secondly, suppose that there is an element $x_k + I \in R/I$ such that $o(x_k + I) \neq o(x_1 + I)$. Then, $\mathrm{ann}(x_k) \neq ann(x_1)$ (follows similarly as in Case 1) and $\mathrm{ann}(x_k) \neq \mathrm{ann}(x_2)$. Now, let $I = \mathrm{ann}(x_k)$. Then, by the same argument as in Case 1, the result follows.

Theorem 13. *Let R be a commutative local ring with identity 1 such that X is a union of a finite number of orbits under the regular action on X by G, then $|\{av(x)\colon x \in X\}| = |\{o(x)\colon x \in X\}|$.*

Proof. Let R be a commutative local ring with identity, then there exists an element $x \in R$, such that $x^{n+1} = 0, x^n \neq 0$ (by Lemma 2) and $X = o(x) \cup o(x^2) \cup o(x^3) \cup \cdots \cup o(x^n)$ (by Lemma 1). Since for every i $(i = 1, 2, \ldots, n)$ $x^{n-i+1}, x^{n-i+2}, \ldots, x^n \in av(x^i)$, then $\mathrm{ann}(x^i) = o(x^{n-i+1}) \cup o(x^{n-i+2}) \cup \cdots \cup o(x^n)$. So, we have

$$\begin{aligned}
\mathrm{ann}(x) &= o(x^n),\\
\mathrm{ann}(x^2) &= o(x^{n-1}) \cup o(x^n),\\
\mathrm{ann}(x^3) &= o(x^{n-2}) \cup o(x^{n-1}) \cup o(x^n),\\
&\vdots\\
\mathrm{ann}(x^n) &= o(x) \cup o(x^2) \cup \cdots \cup o(x^n).
\end{aligned} \tag{3}$$

Moreover, for all $y \in X$, there exists an orbit $o(x^i)$, such that $y \in o(x^i)$. Therefore, there exists $g \in G$ such that $y = gx^i$ and thus $\mathrm{ann}(y) = \mathrm{ann}(x^i)$. Hence, $|\{av(x)\colon \forall x \in X\}| = |\{o(x)\colon \forall x \in X\}| = n$. □

Theorem 14. *Let R be a commutative ring with identity 1 and $X = o(x_1) \cup o(x_2) \cup \cdots \cup o(x_n)$. If for every $x \in X$, there exists an element $x_j \in X$ such that $x \in ann(x_j)$, and then, R is a local ring.*

Proof. Since $x \in X$, then there exists an orbit $o(x_i)$ such that $x \in o(x_i)$. So, there exists an element $g \in G$ such that $x = gx_i$, also $x \in \mathrm{ann}(x_j)$. Therefore, $x_i \in \mathrm{ann}(x_j)$ and $o(x_i) \subseteq \mathrm{ann}(x_j)$, for every $1 \leq i \leq n$. Thus, $\mathrm{ann}(x_j) = X$ is a maximal ideal, and this proves that R is a local ring. □

4. When Does the Equality $o(1-e) = av(e)$ Hold?

Before stating our main theorem in this section, we need to recall some definitions and lemmas [14].

Definition 15. Let R be a finite commutative ring with identity, R is called irreducible if it does not contain nontrivial idempotent elements. Therefore, R is irreducible if and only if it is not isomorphic to the product of some other nontrivial commutative rings with identity.

Lemma 16. *Let R be a commutative regular ring with identity. If R contains exactly two nontrivial idempotent elements, then X contains exactly two orbits.*

Proof. Since $(1-e)^2 = 1 - e - e + e^2 = 1 - e$, then e and $1 - e$ are two idempotent elements in R. We show that $e \notin o(1-e)$. Let $e \in o(1-e)$, then there exists an element $g \in G$ such that

$$e = g(1-e) \Rightarrow e = g - ge \Rightarrow e(1+g) = g. \tag{4}$$

Therefore, $g^{-1}(1+g)e = 1$. Thus, e is an inverse element of R which is a contradiction. By Lemma 1 in [8], X contains exactly two orbits. □

Lemma 17. *Let R be a commutative regular ring with identity. Then, R exactly contains two nontrivial idempotent elements if and only if $R = F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof. Let $R = F_1 \times F_2$. Then, R contains exactly two nontrivial idempotent elements. Conversely, if R contains two idempotent elements, we show that $R = F_1 \times F_2$. It is clear that $R = \mathrm{Re} \times R(1-e)$ (see Lemma 4) and for each orbit which contains x, there exist an idempotent element e such that $o(x) = o(e)$ (by Lemma 17). Thus, for $x \in X$, there exists an element $g \in G$ such that $x = ge$ or $x = g(1-e)$. Hence,

$$\begin{aligned}
\mathrm{Re} &= \{ge \mid g \in G\} \cup \{0\},\\
R(1-e) &= \{g(1-e) \mid g \in G\} \cup \{0\}.
\end{aligned} \tag{5}$$

We show that Re is a commutative regular ring. For $ge \in \mathrm{Re}$, there exists an element $g^{-1}e \in \mathrm{Re}$ such that $geg^{-1}ege = ge$. In a similar way, $R(1-e)$ is a regular commutative ring. Then, by Lemma 4, it follows that Re and $R(1-e)$ are fields. □

Corollary 18. *Let R be a regular commutative ring and R has two nontrivial idempotent elements. Then, the zero-divisor graph R is a complete bipartite graph.*

Now, we have the following theorem.

Theorem 19. *Let R be a regular commutative ring. For any idempotent $\in R$ $(e \neq 0, 1)$, $o(1-e) = av(e)$ if and only if $R = F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof. If $R = F_1 \times F_2$, then $o(1-e) = av(e)$. Conversely, if e_1 is an idempotent element of R, by Lemma 4, we get $R = Re_1 \times R(1-e_1)$. If e_2 be another idempotent element of R, then we show that Re_1 does not contain e_2. If $e_2 \in Re_1$, then $e_2(1-e_1) = 0$. Therefore, $e_2 \in o(e_1)$. So there is an element $g \in G$ such that $e_2 = ge_1$. Thus, $e_2^2 = e_2 = g^2 e_1 = ge_1$. So $g(g-1)e_1 = 0$ implies $(g-1) \in o(1-e_1)$. Now, there is an element $g' \in G$ such that

$$\begin{aligned} g-1 = g'(1-e_1) &\Rightarrow g = g'(1-e_1)+1 \\ &\Rightarrow e_2 = ge_1 = g'(1-e_1)e_1 + e_1 = e_1. \end{aligned} \tag{6}$$

We conclude $e_2 = ge_1 = e_1$. In the same way, we can show that $R(1-e_1)$ does not contain e_2. Now, suppose that $e_2 = xe_1 + y(1-e_1)$. Then, $e_2 e_1 = xe_1$ and $e_2(1-e_1) = y(1-e_1)$. Since $e_2 e_1$ is idempotent and by the above argument, $e_2 e_1 = e_1$ and $e_2(1-e_1) = 1-e_1$. Therefore, $e_2 e_1 - e_2(1-e_1) = 1 - e_1 + e_1 = 1$. Then, $e_2 = 1$. Thus, R contains exactly two idempotent elements. So, by Lemma 17, $R = F_1 \times F_2$. □

5. Discussion and Conclusion

In this paper, we investigated the cardinality of the set $\{av(x) \mid x \in X\}$ equal to the cardinality of the set $\{o(x) \mid x \in X\}$ for a ring R, where $av(x)$ is the set of $\Gamma(R)$ of R which are adjacent to x, for all $x \in X$, and $o(x)$ is the orbit of x under the group action. Furthermore, we discussed some properties of regular action in noncommutative rings with X as a finite union of orbits under the regular action. In [12], we investigated the question on zero-divisor graphs, denoted by $\Gamma(R)$ posed by Han in 2010 [13], when the equality $o(1-e) = av(e)$ holds for any idempotent element e ($\neq 0, 1$) in a commutative regular ring R with identity. The researchers show that there are several notions of zero-divisor graphs for commutative rings in [1], noncommutative rings in [5], and commutative semigroups in [6, 7] which link between algebraic structure and graph theory and motivate others to focus on the same method.

Data Availability

No data were used to support this study.

Disclosure

This article was submitted as a preprint in the link https://assets.researchsquare.com/files/rs-2782623/v1-covered.pdf?c=1681360249.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally to this article. They have all read and approved the final manuscript.

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