Hindawi
Journal of Mathematics
Volume 2023, Article ID 9740245, 9 pages
https://doi.org/10.1155/2023/9740245

Research Article

Covers and Envelopes by Submodules or Quotient-Modules

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Received 26 October 2022; Revised 10 March 2023; Accepted 16 March 2023; Published 15 April 2023

Academic Editor: Faranak Farshadifar

Let R be a ring, $\mathscr{X}$ a class of left R-modules, $\mathscr{S}$ the class of submodules of $\mathscr{X}$, and $\mathscr{Q}$ the class of quotient-modules of $\mathscr{X}$. It is shown that $\mathscr{S}$ ($\mathscr{Q}$) is precovering (preenveloping) if and only if every injective (projective) left R-module has an $\mathscr{X}$-precover ($\mathscr{X}$-preenvelope). Both epic and monic $\mathscr{S}$-(pre) covers ($\mathscr{Q}$-(pre) envelopes) are studied. Moreover, some applications are given. In particular, it is proven that the injective envelope of any projective left R-module is projective if and only if the class of quotient-modules of projective and injective left R-modules is monic preenveloping.

1. Introduction

Throughout this paper, R is an associative ring with identity, and modules are unitary. For a left R-module M, $E(M)$ stands for the injective envelope. The character module M^+ is defined by $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. The class of projective (injective) left R-modules is denoted by ($\mathscr{I}nj$).

Let $\mathscr{X}$ be a class of left R-modules and M a left R-module. Following [1], we say that a homomorphism $\varphi\colon M \longrightarrow C$ is a $\mathscr{X}$-preenvelope of M if $C \in \mathscr{X}$ and the abelian group homomorphism Hom (φ, C'): Hom $(C, C') \longrightarrow$ Hom (M, C') is surjective for each $C' \in \mathscr{X}$. A $\mathscr{X}$-preenvelope $\varphi\colon M \longrightarrow C$ is called a $\mathscr{X}$-envelope if every endomorphism $f\colon C \longrightarrow C$ such that $f\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of $\mathscr{X}$-precovers and $\mathscr{X}$-covers. $\mathscr{X}$-envelopes ($\mathscr{X}$-covers) may not exist in general, but if they exist, they are unique up to isomorphisms. Hence, we will always assume that the classes of left R-modules are closed under isomorphisms in this paper.

Note that the class of submodules of injective left R-modules or the class of quotient-modules of projective left R-modules is the class of left R-modules. It is clear that this class is precovering (preenveloping). Hence, this paper is motivated by the following questions:

Let $\mathscr{X}$ be a class of left R-modules.

Question 1. When is the class $\mathscr{S}$ of submodules of $\mathscr{X}$ (pre) covering?

Question 2. When is the class $\mathscr{Q}$ of quotient-modules of $\mathscr{X}$ (pre) enveloping?

In Section 2, it is shown that $\mathscr{S}$ is precovering if and only if every injective left R-module has an $\mathscr{X}$-precover. It is also proven that $\mathscr{S}$ is epic precovering if and only if every injective left R-module has an epic $\mathscr{X}$-precover. There are many applications. It is shown that:

(1) The class of submodules of projective left R-modules is epic precovering.
(2) The class of submodules of flat left R-modules is epic precovering.
(3) Let R be a right coherent ring. The class of submodules of pure-injective flat left R-modules is epic precovering.
(4) Let R be a commutative noetherian ring. The class of submodules of flat-cotorsion R-modules is epic precovering.

It is also proven that if any injective left R-module has a projective cover (e.g., the ring R is a perfect ring), then the following are equivalent:

(1) The class of submodules of projective and injective left R-modules is epic precovering.

(2) The projective cover of any injective left R-module is injective.

Moreover, suppose that the class $\mathcal{X}$ is closed under pure submodules, direct products, and direct limits, the class of submodules of $\mathcal{X}$ is covering if and only if every injective left R-module has an $\mathcal{X}$-cover (see Theorem 2). There are many examples. It is proven that:

(1) The class of submodules of fp-flat left R-modules is epic covering.

(2) If R is right (m, n)-coherent, then the class of submodules of (m, n)-flat left R-modules is epic covering.

In Section 3, it is shown that $\mathcal{Q}$ is preenveloping if and only if every projective left R-module has an $\mathcal{X}$-preenvelope. It is also shown that $\mathcal{Q}$ is monic preenveloping if and only if every projective left R-module has a monic $\mathcal{X}$-preenvelope. There are many corollaries. It is proven that:

(1) The class of quotient-modules of injective left R-modules is monic preenveloping.

(2) The class of quotient-modules of pure-injective left R-modules is monic preenveloping.

(3) The class of quotient-modules of FP-injective left R-modules is monic preenveloping.

(4) The class of quotient-modules of fp-injective left R-modules is monic preenveloping.

(5) The class of quotient-modules of (m, n)-injective left R-modules is monic preenveloping.

(6) Let R be a right coherent ring. Then the class of quotient-modules of pure-injective flat left R-modules is monic preenveloping.

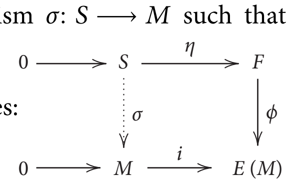
It is well known that each module has injective envelope. It is also proven that the injective envelope of any projective left R-module is projective if and only if the class of quotient-modules of projective and injective left R-modules is monic preenveloping.

2. Precovers by Submodules

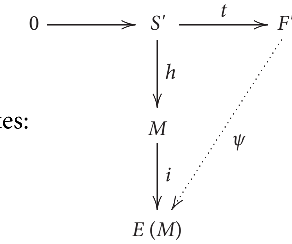
In this section, we study Question 1.

Lemma 1. *Let $\mathcal{X}$ be a class of left R-modules, $\mathcal{S}$ the class of submodules of $\mathcal{X}$, and M a left R-module. If $E(M)$ has an $\mathcal{X}$-precover φ: $F \longrightarrow E(M)$, then M has an $\mathcal{S}$-precover $S \longrightarrow M$ with $S \subseteq F$.*

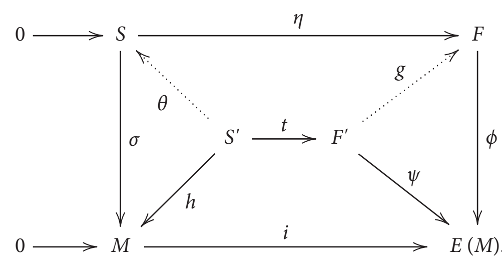
Proof. Let i: $M \longrightarrow E(M)$ be the injective envelope (we may regard i as the inclusion). Set $S = \varphi^{-1}(i(M))$. Then, there is a morphism σ: $S \longrightarrow M$ such that the following diagram commutes:



where η: $S \longrightarrow F$ is the inclusion. Thus, S with η and σ is the pullback for φ and i by [[2], Chap IV, §5]. For any left R-module $S' \subseteq F'$ with $F' \in \mathcal{X}$ and any homomorphism h: $S' \longrightarrow M$, there is a morphism ψ: $F' \longrightarrow E(M)$ such that the following diagram commutes:



where t is the inclusion map. Moreover, there exists a morphism g: $F' \longrightarrow F$ such that $\psi = \varphi g$ since φ is an $\mathcal{X}$-precover. Thus, $ih = \psi t = \varphi(gt)$. By the factorization over S (see [[2], Chap IV, §5]), there is a left R-homomorphism θ: $S' \longrightarrow S$ $(\theta(s') = gt(s'), \forall s' \in S)$ such that $\eta\theta = gt$ and $h = \sigma\theta$. Hence, there exists the following commutative diagram with exact rows:



Therefore, σ is an $\mathcal{S}$-precover of M. □

Theorem 1. *Let $\mathcal{X}$ be a class of left R-modules and $\mathcal{S}$ the class of submodules of $\mathcal{X}$. The following are equivalent:*

(1) The class $\mathcal{S}$ is precovering

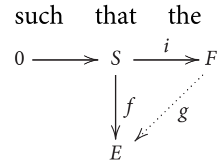
(2) Every injective left R-module has an $\mathcal{S}$-precover

(3) Every injective left R-module E has an $\mathcal{S}$-precover $S \longrightarrow E$ with $S \in \mathcal{X}$

(4) Every injective left R-module has an $\mathcal{X}$-precover

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (2) are trivial

(2) $\Longrightarrow$ (3) Let E be any injective left R-module and f: $S \longrightarrow E$ an $\mathcal{S}$-precover of E. Note that $S \in \mathcal{S}$. There is a left R-module $F \in \mathcal{X}$ such that S is a submodule of F. Since E is injective, there is a morphism g: $F \longrightarrow E$ such that the following diagram commutes:



where i: $S \longrightarrow F$ is the canonical inclusion. Obviously, $F \in \mathcal{S}$. Then, g is an $\mathcal{S}$-precover of E.

(3) $\Longrightarrow$ (4) Since $\mathcal{X} \subseteq \mathcal{S}$ and $S \in \mathcal{X}$, $S \longrightarrow E$ is an $\mathcal{X}$-precover of E

(4) $\Longrightarrow$ (1) follows by Lemma 1 □

Corollary 1. *Suppose that the class $\mathscr{X}$ is closed under submodules. Then, $\mathscr{X}$ is precovering if and only if every injective left R-module has an $\mathscr{X}$-precover.*

Recall that a torsion theory (see [2], I 2) $\tau = (\mathscr{T}, \mathscr{F})$ for left R-modules consists of two classes $\mathscr{T}$ and $\mathscr{F}$, the torsion class and the torsion-free class, respectively, such that Hom $(T, F) = 0$, whenever $T \in \mathscr{T}$ and $F \in \mathscr{F}$. Then, the class $\mathscr{T}$ is closed under quotient-modules, extensions, and direct sums (see [2], Proposition 2.1), and the class $\mathscr{F}$ is closed under submodules, extensions, and direct products (see [2], Proposition 2.2).

Example 1. Let $\tau = (\mathscr{T}, \mathscr{F})$ be a torsion theory. Then, $\mathscr{F}$ is precovering if and only if every injective left R-module has an $\mathscr{F}$-precover.

A torsion theory $\tau = (\mathscr{T}, \mathscr{F})$ is called hereditary (see [2], I 3) if $\mathscr{T}$ is closed under submodules.

Example 2. Let $\tau = (\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory. Then, $\mathscr{T}$ is precovering if and only if every injective left R-module has a $\mathscr{T}$-precover.

Now, we consider epic precover in Theorem 1. If every left R-module has an epic $\mathscr{X}$-(pre) cover, we write that $\mathscr{X}$ is epic (pre) covering.

Lemma 2. *Let $\mathscr{X}$ be a class of left R-modules, $\mathscr{S}$ the class of submodules of $\mathscr{X}$, and M a left R-module. If $E(M)$ has an epic $\mathscr{X}$-precover $\varphi\colon F \longrightarrow E(M)$, then M has an epic $\mathscr{S}$-precover $S \longrightarrow M$ such that $S \subseteq F$.*

Proof. In view of the proof of Lemma 1 and [[2], Chap IV, Proposition 5.1], there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0 & \longrightarrow & S & \xrightarrow{\eta} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow \sigma & & \downarrow i & & \\
0 & \longrightarrow & C_0 & \longrightarrow & F & \xrightarrow{\phi} & E(M) & \longrightarrow & 0,
\end{array}$$

where $i\colon M \longrightarrow E(M)$ is the injective envelope of $M$and the right square is a pullback diagram. Thus, σ is an epic $\mathscr{S}$-precover of M by Lemma 1. □

Proposition 1. *Let $\mathscr{X}$ be a class of left R-modules and $\mathscr{S}$ the class of submodules of $\mathscr{X}$. The following are equivalent:*

(1) Every left R-module has an epic $\mathscr{S}$-precover

(2) Every injective left R-module has an epic $\mathscr{S}$-precover

(3) Every injective left R-module E has an epic $\mathscr{S}$-precover $S \longrightarrow E$ with $S \in \mathscr{X}$

(4) Every injective left R-module has an epic $\mathscr{X}$-precover.

Proof. (1) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (2), and (3) $\Longrightarrow$ (4) are trivial

(2) $\Longrightarrow$ (3). Let E be any injective left R-module and $f\colon S \longrightarrow E$ be an epic $\mathscr{S}$-precover of E. Note that $S \in \mathscr{S}$. There is a left R-module $F \in \mathscr{X}$ such that S is a submodule of F. Since E is injective, there is a morphism $g\colon F \longrightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccccc}
0 & \longrightarrow & S & \xrightarrow{i} & F \\
 & & \downarrow f & \swarrow g & \\
 & & E & &
\end{array}$$

where $i\colon S \longrightarrow F$ is the canonical inclusion. Since f is epic, g is epic too. It follows that g is an epic $\mathscr{S}$-precover of E since $F \in \mathscr{X} \subseteq \mathscr{S}$.

(4) $\Longrightarrow$ (1) follows by Lemma 2. □

Example 3

(1) The class of submodules of projective left R-modules is epic precovering

(2) The class of submodules of flat left R-modules is epic precovering.

Proof

(1) Obviously, any module has an epic projective precover. So, (1) follows from Lemma 2

(2) By [[3], Theorem 3], any module has an epic flat cover. So (2) follows from Lemma 2.

Set $\mathscr{X}$ = {flat R-modules} ∩ {cotorsion R-modules}. The flat-cotorsion class has been studied by many authors ([4–7] etc.). □

Example 4. Let R be a commutative noetherian ring. Then, the class of submodules of flat-cotorsion R-modules is epic covering.

Proof. Let E be an injective R-module. Then E has an epic flat cover $f\colon F \longrightarrow E$ by [3]. It follows that F is flat-cotorsion by [[4], Theorem 5.3.28]. For any flat-cotorsion R-module F' and any homomorphism $g\colon F' \longrightarrow E$, there is a morphism $h\colon F' \longrightarrow F$ such that $g = fh$. Thus, f is an epic flat-cotorsion cover of E. The result follows from Lemma 2.

Note that any pure-injective left R-module is cotorsion. Set $\mathscr{X}$ = {flat left R-modules} ∩ {pure-injective left R-modules}.

Recall that a ring R is said to be right coherent (see [8]) in case each finitely generated right ideal of R is finitely presented. We have the following. □

Example 5. Let R be a right coherent ring. Then, the class of submodules of pure-injective flat left R-modules is epic precovering.

Proof. Let E be an injective R-module. By [[4], Theorem 5.3.11], we have that every injective left R-module has a flat cover $f\colon F \longrightarrow E$ with F flat and pure-injective. Clearly, f is an epimorphism. Thus, f is an epic pure-injective flat cover of E. The result follows from Lemma 2. □

Corollary 2. *Let $\mathscr{X}$ be a class of left R-modules such that $\mathscr{X} \subseteq \mathscr{P}roj$, and $\mathscr{S}$ be the class of submodules of $\mathscr{X}$. If any injective left R-module has a projective cover (e.g., the ring R is a left perfect ring) and $\mathscr{X}$ is closed under direct summands, then the following are equivalent:*

(1) Every left R-module has an epic $\mathscr{S}$-precover

(2) Every injective left R-module has an epic $\mathscr{S}$-precover

(3) Every injective left R-module has an epic $\mathscr{S}$-precover $S \longrightarrow E$ with $S \in \mathscr{X}$

(4) Every injective left R-module has an epic $\mathscr{S}$-cover $S \longrightarrow E$ with $S \in \mathscr{X}$

(5) Every injective left R-module has an epic $\mathscr{X}$-precover

(6) Every injective left R-module has an epic $\mathscr{X}$-cover

(7) The projective cover of any injective left R-module is in $\mathscr{X}$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (5) follow from Proposition 1

(4) $\Longrightarrow$ (3) and (6) $\Longrightarrow$ (5) are trivial

(7) $\Longrightarrow$ (6) Let E be an injective left R-module and g: $P \longrightarrow E$ be the projective cover of E. Clearly, g is epic. By (7), $P \in \mathscr{X}$.Because $\mathscr{X} \subseteq \mathscr{P}roj$, g: $P \longrightarrow E$ is an epic $\mathscr{X}$-cover of E.

(5) $\Longrightarrow$ (6), (7) Let E be an injective left R-module, f: $X \longrightarrow E$ an epic $\mathscr{X}$-precover of E and g: $P \longrightarrow E$ the projective cover of E. Note that X and P are both projective. There exist morphism h: $P \longrightarrow X$ and ϕ: $X \longrightarrow P$ such that $g = fh$ and $f = g\phi$. Hence, $g = g\phi h$. Since g is a cover, ϕh is an isomorphism. Thus, P is a direct summand of X. It follows that P is in $\mathscr{X}$. Thus, g is an epic $\mathscr{X}$-cover.

(6) $\Longrightarrow$ (4) Let E be an injective left R-module and f: $X \longrightarrow E$ an epic $\mathscr{X}$-cover of E. Note that (6) $\Longrightarrow$ (3). Then, there is an epic $\mathscr{S}$-precover g: $S \longrightarrow E$ with $S \in \mathscr{X}$. And so there is a morphism h: $S \longrightarrow X$ such that $g = fh$. Thus, f is an epic $\mathscr{S}$-cover of E.

Let $\mathscr{X}$ = {injective left R-modules} $\cap$ {projective left R-modules}. □

Example 6. If any injective left R-module has a projective cover, then the following are equivalent.

(1) The class of submodules of projective and injective left R-modules is epic precovering.

(2) The projective cover of any injective left R-module is injective.

Proof

(1) $\Longrightarrow$ (2). Let E be an injective left R-module and g: $P \longrightarrow E$ be the projective cover of E. By Corollary 2, we get that $P \in \mathscr{X}$ = {injective left R-modules} $\cap$ {projective left R-modules}. Thus, P is injective.

(2) $\Longrightarrow$ (1) is trivial by Corollary 2.

A left R-module M is called *FP-injective* (or absolutely pure) [9, 10] if $\text{Ext}^1(F, M) = 0$ for all finitely presented left R-modules F. Let $\mathscr{X}$ = {FP-injective left R-modules} $\cap$ {projective left R-modules}. □

Example 7. If any injective left R-module has a projective cover, then the following are equivalent:

(1) The class of submodules of projective and FP-injective left R-modules is epic precovering.

(2) The projective cover of any injective left R-module is FP-injective.

Next, we consider the monic precover.

Lemma 3 *Let $\mathscr{X}$ be a class of left R-modules, $\mathscr{S}$ the class of submodules of $\mathscr{X}$, and M a left R-module. If E (M) has a monic $\mathscr{X}$-cover φ: $F \longrightarrow E(M)$, then M has a monic $\mathscr{S}$-cover $S \longrightarrow M$ with $S \subseteq F$.*

Proof. According to the proof of Lemma 1, we get that S with η and σ is the pullback:

$$\begin{array}{ccccc} & & S & \xrightarrow{\sigma} & M \\ & & \downarrow{\scriptstyle\eta} & & \downarrow \\ 0 & \longrightarrow & F & \xrightarrow[\phi]{} & E(M). \end{array}$$

It follows that σ is monic by [[2], Chap IV, Proposition 5.1(i)]. Thus, σ is a monic $\mathscr{S}$-precover of M by Lemma 1.

The following example shows that the necessary and sufficient conditions for epic $\mathscr{S}$-precover (in Theorem 1) do not apply to monic $\mathscr{S}$-(pre) cover. □

Example 8. Let R be a semisimple ring. If ${}_RR = P_0 \oplus P_1$, where P_0 and P_1 are two nonisomorphic simple left R modules. Now, let $\mathscr{X} = \{\oplus X_i \mid X_i \cong {}_RR\}$ and $\mathscr{S}$ be the class of submodules of $\mathscr{X}$. Since R is semisimple, every left R-module has a monic $\mathscr{S}$-precover by [[11], Proposition 13.9]. Note that P_0 is injective. But P_0 has an $\mathscr{X}$-precover π: $R \longrightarrow P_0$, where π is the canonical projection. And, monic $\mathscr{X}$-cover of P_0 does not exist.

Finally, we consider when is the class of submodules of $\mathscr{X}$ covering.

Lemma 4. *Suppose that the class $\mathscr{X}$ is closed under pure submodules, direct products, and direct limits. Then, the class $\mathscr{S}$ of submodules of $\mathscr{X}$ is closed under direct limits.*

Proof. The proof is similar to the proof of [[4], Lemma 5.3.12].

Let $((S_i), (\varphi_{ji}))$ be a well ordered inductive system with each S_i a submodule of a left R-module $X_i \in \mathscr{X}$. We need to show that $\varinjlim S_i$ is also a submodule of a left R-module in $\mathscr{X}$.

By [[4], Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ (dependent on Card S_i and Card R) such that if f: $S_i \longrightarrow G$

is any morphism with $G \in \mathcal{X}$, then there is a pure submodule $F_i \subset G$ with $f(S_i) \subset F_i$ and Card $F_i \leq \aleph_\alpha$. Note that $\mathcal{X}$ is closed under pure submodules, then $F_i \in \mathcal{X}$.

Let $\mathcal{M}$ be a set of $\{F\colon F \in \mathcal{X}$ and Card $F \leq \aleph_\alpha\}$. For each i, we consider all morphisms $f\colon S_i \longrightarrow F$ with $F \in \mathcal{M}$. Let $F_f = F$, $F_i = \prod F_f$ over all such f, and $S_i \longrightarrow F_i$ be the morphism $x \mapsto (f(x))$. Then, $F_i \in \mathcal{X}$ since $\mathcal{X}$ is closed under direct products. Note that S_i is a submodule of a left R-module $X_i \in \mathcal{X}$. There is a monic morphism $\tau_i\colon S_i \longrightarrow X_i$. Hence, there is a left R-module $F' \in \mathcal{M}$ such that $S_i \longrightarrow F'$ is the inclusion. So, $S_i \longrightarrow F_i$ is an injection.

Let $F_j = \prod G_g$ (over morphisms $g\colon S_j \longrightarrow G_g$ described above). If $\varphi\colon S_i \longrightarrow S_j$ is a morphism, the decomposition $S_i \longrightarrow S_j \longrightarrow \prod G_g \longrightarrow G_{g'}$ (the last map being the projection map) is one of the morphisms f', that is, $G_{g'} = F_{f'}$ and $S_i \longrightarrow G_{g'}$ is the morphism f'. So, let $\prod F_f \longrightarrow G_{g'}$ be the projection map corresponding to f'. Then, we see that

$$\begin{array}{ccc} S_i & \longrightarrow & F_i \\ \downarrow & & \downarrow \\ S_j & \longrightarrow & F_j \end{array}$$

is commutative and the morphisms $F_i \longrightarrow F_j$ are functorial in the obvious sense. So, we can define an direct limit $\lim_{\longrightarrow} F_i$. Note that $S_i \longrightarrow F_i$ is an injection. So, $\lim_{\longrightarrow} S_i \longrightarrow \lim_{\longrightarrow} F_i$ is also an injection. Thus, we are done since $\lim_{\longrightarrow} F_i \in \mathcal{X}$.

From [[4], Corollary 5.2.7] and Theorem 1, we get the following theorem immediately. □

Theorem 2. *Suppose that the class $\mathcal{X}$ is closed under pure submodules, direct products, and direct limits. The class $\mathcal{S}$ of submodules of $\mathcal{X}$ is covering if and only if every injective left R-module has an $\mathcal{X}$-cover.*

As applications, we have the following examples.

Recall that a left R-module M is said to be fp-flat [12] if for every monomorphism $v\colon A \longrightarrow B$ with A and B finitely presented right R-modules, $A \otimes M \longrightarrow B \otimes M$ is a monomorphism. A left R-module M is said to be fp-injective [12] if for every monomorphism $\mu\colon K \longrightarrow L$ with K and L finitely presented left R-modules, Hom $(L, M) \longrightarrow$ Hom(K, M) is an epimorphism.

Lemma 5 (see [13]). *Theorem 3.8 and Proposition 3.11.*

(1) The class of fp-flat left R-modules is closed under direct products, direct sums, direct summands, and direct limits.

(2) The class of fp-injective left R-modules is closed under direct products, direct sums, direct summands, and direct limits.

Lemma 6 [(see [13]), Theorem 3.3]. *A left R-module M is fp-injective (fp-flat) if and only if M^+ is fp-flat (fp-injective).*

Corollary 3

(1) The class of fp-flat left R-modules is closed under pure submodules and pure quotient-modules.

(2) The class of fp-injective left R-modules is closed under pure submodules and pure quotient-modules.

Proof. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a pure exact sequence. This induces a split exact sequence: $0 \longrightarrow C^+ \longrightarrow B^+ \longrightarrow A^+ \longrightarrow 0$. By Lemma 6, if B is fp-flat (fp-injective), then B^+ is fp-injective (fp-flat). This means that A^+ and C^+ are fp-injective (fp-flat) by Lemma 5. Thus, A and C are fp-flat (fp-injective) by Lemma 6. Hence, the result follows. □

Proposition 2

(1) The class of fp-flat left R-modules is covering

(2) The class of fp-injective right R-modules is covering

Proof. Clearly, the class of fp-flat left R-modules or the class of fp-injective right R-modules is closed under direct sums and pure quotient modules by Lemma 5 and Corollary 3. Thus, the result follows from [[14], Theorem 2.5]. □

Example 9. The class of submodules of fp-flat left R-modules is epic covering.

Proof. It follows from Lemma 5, Proposition 2, and Theorem 2. □

Lemma 7 [(see [12]), Theorem 2.4]. *A ring R is right coherent if and only if every fp-flat left R-module is flat.*

Example 10 [(see [4]), Theorem 5.3.14]. If R is right coherent, then the class of submodules of flat left R-modules is epic covering.

Let m and n be fixed positive integers. A right R-module P is said to be (m, n)-presented [15] if there exists an exact sequence $0 \longrightarrow K \longrightarrow R^m \longrightarrow P \longrightarrow 0$ of right R-modules, where K is n-generated. A ring R is called right (m, n)-coherent [15] in case each n-generated submodule of the right R-module R^m is finitely presented. A right R-module M is said to be (m, n)-injective [16] if $\mathrm{Ext}^1(P, M) = 0$ for any (m, n)-presented right R-module P; a left R-module N is said to be *(m, n)-flat* [15] if $\mathrm{Tor}_1(P, N) = 0$ for any (m, n)-presented right R-module P. From the definitions, it is easy to see that:

FP-injective = (m, n)-injective for all positive integers m and n,

Flat = $(1, n)$-flat for all positive integers $n = (m, n)$-flat for all positive integers m and n,

Coherent = $(1, n)$-coherent for all positive integers $n = (m, n)$-coherent for all positive integers m and n.

A ring R is said to be right J-coherent [17] if $J(R)$ is a coherent right R-module, where $J(R)$ is the Jacobson radical of R. A left R-module N is said to be J-flat [17] if $\mathrm{Tor}_1(R/I, N) = 0$ for every finitely generated right ideal I in $J(R)$. A right R-module M is called J-injective [17] if $\mathrm{Ext}^1(R/I, M) = 0$ for every finitely generated right ideal I in $J(R)$.

A ring R is said to be right N-coherent [18] if $N(R)$ is a coherent left R-module, where $N(R)$ is the intersection of all prime ideals of R. A left R-module N is said to be N-flat [18] if $\mathrm{Tor}_1(R/I, N) = 0$ for every finitely generated right ideal I in $N(R)$. A right R-module M is called J-injective [17] if $\mathrm{Ext}^1(R/I, M) = 0$ for every finitely generated right ideal I in $N(R)$.

Remark 1. By definitions, the class of (m,n)-flat (J-flat, N-flat) left R-modules is closed under direct limits, direct summands, direct sums, pure submodules, and pure quotient-modules. Then every left R-module has an epic (m,n)-flat (J-flat, N-flat) cover by [[14], Theorem 2.5]. Hence, the class of submodules of (m,n)-flat (J-flat, N-flat) left R-module is precovering by Theorem 1.

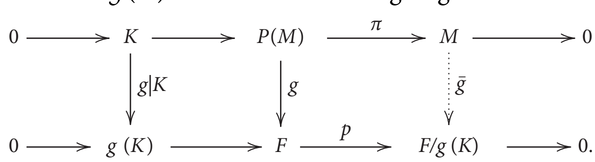
If R is right (m,n)-coherent (J-coherent, N-coherent), then the class of (m,n)-flat (J-flat, N-flat) left R-modules is closed under direct product (see [[15], Theorem 5.6], [[17], Theorem 2.13], [[18], Theorem 2.13]). Hence, the class of submodules of (m,n)-flat (J-flat, N-flat) left R-modules is covering by Theorem 2.

3. Preenvelopes by Quotient-Modules

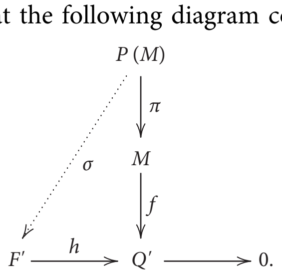
In this section, we study Question 2.

Lemma 8. *Let $\mathscr{X}$ be a class of left R-modules, $\mathscr{Q}$ the class of quotient-modules of $\mathscr{X}$, and M a left R-module. If $\pi: P(M) \longrightarrow M \longrightarrow 0$ is a projective resolution of M and $P(M)$ has an $\mathscr{X}$-preenvelope $g: P(M) \longrightarrow F$, then M has a $\mathscr{Q}$-preenvelope.*

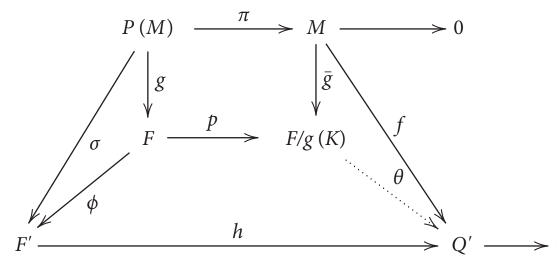
Proof. Let $K = \ker(\pi)$, $Q = F/g(K)$, and $p: F \longrightarrow F/g(K)$ be the natural epimorphism. Then there is a homomorphism $\overline{g}: M \longrightarrow F/g(K)$ such that the following diagram commutes:



Note that $P(M)$ is projective. For any epimorphism $h: F' \longrightarrow Q'$ with $F' \in \mathscr{X}$ and any homomorphism $f: M \longrightarrow Q'$, there is a morphism $\sigma: P(M) \longrightarrow F'$ such that the following diagram commutes:



Moreover, there exists a morphism $\varphi: F \longrightarrow F'$ such that $\sigma = \varphi g$ since g is an $\mathscr{X}$-preenvelope. Thus, $f\pi = h\sigma = h\varphi g$. This implies that $g(K) \subset \ker(h\varphi)$. Hence, there is a induced morphism $\theta: F/g(K) \longrightarrow Q'$ such that $\theta p = h\varphi$. This means that $f\pi = h\varphi g = \theta p g = \theta\overline{g}\pi$. Since π is epic, $f = \theta\overline{g}$. And so we have the following commutative diagram:



It follows that $\overline{g}$ is a $\mathscr{Q}$-preenvelope of M. □

Theorem 3. *Let $\mathscr{X}$ be a class of left R-modules and $\mathscr{Q}$ the class of quotient-modules of $\mathscr{X}$. The following are equivalent:*

(1) The class $\mathscr{Q}$ is preenveloping.

(2) Every projective left R-module has a $\mathscr{Q}$-preenvelope.

(3) Every projective left R-module P has a $\mathscr{Q}$-preenvelope $P \longrightarrow Q$ with $Q \in \mathscr{X}$.

(4) Every projective left R-module has an $\mathscr{X}$-preenvelope.

Proof

(1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (2) are trivial.

(2) $\Longrightarrow$ (3) Let P be any projective left R-module and $f: P \longrightarrow Q$ be a $\mathscr{Q}$-preenvelope of P. Note that $Q \in \mathscr{Q}$, there is an epimorphism $\pi: F \longrightarrow Q$ with $F \in \mathscr{X}$. Since P is projective, there is a morphism $g: P \longrightarrow F$ such that $f = \pi g$. Clearly, $F \in \mathscr{X} \subseteq \mathscr{Q}$. It follows that g is a $\mathscr{Q}$-preenvelope of P with $F \in \mathscr{X}$.

(3) $\Longrightarrow$ (4) Since $\mathscr{X} \subseteq \mathscr{Q}$ and $Q \in \mathscr{X}$, $P \longrightarrow Q$ is an $\mathscr{X}$-preenvelope of P.

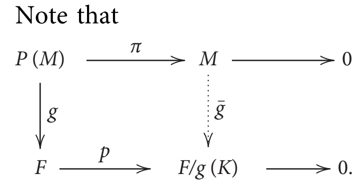
(4) $\Longrightarrow$ (1) follows from Lemma 8. □

Corollary 4. *Suppose that the class $\mathscr{X}$ is closed under quotient-modules. Then, $\mathscr{X}$ is preenveloping if and only if every projective left R-module has an $\mathscr{X}$-preenvelope.*

Example 11. Let $\tau = (\mathscr{T}, \mathscr{F})$ be a torsion theory. Then, $\mathscr{T}$ is preenveloping if and only if every projective left R-module has a $\mathscr{T}$-preenvelope.

If every left R-module has a monic $\mathscr{X}$-(pre) envelope, we write that $\mathscr{X}$ is monic (pre) enveloping.

Note that



In Lemma 8 is the pushout for π and g. Dual to Lemma 2, we get the following.

Lemma 9. *Let $\mathcal{X}$ be a class of left R-modules, $\mathcal{Q}$ the class of quotient-modules of $\mathcal{X}$, and M a left R-module. If $\pi\colon P(M) \longrightarrow M \longrightarrow 0$ is a projective resolution of M and $P(M)$ has a monic $\mathcal{X}$-preenvelope $g\colon P(M) \longrightarrow F$, then M has a monic $\mathcal{Q}$-preenvelope.*

Proposition 3. *Let $\mathcal{X}$ be a class of left R-modules and $\mathcal{Q}$ the class of quotient-modules of $\mathcal{X}$. The following are equivalent:*

(1) Every left R-module has a monic $\mathcal{Q}$-preenvelope

(2) Every projective left R-module has a monic $\mathcal{Q}$-preenvelope

(3) Every projective left R-module P has a monic $\mathcal{Q}$-preenvelope $P \longrightarrow Q$ with $Q \in \mathcal{X}$

(4) Every projective left R-module has a monic $\mathcal{X}$-preenvelope.

Proof

(1) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (2), and (3) $\Longrightarrow$ (4) are trivial

(2) $\Longrightarrow$ (3) follows from Theorem 3

(4) $\Longrightarrow$ (1) follows from Lemma 9 □

Example 12

(1) The class of quotient-modules of injective left R-modules is monic preenveloping

(2) The class of quotient-modules of pure-injective left R-modules is monic preenveloping

(3) The class of quotient-modules of FP-injective left R-modules is monic preenveloping

(4) The class of quotient-modules of fp-injective left R-modules is monic preenveloping

(5) The class of quotient-modules of (m, n)-injective left R-modules is monic preenveloping

(6) The class of quotient-modules of N-injective left R-modules is monic preenveloping

(7) The class of quotient-modules of J-injective left R-modules is monic preenveloping

Proof

(1) Obviously, any module has a monic injective envelope. So, (1) follows from Lemma 9.

(2) By [[4], Example 6.6.5], we get that any left R-module module has a monic pure-injective envelope. So, (2) follows from Lemma 9.

(3) By [[4], Theorem 6.2.4], we get that any left R-module module has a monic FP-injective preenvelope. So, (3) follows from Lemma 9.

(4) By Lemma 5, Corollary 3 and [[4], Lemma 5.3.12, and Theorem 6.1.2], we get that any left R-module has a monic fp-injective preenvelope. So, (4) follows from Lemma 9.

(5) By [[19], Theorem 3.1], we get that any module has a monic (m, n)-injective preenvelope. So, (5) follows from Lemma 9.

(6) By [[17], Lemma 2.4] and [[4], Lemma 5.3.12, and Theorem 6.1.2], we get that any module has a monic J-injective preenvelope. So, (6) follows from Lemma 9.

(7) By [[18], Remark 3.11], we get that every left R-module has a monic N-injective preenvelope. So, (7) follows from Lemma 9.

Set $\mathcal{X} = \{$pure-injective left R-modules$\} \cap \{$flat left R-modules$\}$. □

Example 13. Let R be a right coherent ring. Then, the class of quotient-modules of pure-injective flat left R-modules is monic preenveloping.

Proof. Let P be a projective left R-module. There is a monic pure-injective envelope $f\colon P \longrightarrow E$ with E pure-injective by [[4], Example 6.5.5(2)]. It follows that E is flat by [[4], Proposition 6.7.1]. For any pure-injective flat left R-module E' and any homomorphism $g\colon P \longrightarrow E'$, there is a morphism $h\colon E \longrightarrow E'$ such that $g = hf$. Thus, f is a monic pure-injective flat envelope of P. The result follows from Lemma 9. □

Corollary 5. *Let $\mathcal{X}$ be a class of left R-modules and $\mathcal{Q}$ be the class of quotient-modules of $\mathcal{X}$. If $\mathcal{X}$ is closed under direct summands and $\mathcal{X} \subseteq \mathcal{I}nj$, then the following are equivalent:*

(1) Every left R-module has a monic $\mathcal{Q}$-preenvelope

(2) Every projective left R-module has a monic $\mathcal{Q}$-preenvelope

(3) Every projective left R-module P has a monic $\mathcal{Q}$-preenvelope $P \longrightarrow Q$ with $Q \in \mathcal{X}$

(4) Every projective left R-module P has a monic $\mathcal{Q}$-envelope $P \longrightarrow Q$ with $Q \in \mathcal{X}$

(5) Every projective left R-module has a monic $\mathcal{X}$-preenvelope

(6) Every projective left R-module has a monic $\mathcal{X}$-envelope

(7) The injective envelope of any projective left R-module is in $\mathcal{X}$.

Proof

(1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (5) follow from Proposition 3.

(4) $\Longrightarrow$ (3) and (6) $\Longrightarrow$ (5) are trivial.

(7) $\Longrightarrow$ (6) Let P be a projective left R-module and $g\colon P \longrightarrow E$ be the injective envelope of P. Clearly, g is

monic. By (7), $E \in \mathcal{X}$. Because $\mathcal{X} \subseteq \mathcal{I}nj$ by hypothesis, $g\colon P \longrightarrow E$ is a monic $\mathcal{X}$-envelope of P.

(5) $\Longrightarrow$ (6), (7) Let P be a projective left R-module, $f\colon P \longrightarrow X$ a monic $\mathcal{X}$-preenvelope of P and $g\colon P \longrightarrow E$ the injective envelope of P. Note that X and E are both injective. Then there exist morphism $h\colon E \longrightarrow X$ and $\phi\colon X \longrightarrow E$ such that $g = \phi f$ and $f = hg$. Hence, $g = \phi hg$. Since g is an envelope, ϕh is an isomorphism. Thus, E is a direct summand of X. It follows that E is in $\mathcal{X}$. Thus, g is a monic $\mathcal{X}$-envelope.

(6) $\Longrightarrow$ (4) Let P be a projective left R-module and $f\colon P \longrightarrow X$ be a monic $\mathcal{X}$-envelope of P. Since (6) $\Longrightarrow$ (3), there is a monic $\mathcal{Q}$-preenvelope $g\colon P \longrightarrow Q$ with $Q \in \mathcal{X}$. This implies that there is a morphism $h\colon X \longrightarrow Q$ such that $g = hf$. Thus, f is a monic $\mathcal{Q}$-enveloper of P.

Let $\mathcal{X} =$ {injective left R-modules} $\cap$ {projective left R-modules}. □

Corollary 6. *The following are equivalent:*

(1) The class of quotient-modules of projective and injective left R-modules is monic preenveloping.

(2) The injective envelope of any projective left R-module is projective.

Proof

(1) $\Longrightarrow$ (2) Let P be a projective left R-module and $g\colon P \longrightarrow E(P)$ be the injective envelope of P. By Corollary 5, we get that $E(P) \in \mathcal{X}$ (={injective left R-modules} $\cap$ {projective left R-modules}). Thus, $E(P)$ is projective.

(2) $\Longrightarrow$ (1) is trivial by Corollary 5.

Let $\mathcal{X} =$ {injective left R-modules} $\cap$ {flat left R-modules}. □

Corollary 7. *The following are equivalent:*

(1) The class of quotient-modules of flat and injective left R-modules is monic preenveloping.

(2) The injective envelope of any projective left R-module is flat.

Dual to Lemma 3, we get the following.

Lemma 10. *Let $\mathcal{X}$ be a class of left R-modules, $\mathcal{Q}$ the class of quotient-modules of $\mathcal{X}$, and M a left R-module. If $\pi\colon P(M) \longrightarrow M$ is a projective resolution of M and $P(M)$ has an epic $\mathcal{X}$-envelope $g\colon P(M) \longrightarrow F$, then M has an epic $\mathcal{Q}$-envelope.*

The following example shows that the necessary and sufficient conditions for monic $\mathcal{Q}$-precover (Proposition 3) do not apply to epic $\mathcal{Q}$-preenvelope.

Example 14. Let R be a semisimple ring. If ${}_R R = P_0 \oplus P_1$, where P_0 and P_1 are two nonisomorphic simple left R modules. Now, let $\mathcal{X} = \{\prod X_i \mid X_i \cong_R R\}$ and $\mathcal{Q}$ be the class of quotient-modules of $\mathcal{X}$. Since R is semisimple, every left R-module has an epic $\mathcal{Q}$-preenvelope by [[11], Proposition 13.9]. Note that P_0 is projective. But P_0 has an $\mathcal{X}$-preenvelope $\eta\colon P_0 \longrightarrow R$, where η is the canonical injection. And, epic $\mathcal{X}$-preenvelope of P_0 does not exist.

Remark 2. It would be interesting to study pure-submodules. Let PE (M) be the pure-injective envelope of M. According to the proof of Lemma 1, we may get the following Proposition.

Let $\mathcal{X}$ be a class of left R-modules, $\mathcal{S}$ the class of pure-submodules of $\mathcal{X}$, and M a left R-module. If PE (M) has an $\mathcal{X}$-precover $\varphi\colon F \longrightarrow$ PE (M), then M has an $\mathcal{S}$-precover $S \longrightarrow M$ with S pure in F.

Therefore, we can get the corresponding results on precovers by pure-submodules. Preenvelopes by pure-quotient-modules may also be studied dually.

Data Availability

The data supporting the results of this study and the PDF file are the same.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The work was supported by Natural Science Foundation of Fujian Province, China (2020J01908 and 2020J05206).

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