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Research Article

Bilinear Minimax Optimal Control Problems of the Velocity Term in an Extensible Beam Equation with Rotational Inertia

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The author studies the minimax optimal control problem for an extensible beam equation that takes into account rotational inertia effects. A velocity term multiplied by the bilinear control and disturbance functions to construct a minimax optimal control strategy is added. The existence of an optimal pair, meaning optimal control and perturbation, is proved by assuming some conditions on the considered quadratic cost function. The optimal conditions for optimal pairs are provided by the adjoint systems that correspond to some physically meaningful observation cases.

1. Introduction

This study is devoted to the study of the minimax optimal control problem for the following model of an extensible beam with rotational force:

$$(1-\gamma\Delta)y'' + \Delta^2 y - \left(1 + \int_\Omega |\nabla y|^2 \mathrm{d}x\right)\Delta y + g(y) = \mathbf{U}y' + f \text{ in } Q \coloneqq \Omega \times (0,T), \tag{1}$$

where $\prime = (\partial/\partial t), \Omega$ is a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$, where $N \in \{1,2,3\}$ (mainly $N = 3$), the positive constant γ represents the effect of the rotational inertia of the beam, $g(y)$ is a nonlinear term that is explicitly explained later, $\mathbf{U}$ is a control input function, and f is a forcing term. We consider either hinged boundary condition

$$y = \Delta y = 0 \text{ on } \Sigma \coloneqq \partial\Omega \times (0,T), \tag{2}$$

or clamped boundary condition

$$y = \frac{\partial y}{\partial \nu} = 0 \text{ on } \Sigma, \tag{3}$$

where ν is the outward unit normal vector tailing on $\partial\Omega$. And we consider the initial condition

$$\begin{aligned} y(x,0) &= y_0(x), \\ y'(x,0) &= y_1(x) \text{ in } \Omega. \end{aligned} \tag{4}$$

Equation (1) with $\gamma = 0$ has been extensively studied in many articles. Initially, Woinowsky [1] proposed a one-dimensional version with the goal of describing the transverse deflection of an extensible beam. From a more physical point of view, including mathematical analysis of the extensible beam model, we can refer to Ball [2], Dickey [3], and Eisley [4].

We can also find many pioneering studies on the same equation, such as the well posedness of the equation with an additional damping term, or the stability of a solution to the equation (see [5–9]).

By the way, the case where $\gamma > 0$ in Equation (1) is called a Rayleigh beam that is introduced to consider the effect of rotational inertia on a large deflection beam model. For the introduction and studies on Rayleigh beam models, we can refer to Chueshov and Lasiecka [10, 11].

As a contribution to control theory to nonlinear beam equations without rotational inertia $\gamma = 0$, we studied in [12] the optimal control problem by the framework of Lions [13] (cf. [14], [15]) using distributed forced control variables. In [12], the optimal control problems were studied for Equation (1) in which $\gamma = 0$ and without $g(y)$ and $\mathbf{U}$ under the framework of Lions [13]. And quite recently, we studied in [16] that the nonlinear solution map of equation (1) with $\gamma = 0$ is Fréchet differentiable and applied our results to a bilinear robust control problem in which the control variable is taken as a multiplier of the displacement term rather than the velocity term. That is to say, $\mathbf{U}y'$ is replaced by $\mathbf{U}y$ in Equation (1). By the minimax optimal control strategy (cf. [17]), the existence of an optimal pair was shown and the necessary optimality condition of the optimal pair was studied, which satisfies

$$J(\bar{c}, d) \le J(\bar{c}, \bar{d}) \le J(c, \bar{d}) \forall (c, d) \in \mathscr{U}_{ad} \times \mathscr{V}_{ad}, \tag{5}$$

where J is a quadratic cost, c and d are distributed control and disturbance, respectively, and $\mathscr{U}_{ad}$ and $\mathscr{V}_{ad}$ are admissible sets.

As is noted in [18], [19], in a hyperbolic control system, a control strategy with a velocity term rather than a displacement term seems preferable. The author in this study has already pointed this out and tried to apply it to Equation (1) with $\gamma = 0$. However, we found that this requires more regularity of the control variable or solution, which is not favorable in control theory. With this in mind, in this study, the author is motivated to study the minimax optimal control problems for the bilinear control input of the velocity term of Equation (1) with $\gamma > 0$.

A description of the minimax optimal control strategy is given in [16]. To set up the control problem, the multiplier function $\mathbf{U}$ in Equation (1) is replaced with $c + d$. Here, c and d are control and disturbance variables, respectively, and c and d belong to the set of admissible control and disturbance sets $\mathscr{U}_{ad}$ and $\mathscr{V}_{ad}$, respectively. The following quadratic cost function is considered:

$$J(c, d) = \frac{1}{2}\left\|\mathscr{C}y - Y_d\right\|_M^2 + \frac{\alpha}{2}\|c\|_{\mathscr{U}}^2 - \frac{\beta}{2}\|d\|_{\mathscr{V}}^2, \tag{6}$$

where y is a solution satisfying Equation (1), $\mathscr{C}$ is an observation operator, $M, \mathscr{U}, \mathscr{V}$ are the Hilbert spaces of observation, control, and disturbance (or noise) variables, respectively, $Y_d \in M$ is the aiming value, and α and β are the positive constants related to the weights of the second and third terms of (6).

In this study, the author tries to find and characterize control and noise variables to minimize and maximize the quadratic cost (6) within $\mathscr{U}_{ad}$ and $\mathscr{V}_{ad}$, respectively. In other words, it is a problem of finding and characterizing a saddle point that satisfies (5). As stated in [16], without any confusion, the term optimal pair is still used to denote these saddle points $(\bar{c}, \bar{d})$ in (5). For the study of the existence of an optimal pair $(\bar{c}, \bar{d})$ satisfying (5), the minimax theorem in infinite dimensions proposed by Barbu and Precupanu [20] (cf. [17]) is used. To do this, we need to ensure that the solution map is differentiable and that its derivative is continuous in the Hilbert norm topology. Therefore, the Fréchet differentiability of the solution map was verified from $\mathbf{U}$ in Equation (1) to the solution of Equation (1), and the local Lipschitz continuity of the Fréchet derivative was proved by imposing some conditions on the nonlinear function g in Equation (1).

Next, the necessary optimality conditions for the optimal pairs $(\bar{c}, \bar{d})$ are derived, corresponding to physically meaningful observation cases. In this paper, the author mainly considers two observation cases: The first observation case is the distribution of velocities, and the other is the distributive and terminal value observation case. In order to derive the optimal conditions for the optimal pair, we need to find and use the relevant adjoint equation corresponding to the observed case. The optimal pair can then be given explicitly through the adjoint system.

The novelty of this paper is summarized as follows: In the case of velocity distribution observation, the author is faced with the difficulties of regularity in the process of deriving the optimal condition for the optimal pair through the adjoint equation, but this problem is overcome by the double regularization method. For the distributive and terminal value observation case, due to the lack of regularity of the control and noise variables, it is not possible to explicitly construct the related adjoint system. In general, control can be irregular; instead of assuming regularity in admissible control and noise sets, the transposition method is used to derive the necessary optimality condition for the optimal pair.

2. Notations and Preliminaries

Given a Banach space X, its topological dual is denoted by X', and the duality pairing between X' and X by $\langle \cdot, \cdot \rangle_{X', X}$. For simplicity, the following abbreviations are used:

$$\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}, H = L^2(\Omega), \tag{7}$$

where $p \ge 1$. $(\cdot, \cdot)_2$ and $\|\cdot\|_2$ represent the scalar product and norm on H (or $[H]^N$ $(N \le 3)$). As is known, it is denoted by $H^k(\Omega)$ that the Sobolev space of order $k \ge 1$ on Ω with the L^2-based Sobolev norm. $H_0^k(\Omega)$ is the completion of $C_0^\infty(\Omega)$ for the $H^k(\Omega)$ norm.

It is denoted as

$$V_2 = \begin{cases} H^2(\Omega) \cap H_0^1(\Omega), & \text{for condition,} \\ H_0^2(\Omega), & \text{for condition,} \end{cases} \tag{8}$$

and the operator A: $V_2 \longrightarrow V_2'$ is defined by

$$\langle A\phi, \psi\rangle_{V_2' V_2} = (\Delta\phi, \Delta\psi)_2, \forall \phi, \psi \in V_2, \tag{9}$$

$$D(A) = \{\phi \in V_2 | A\phi \in H\}. \tag{10}$$

Since A is self-adjoint from V_2 into V_2', it is strictly positive on V_2 due to (9), and the injection of V_2 in H is compact. Thus, from the spectral theory of self-adjoint compact operators in the Hilbert space as given in ([27], Theorem 7.7) (cf. [21]), a complete orthonormal basis of H, $\{w_k\}_{k=1}^{\infty}$, can be found,which consists of eigenvectors of A, such that

$$\begin{cases} Aw_k = \lambda_k w_k \forall k, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots, \lambda_k \longrightarrow \infty \text{ as } k \longrightarrow \infty. \end{cases} \tag{11}$$

Thus, the powers A^s of A can be defined for any $s \in \mathbb{R}$, such that for $s \ge 0$,

$$D(A^s) = \left\{ \phi \in H | \sum_{k=1}^{\infty} \lambda_k^{2s} (\phi, w_k)_2^2 < \infty \right\}. \tag{12}$$

For $s < 0$, $D(A^s)$ is the completion of H for the following norm:

$$\left\{ \sum_{k=1}^{\infty} \lambda_k^{2s} (\phi, w_k)_2^2 \right\}^{(1/2)}. \tag{13}$$

For $s \in \mathbb{R}$, the scalar product and norm of $D(A^s)$ can be written alternatively as

$$\begin{aligned} (\phi, \psi)_{D(A^s)} &= (A^s \phi, A^s \psi)_2 \\ &= \sum_{k=1}^{\infty} \lambda_k^{2s} (\phi, w_k)_2 (\psi, w_k)_2, \\ \|\phi\|_{D(A^s)} &= \left\{ (\phi, \phi)_{D(A^s)} \right\}^{(1/2)} \\ &= \left\{ \sum_{k=1}^{\infty} \lambda_k^{2s} (\phi, w_k)_2^2 \right\}^{(1/2)}. \end{aligned} \tag{14}$$

Embedding is as follows:

$$\begin{aligned} &D(A^{s_1}) \hookrightarrow D(A^{s_2}) \text{is continuous if} \\ &\quad s_1 \ge s_2 \text{ and compact if } s_1 > s_2. \end{aligned} \tag{15}$$

Furthermore, for $s_1 \ge s_2$,

$$\|\phi\|_{D(A^{s_2})} \le \lambda_1^{s_2 - s_1} \|\phi\|_{D(A^{s_1})} \forall \phi \in D(A^{s_1}). \tag{16}$$

For simplicity, throughout the paper, the author denotes $V_s = D(A^{(s/4)})$. Then, for $s \in \mathbb{R}$, V_s are Hilbert spaces with the following scalar products and norms:

$$\begin{aligned} ((\phi, \psi))_s &\coloneqq (\phi, \psi)_{V_s} = \left(A^{(s/4)} \phi, A^{(4/s)} \psi \right)_2, \\ \|\phi\|_{V_s} &= \left\| A^{s/4} \phi \right\|_2. \end{aligned} \tag{17}$$

For all $\phi, \psi \in V_s$. Thus,

$$\begin{aligned} \|\phi\|_{V_2} &= \left\| A^{(1/2)} \phi \right\|_2 = \|\Delta\phi\|_2, \\ \|\psi\|_{V_1} &= \left\| A^{(1/4)} \psi \right\|_2 = \|\nabla\psi\|_2. \end{aligned} \tag{18}$$

For all $(\phi, \psi) \in V_2 \times V_1$.

From the well-known embedding theorem, as given by Adams [22], the following embedding

$$V_2 \hookrightarrow C_0(\Omega), \tag{19}$$

is compact when $N \le 3$.

The following assumptions for $g(y)$ in Equation (1) are given as follows. $g(\cdot)$ in Equation (1) is a C^1 function with $g(0) = 0$:

(A1) Let G be the function given by $G(s) = \int_0^s g(r) dr$, and we make the following assumption:

$$\liminf_{|s| \longrightarrow \infty} \frac{G(s)}{s^2} \ge 0. \tag{20}$$

(A2) There exists a constant $c_1 > 0$ such that

$$|g'(s)| \le c_1 (1 + |s|^r) (0 \le r < \infty). \tag{21}$$

It is deduced from (20) that, for every $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$G(s) + \eta s^2 > -C_\eta \forall s \in \mathbb{R}. \tag{22}$$

We can convert Equation (1) with the boundary condition (2) or (3) and the initial condition (4) to an absolute evolution equation, which is given by

$$\begin{cases} M_\gamma y'' + Ay + \left(1 + \|y\|_{V_1}^2\right) A^{(1/2)} y + g(y) = \mathbf{U} y' + f \text{ in } (0, T), \\ y(0) = y_0, \\ y'(0) = y_1, \end{cases} \tag{23}$$

where $M_\gamma = I + \gamma A^{(1/2)}$.

Remark 1. From (A1), (A2), and (19), the following conditions are deduced which are suitable for the nonlinear operator g in Equation (23):

(i) It follows from (21) and (19) that the nonlinear operator g in Equation (23) is a C^1 bounded operator from V_2 into H, a Fréchet differentiable with the differential g', and Lipschitzian from the bounded sets of V_2 into H. Indeed, for every $R > 0$, there exists $c(R)$ such that

$$\|g(\phi_1) - g(\phi_2)\|_2 \le c(R) \|\phi_1 - \phi_2\|_2 \forall \phi_i \in V_2 (i = 1, 2), \tag{24}$$

where $\|\phi_i\|_{V_2} \le R (i = 1, 2)$.

(ii) As a consequence of (A1) and (22), it is deduced that there exists $G \in C^1(V_2, \mathbb{R}), G(0) = 0$, such that $g(\phi) = G'(\phi), \forall \phi \in V_2$ and that for every $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$G(\phi) + \eta \|\phi\|_2^2 > -C_\eta \forall \phi \in V_2. \tag{25}$$

The Hilbert space $W(0, T)$ is defined by

$$W(0, T) = \left\{ \phi | \phi \in L^2(0, T; V_2), \phi' \in L^2(0, T; V_1), \phi'' \in L^2(0, T; H) \right\}, \tag{26}$$

equipped with the norm

$$\|\phi\|_{W(0,T)} = \left(\|\phi\|^2_{L^2(0,T;V_2)} + \|\phi'\|^2_{L^2(0,T;V_1)} + \|\phi''\|^2_{L^2(0,T;H)}\right)^{(1/2)}, \quad (27)$$

where ϕ' and ϕ'' represent the distributional derivatives of ϕ with respect to time variables.

In the following, for simplicity, C is frequently used to denote the generic constant and the integral variable is omitted from all definite integrals.

Definition 1. The author calls y a weak solution of Equation (23) if $y \in W(0,T)$, and the following holds

$$\begin{cases} \langle M_\gamma y''(\cdot), \phi\rangle_{V_2',V_2} + ((y(\cdot), \phi))_2 + \left(1 + \|y(\cdot)\|^2_{V_1}\right)((y(\cdot), \phi))_1 + (g(y(\cdot)), \phi)_2 \\ = (\mathbf{U}(\cdot)y'(\cdot) + f(\cdot), \phi)_2 \\ \forall \phi \in V_2 \text{ in the sense of } \mathscr{D}'(0,T), \\ y(0) = y_0, \\ y'(0) = y_1. \end{cases} \quad (28)$$

The following existence theorem can be given by referring to Chueshov and Lasiecka [10, 11].

Theorem 1. *Let (A1) and (A2) be fulfilled and $(y_0, y_1, f) \in V_2 \times V_1 \times L^2(0,T;V_1')$, and $\mathbf{U} \in L^2(0,T;H)$. Then, there exists a weak solution y of Equation (23), satisfying*

$$y \in W(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;V_1). \quad (29)$$

To prove the regularity and uniqueness of a weak solution of Equation (23), the steps mentioned by Lions and Magenes are followed ([23], pp. 275–278). First, the following lemma provided by Lions and Magenes is exploited [23].

Lemma 1. *Let X, Y be two Banach spaces, $X \subset Y$ with dense and X being reflexive.*

$$C_w([0,T];Y) = \left\{f \in L^\infty(0,T;Y) | \langle f(\cdot), \varphi\rangle_{Y,Y'} \in C([0,T]), \forall \varphi \in Y'\right\}, \quad (30)$$

and then,

$$L^\infty(0,T;X) \cap C_w([0,T];Y) = C_w([0,T];X). \quad (31)$$

Proof. From Dautray and Lions ([26], p. 480), it is clear that $W(0,T) \hookrightarrow C([0,T];V_1) \cap C^1([0,T];H)$. Therefore, since $y \in W(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;V)_1$, the proof is the immediate consequence of Lemma 1 obtained by setting $X = V_2, Y = V_1$ to have $y \in C_w([0,T];V_2)$ and by setting $X = V_1, Y = H$ to have $y' \in C_w([0,T];V_1)$.

Hence, the proof is completed.

The following lemma is frequently used to obtain many estimates throughout the study. □

Then, the following improved regularity for the weak solution y of Equation (23) can be proved.

Corollary 1. *Let y be a weak solution of Equation (23). Then, it can be seen that*

$$\begin{aligned} y &\in C_w([0,T];V_2), \\ y' &\in C_w([0,T];V_1). \end{aligned} \quad (32)$$

Lemma 2. *If y is the weak solution of Equation (23), then the equation is obtained as follows:*

$$\begin{aligned} &\|y'(t)\|_2^2 + \gamma\|y'(t)\|^2_{V_1} + \|y(t)\|^2_{V_2} + \frac{1}{2}\left(1 + \|y(t)\|^2_{V_1}\right)^2 + 2G(y(t)) \\ &= 2\int_0^t (\mathbf{U}y', y')_2 ds + 2\int_0^t \langle f, y'\rangle_{V_1',V_1} ds + \|y_1\|_2^2 + \gamma\|y_1\|^2_{V_1} + \|y_0\|^2_{V_2} + \frac{1}{2}\left(1 + \|y_0\|^2_{V_1}\right)^2 + 2G(y_0). \end{aligned} \quad (33)$$

Proof. Based on (32), we follow the proof given in Lions and Magenes ([23], pp. 276–279). By regarding f in ([23], pp. 276–279) as $(1+\|y\|_{V_1}^2)A^{(1/2)}y - g(y) + \mathbf{U}y' + f$, the following equation is obtained through the double regularization, as given in ([23], pp. 276–279):

$$\begin{aligned}\|y'(t)\|_2^2 + \gamma\|y'(t)\|_{V_1}^2 + \|y(t)\|_{V_2}^2 = \|y_1\|_2^2 + \gamma\|y_1\|_{V_1}^2 + \|y_0\|_{V_2}^2 + 2\int_0^t (\mathbf{U}y', y')_2 \mathrm{d}s + 2\int_0^t \langle f, y'\rangle_{V_1', V_1}\mathrm{d}s \\ - 2\int_0^t (g(y), y')_2 \mathrm{d}s - 2\int_0^t \left(1+\|y\|_{V_1}^2\right)\left(A^{(1/2)}y, y'\right)_2 \mathrm{d}s.\end{aligned} \tag{34}$$

Since

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(1+\|y(t)\|_{V_1}^2\right)^2 = \left(1+\|y(t)\|_{V_1}^2\right)\frac{d}{dt}\|y(t)\|_{V_1}^2 = 2\left(1+\|y(t)\|_{V_1}^2\right)\left(A^{(1/2)}y(t), y'(t)\right)_2, \\ &\frac{d}{dt}G(y(t)) = \left(g(y(t)), y'(t)\right)_2,\end{aligned} \tag{35}$$

(33) can be obtained by combining (34) with (35).

Hence, the proof is completed.

Now, it paves the way to state the following theorem. □

Theorem 2. *Let y be the weak solution of Equation (23). Then, $y \in C([0,T]; V_2) \cap C^1([0,T]; V_1)$. Moreover, the mapping $p = (y_0, y_1, f, \mathbf{U}) \longrightarrow y(p)$ of $\mathscr{P}: = V_2 \times V_1 \times L^2(0,T; V_1') \times L^2(0,T; H)$ into $\mathscr{S}(0,T) = W(0,T) \cap C([0,T]; V_2) \cap C^1([0,T]; V)_1$ is locally Lipschitz continuous. Let $p_i = (y_0^i, y_1^i, f_i, \mathbf{U}_i) \in \mathscr{P}(i = 1,2)$. Then, the equation is obtained as follows:*

$$\|y(p_1) - y(p_2)\|_{\mathscr{S}(0,T)} \le C\left(\left\|y_0^1 - y_0^2\right\|_{V_2}^2 + \left\|y_1^1 - y_1^2\right\|_{V_1}^2 + \left\|f_1 - f_2\right\|_{L^2(0,T;V_1')}^2 + \left\|\mathbf{U}_1 - \mathbf{U}_2\right\|_{L^2(0,T;H)}^2\right)^{(1/2)} \equiv C\|p_1 - p_2\|_{\mathscr{P}}. \tag{36}$$

Proof. Using the energy equality method by Dautray and Lions ([26], pp. 578–581) combined with (32) and (34); then, the equation is obtained as follows:

$$y \in C([0,T]; V_2) \cap C^1([0,T]; V_1). \tag{37}$$

Therefore, the author focuses on showing (36). For each $p_i = (y_0^i, y_1^i, f_i, \mathbf{U}_i) \in \mathscr{P}(i = 1,2)$, the author denotes $y_1 - y_2 \equiv y(p_1) - y(p_2)$ by y. Then, it is deduced from Equation (23) that y satisfies

$$\begin{cases} M_\gamma y'' + Ay + \left(1+\|y_1\|_{V_1}^2\right)A^{(1/2)}y + g(y_1) - g(y_2) = \eta(y) + \mathbf{U}_1 y' \\ + (\mathbf{U}_1 - \mathbf{U}_2)y_2' + f_1 - f_2 \text{ in } (0,T), \\ y(0) = y_0^1 - y_0^2, \\ y'(0) = y_1^1 - y_1^2, \end{cases} \tag{38}$$

in the weak sense where

$$\eta(y) = -\left(\|y_1\|_{V_1}^2 - \|y_2\|_{V_1}^2\right)A^{(1/2)}y_2 = -((y, y_1 + y_2))_1 A^{(1/2)}y_2. \quad (39)$$

By analogy with (34), the energy equality for the weak solution y of Equation (38) can be deduced as follows:

$$\begin{aligned}\|y'(t)\|_2^2 + \gamma\|y'(t)\|_{V_1}^2 + \|y(t)\|_{V_2}^2 &= \|y'(0)\|_2^2 + \gamma\|y'(0)\|_{V_1}^2 + \|y'(0)\|_{V_2}^2 + 2\int_0^t (\eta(y) + \mathbf{U}_1 y', y')_2 \mathrm{d}s + 2\int_0^t ((\mathbf{U}_1 - \mathbf{U}_2)y_2', y')_2 \mathrm{d}s \\ &+ 2\int_0^t \langle f_1 - f_2, y'\rangle_{V_1', V_1}\mathrm{d}s - 2\int_0^t (g)((y_1) - g(y_2), y')_2 \mathrm{d}s - 2\int_0^t \left(1 + \|y_1\|_{V_1}^2\right)\left(A^{(1/2)}y, y'\right)_2 \mathrm{d}s.\end{aligned} \quad (40)$$

From (24) and the fact that $\mathcal{S}(0,T) \hookrightarrow C(\overline{Q})$,

$$\|g(y_1) - g(y_2)\|_2^2 \le C\|y\|_2^2, \quad (41)$$

where C depends only on $p_i\,(i = 1, 2)$. Thanks to (41),

$$\left|2\int_0^t (g(y_1) - g(y_2), y')_2 \mathrm{d}s\right| \le 2\int_0^t \|g(y_1) - g(y_2)\|_2\|y'\|_2 \mathrm{d}s \le C\int_0^t \|y\|_2\|y'\|_2 \mathrm{d}s \le C\int_0^t \|y\|_{V_2}\|y'\|_2 \mathrm{d}s \le C\int_0^t \left(\|y\|_{V_2}^2 + \|y'\|_2^2\right)\mathrm{d}s. \quad (42)$$

Since $V_1 \hookrightarrow L^4(\Omega)$ when $N \le 3$,

$$\begin{aligned}&\left|2\int_0^t ((\mathbf{U}_1 - \mathbf{U}_2))y_2', y')_2 \mathrm{d}s\right| \\ &\le 2\int_0^t \|\mathbf{U}_1 - \mathbf{U}_2\|_2\|y_2'\|_4\|y'\|_4 \mathrm{d}s \\ &\le C\int_0^t \|\mathbf{U}_1 - \mathbf{U}_2\|_2\|y_2'\|_{V_1}\|y'\|_{V_1}\mathrm{d}s \\ &\le C\int_0^t \|\mathbf{U}_1 - \mathbf{U}_2\|_2\|y'\|_{V_1}\mathrm{d}s \\ &\le C\int_0^t \left(\|\mathbf{U}_1 - \mathbf{U}_2\|_2^2 + \|y'\|_{V_1}^2\right)\mathrm{d}s,\end{aligned} \quad (43)$$

where C depends only on p_2. Then, for the other terms to the right of (40), the author refers to [16] and uses Gronwall's lemma to obtain

$$\|y\|_{C([0,T];V_2)\cap C^1([0,T];V_1)} \le C\|p_1 - p_2\|_{\mathcal{P}}. \quad (44)$$

Since $A(\in \mathcal{L}(V_2, V_2'))$ is isomorphism, it is inferred from Equations (38) and (44) that

$$\left\|M_\gamma y''\right\|_{L^2(0,T;V_2')} \le C\|p_1 - p_2\|_{\mathcal{P}}, \quad (45)$$

where $M_\gamma(\in \mathcal{L}(V_2, H) \cap \mathcal{L}(H, V_2'))$ is also isomorphism. Thus, finally from (44) and (45), the following is obtained:

$$\|y\|_{\mathcal{S}(0,T)} \le C\|p_1 - p_2\|_{\mathcal{P}}. \quad (46)$$

Hence, the proof is completed. □

3. Statement of the Main Results

The goal of this section is to describe the minimax optimal control problem for the proposed equation and to state the main results. As stated before, the control problem is set up as follows: $\mathbf{U}$ in equation (23) is replaced by $c + d$, where c and d belong to the admissible control set $\mathcal{U}_{ad}$ and the admissible disturbance (or noise) set $\mathcal{V}_{ad}$, respectively.

The assumptions for $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ are given as follows:

(A3) The control set of u is given by

$$\mathcal{U}_{ad} = \{c \in L^\infty(Q) | c_1 \le c \le c_2 \text{ a.e. in } Q\}. \quad (47)$$

Here, $c_i \in \mathbb{R}, i = 1, 2$.

(A4) The set of d (noises) is given by

$$\mathcal{V}_{ad} = \{d \in L^\infty(Q) | d_1 \le d \le d_2 \text{ a.e. in } Q\}. \quad (48)$$

Here, $d_i \in \mathbb{R}, i = 1, 2$.

For simplicity, the author denotes $\mathcal{W}_{ad} = \mathcal{U}_{ad} \times \mathcal{V}_{ad}$. Thanks to Theorem 1, for fixed $(y_0, y_1, f) \in V_2 \times V_1 \times L^2(0,T;V_1')$, there exists a unique weak solution $y(q) \in \mathcal{S}(0,T)$ satisfying

$$\begin{cases} M_\gamma y''(q) + Ay(q) + \left(1 + \|y(q)\|_{V_1}^2\right)A^{1/2}y(q) + g(y(q)) \\ = (c + d)y'(q) + f \text{ in } (0,T), \\ y(0) = y_0, \\ y'(0) = y_1, \end{cases} \quad (49)$$

where $q = (c, d) \in \mathcal{W}_{ad}$. Thus, it is deduced that the solution map $\mathcal{W}_{ad} \longrightarrow \mathcal{S}(0,T)$ is well defined. The quadratic cost function is considered as follows:

$$\begin{aligned}J(q) &= J(c,d) \\ &= \frac{1}{2}\|\mathcal{C}y(q) - Y_d\|_M^2 + \frac{\alpha}{2}\|c\|_{L^2(0,T;H)}^2 - \frac{\beta}{2}\|d\|_{L^2(0,T;H)}^2,\end{aligned} \quad (50)$$

where $Y_d \in M$ is the aiming state value and M and $\mathcal{C} \in \mathcal{L}(\mathcal{W}(0,T), M)$ denote the Hilbert space of observation variables and the observation operator, respectively.

As stated before, the control strategy is to find and characterize optimal control $\overline{c} \in \mathcal{U}_{ad}$ even at the worst disturbance (or noise) $\overline{d} \in \mathcal{V}_{ad}$, satisfying

$$J(\overline{c}, d) \leq J(\overline{c}, \overline{d}) \leq J(c, \overline{d}) \forall (c, d) \in \mathcal{W}_{ad}, \tag{51}$$

where $(\overline{c}, \overline{d})$ is called the saddle point of the function $J(\cdot,\cdot)$ in (50). Then, using the methods given in [16] (cf. [18]), the author characterizes the optimal pair $(\overline{c}, \overline{d})$ by stating the necessary optimal conditions through the adjoint equations related to Equation (49) and the cost (50).

The main results of this paper are as follows.

3.1. Case of $M = L^2(0, T; H)$ and Observe $\mathscr{C} y(q) = y'(q)$. If we take $M = L^2(0, T; H)$ and observe $\mathscr{C} y(q) = y'(q)$ in (50), then the optimal pair $\overline{q} = (\overline{c}, \overline{d}) \in \mathcal{W}_{ad}$ can be characterized as follows.

Theorem 3. *Let (A1)– (A5) be fulfilled. For sufficiently large α and β in the cost (50), there exists an optimal pair $\overline{q} = (\overline{c}, \overline{d}) \in \mathcal{W}_{ad}$ satisfying (50) such that it can be given by*

$$\begin{aligned} \overline{c} &= \max \left\{ c_1, \min \left\{ \frac{y'(\overline{q}) p'}{\alpha}, c_2 \right\} \right\}, \\ \overline{d} &= \max \left\{ d_1, \min \left\{ -\frac{y'(\overline{q}) p'}{\beta}, d_2 \right\} \right\}, \end{aligned} \tag{52}$$

where $y(\overline{q})$ is the weak solution of Equation (49), corresponding to $\overline{q} = (\overline{c}, \overline{d}) \in \mathcal{W}_{ad}$, and p is the weak solution of

$$\begin{cases} M_\gamma p'' + Ap - \displaystyle\int_t^T (\mathscr{G}(y(\overline{q}), p') + g'(0) p') \mathrm{d}s \\ = -(\overline{c} + \overline{d}) p' + y'(\overline{q}) - Y_d \text{ in } (0, T), \\ p(T) = 0, \\ p'(T) = 0, \end{cases} \tag{53}$$

where

$$\mathscr{G}(y(\overline{q}), p') = \left(1 + \|y(\overline{q})\|_{V_1}^2\right) A^{(1/2)} p' + 2((y(\overline{q}), p'))_1 A^{(1/2)} y(\overline{q}). \tag{54}$$

$Y_d \in M = L^2(0, T; H)$ is an aiming observation value.

3.2. Case of $M = L^2(0, T; H) \times H$ and Observe $\mathscr{C} y(q) = (y(q), y(q; T))$. If we take $M = L^2(0, T; H) \times H$ and observe $\mathscr{C} y(q) = y(q), y(q; T)$ in (50), then the optimal pair $\overline{q} = (\overline{c}, \overline{d}) \in \mathcal{W}_{ad}$ can be characterized as follows.

Theorem 4. *Let (A1)–(A5) be fulfilled. For sufficiently large α and β in the cost (50), there exists an optimal pair $\overline{q} = (\overline{c}, \overline{d}) \in \mathcal{W}_{ad}$ satisfying (50) such that it can be given by*

$$\begin{aligned} \overline{c} &= \max \left\{ c_1, \min \left\{ -\frac{y'(\overline{q}) p}{\alpha}, c_2 \right\} \right\}, \\ \overline{d} &= \max \left\{ d_1, \min \left\{ \frac{y'(\overline{q}) p}{\beta}, d_2 \right\} \right\}, \end{aligned} \tag{55}$$

where $y(\overline{q})$ is the weak solution of Equation (49), corresponding to $\overline{q} = (\overline{c}, \overline{d}) \in \mathcal{W}_{ad}$, and p is the weak solution satisfying

$$\begin{cases} \displaystyle\int_0^T \langle p, L(\phi) \rangle_{V_1, V_1'} \mathrm{d}t = \int_0^T (y(\overline{q}) - Y_d, \phi)_2 \mathrm{d}t + (y(\overline{q}; T) - Y_d^T, \phi(T))_2 \\ \forall \phi \text{ such that } L(\phi) \in L^2(0, T; V_1'), \phi(0) = \phi'(0) = 0, \end{cases} \tag{56}$$

where

$$L(\phi) = M_\gamma \phi'' + A\phi + \mathscr{G}(y(\overline{q}), \phi) + g'(y(\overline{q}))\phi - (\overline{c} + \overline{d})\phi'. \tag{57}$$

$\mathscr{G}(y(\overline{q}), \cdot)$is defined in (54), and $(Y_d, Y_d^T) \in M = L^2(0, T; H) \times H$ is an aiming observation value.

Remark 2. (i) Assumption (A5) is specified in the next section. (ii) The weak solution p satisfying (56) is usually called the transposition solution (cf. ([13], pp. 291–295)) and formally satisfies the following equation:

$$\begin{cases} M_\gamma p'' + Ap + \mathscr{G}(y(\overline{q}), p) + g'(y(\overline{q})) p \\ = -((\overline{c} + \overline{d}) p)' + y(\overline{q}) - Y_d \text{ in } (0, T), \\ p(T) = 0, \\ p'(T) = -M_\gamma^{-1} \left(y(\overline{q}; T) - Y_d^T \right). \end{cases} \tag{58}$$

The reason for using the transposition solution p in the necessary optimality conditions is explained later.

4. Differentiability of the Control-to-State Map

The goal of this section is to show the Fréchet differentiability of the nonlinear solution map from $v \in L^2(0,T;H)$ to $y(v) \in \mathcal{S}(0,T)$, where $y(v)$ satisfies

$$\begin{cases} M_\gamma y''(v) + Ay(v) + \left(1 + \|y(v)\|_{V_1}^2\right) A^{(1/2)} y(v) + g(y(v)) = vy'(v) + f \text{ in } (0,T), \\ y(v;0) = y_0, \\ y'(v;0) = y_1, \end{cases} \tag{59}$$

where $(y_0, y_1, f) \in V_2 \times V_1 \times L^2(0,T;V_1')$ is fixed. Then, from Theorem 2, it is deduced that the control-to-state map $L^2(0,T;H) \longrightarrow \mathcal{S}(0,T)$ from the term $v \in L^2(0,T;H)$ of Equation (59) to $y(v) \in \mathcal{S}(0,T)$ is uniquely defined and continuous.

The following definitions of functional differentiation are presented.

Definition 2. The control-to-state map $v \longrightarrow y(v)$ of $L^2(0,T;H)$ into $\mathcal{S}(0,T)$ is said to be Fréchet differentiable at v if there exists an operator $F(v) \in \mathcal{L}(L^2(0,T;H), \mathcal{S}(0,T))$ and a mapping $r(v,\cdot)$: $L^2(0,T;H) \longrightarrow \mathcal{S}(0,T)$ with the following properties. For any $h \in L^2(0,T;H)$, there exists

$$y(v+h) = y(v) + F(v)h + r(v,h), \|r(v,h)\|_{\mathcal{S}(0,T)} \in o\left(\|h\|_{L^2(0,T;H)}\right), \tag{60}$$

as $\|h\|_{L^2(0,T;H)} \longrightarrow 0$, where $y(v)$ and $y(v+h)$ are the weak solutions of Equation (59), corresponding to v and $v+h$, respectively.

Definition 3. Let $y(v)$ be a weak solution of Equation (59), corresponding to $v \in \mathcal{U} \subset L^2(0,T;H)$. Suppose that the first variation $\delta y(v)(h)$ at v in the direction $h \in L^2(0,T;H)$ exists, and there exists a continuous linear operator G: $L^2(0,T;H) \longrightarrow \mathcal{S}(0,T)$ such that

$$\delta y(v)(h) = Gh \forall h \in L^2(0,T;H). \tag{61}$$

Then, the control-to-state map $v \longrightarrow y(v)$ is said to be Gâteaux differentiable at v in the direction h, and we write $G = Dy(v)$.

As is well known, Fréchet differentiability also implies Gâteaux differentiability, and the Gâteaux derivative then coincides with the Fréchet derivative. Therefore, the author gives priority to the study on the Fréchet differentiability of the control-to-state map.

Theorem 5. *Let $y(v)$ be the weak solution of Equation (59). Then, the Fréchet derivative of $y(v)$ at v in the direction of $h \in L^2(0,T;H)$, denoted by $z = Dy(v)h$, is given by the weak solution*

$$\begin{cases} M_\gamma z'' + Az + \left(1 + \|y(v)\|_{V_1}^2\right) A^{(1/2)} z + 2((y(v),z))_1 A^{(1/2)} y(v) + g'(y(v))z \\ = vz' + hy'(v) \text{ in } (0,T), \\ z(0) = 0, z'(0) = 0. \end{cases} \tag{62}$$

Proof. Let

$$\mathcal{G}(y(v),z) := \left(1 + \|y(v)\|_{V_1}^2\right) A^{(1/2)} z + 2((y(v),z))_1 A^{(1/2)} y(v). \tag{63}$$

Then, from Theorem 2 and Theorem 3 in [16], the following equation is obtained:

$$\mathcal{G}(y(v),\cdot) \in \mathcal{L}(V_2,H). \tag{64}$$

Due to (A2) with $y(v) \in \mathcal{S}(0,T) \hookrightarrow C(\overline{Q})$ and (64), it can be seen from Theorem 2 that there exists a unique weak solution $z \in \mathcal{S}(0,T)$ to linearized system Equation (62) and satisfies

$$\|z\|_{\mathcal{S}(0,T)} \leq C\|h\|_{L^2(0,T;H)}. \tag{65}$$

Hence, from (65), the mapping $h \in L^2(0,T;H) \mapsto z(h) \in \mathcal{S}(0,T)$ is linear and bounded. Therefore, it is verified that the existence of an operator $F \in \mathcal{L}((L^2(0,T;H), \mathcal{S}(0,T)))$ with $Fh = z(h)$ for any $h \in L^2(0,T;H)$.

For convenience of calculation, the author denotes $y(v+h) - y(v) - z$ by ϕ. Then, from Theorem 3 given in [16],

$$\begin{aligned} &g(y(v+h)) - g(y(v)) - g'(y(v))z \\ &= G_1(y(v+h), y(v))(y(v+h) - y(v)) - g'(y(v))z \\ &= G_1(y(v+h)), y(v)\phi + (G_1(y(v+h), y(v)) - g'(y(v)))z, \end{aligned} \tag{66}$$

where

$$G_1(y(v+h), y(v)) = \int_0^1 g'(\theta y(v+h) + (1-\theta)y(v))\mathrm{d}\theta. \tag{67}$$

It is known that ϕ satisfies

$$\begin{cases} M_y\phi'' + A\phi + \left(1 + \|y(v)\|_{V_1}^2\right)A^{(1/2)}\phi + ((\phi, y(v+h) + y(v)))_1 A^{(1/2)} y(v+h) \\ + G_1(y(v+h), y(v))\phi = (v+h)\phi' + hz' + I_1 + I_2 + I_3 \text{ in } (0,T), \\ \phi(0) = 0, \\ \phi'(0) = 0. \end{cases} \tag{68}$$

In the weak sense,

$$\begin{aligned} I_1 &= -2((y(v), z))_1\left(A^{(1/2)} y(v+h) - A^{(1/2)} y(v)\right), \\ I_2 &= -(z, y(v+h) - y(v))_1 A^{(1/2)} y(v+h), \\ I_3 &= -(G_1(y(v+h), y(v)) - g'(y(v)))z. \end{aligned} \tag{69}$$

Regarding $f_1 - f_2$ in Equation (38) as $hz' + I_1 + I_2 + I_3$ and estimating the weak solution ϕ of Equation (68) through the energy equation for ϕ, the following inequality is obtained:

$$\|\phi\|_{\mathcal{S}(0,T)} \le C\left\|hz' + I_1 + I_2 + I_3\right\|_{L^2(0,T;V_1')}. \tag{70}$$

First, since $V_1 \hookrightarrow L^4(\Omega)$ when $N \le 3$, the following inequality is obtained with (65) that

$$\left|\int_0^T (hz', \varphi)_2 dt\right| \le \int_0^T \|h\|_2\|z'\|_4\|\varphi\|_4 dt \le C\int_0^T \|h\|_2\|z'\|_{V_1}\|\varphi\|_{V_1} dt \le C\|z\|_{\mathcal{S}(0,T)}\int_0^T \|h\|_2\|\varphi\|_{V_1} dt \le C\|h\|_{L^2(0,T;H)}^2\|\varphi\|_{L^2(0,T;V_1)}. \tag{71}$$

For all $\varphi \in L^2(0,T;V_1)$. Thus,

$$\left\|hz'\right\|_{L^2(0,T;V_1')} \le C\|h\|_{L^2(0,T;H)}^2. \tag{72}$$

By similar arguments in [16] together with Theorem 2 and (65), it can be deduced that

$$\left\|I_i\right\|_{L^2(0,T;H)} \le C\|h\|_{L^2(0,T;H)}^2, \tag{73}$$

where $i = 1, 2$. And also,

$$\begin{aligned} \left\|I_3\right\|_{L^2(0,T;H)} &\le \|z\|_{C([0,T];H)}\left\|G_1(y(v+h), y(v)) - g'(y(v))\right\|_{L^2(0,T;H)} \le C\|z\|_{\mathcal{S}(0,T)}\left\|G_1(y(v+h), y(v)) - g'(y(v))\right\|_{L^2(0,T;H)} \\ &\le C\|w\|_{L^2(0,T;H)}\left\|G_1(y(v+h), y(v)) - g'(y(v))\right\|_{L^2(0,T;H)}. \end{aligned} \tag{74}$$

$$\text{Since } y(u+h) \longrightarrow y(v) \text{ strongly in } \mathcal{S}(0,T)\,(\hookrightarrow C(\overline{Q})). \tag{75}$$

As $\|h\|_{L^2(0,T;H)} \longrightarrow 0$ and g is a C^1- class nonlinear operator, the author infers with (A2) and the Lebesgue-dominated convergence theorem that

$$g \in C^1\left(\mathcal{S}(0,T), L^2(0,T;H)\right). \tag{76}$$

Therefore, the author deduces with (74), (75), and (76) that

$$\left\|I_3\right\|_{L^2(0,T;H)} \in o\left(\|h\|_{L^2(0,T;H)}\right) \quad \text{as } \|h\|_{L^2(0,T;H)} \longrightarrow 0. \tag{77}$$

Hence, from (70)–(77), the following equation can be obtained:

$$\|\phi\|_{\mathcal{S}(0,T)} \in o\left(\|h\|_{L^2(0,T;H)}\right) \quad \text{as } \|h\|_{L^2(0,T;H)} \longrightarrow 0. \tag{78}$$

Hence, the proof is concluded.

The following results are used to prove the existence of optimal pairs in the next section. Another assumption for g is needed to prove the following result. In addition to (A1) and (A2), we give another assumption for $g(y)$ in equation (1) as follows:

(A5) $g(\cdot)$ in Equation (1) is a C^2 function, and there exists a constant $c_2 > 0$ such that

$$\left|g''(s)\right| \le c_2\left(1 + |s|^p\right) (0 \le p < \infty). \tag{79}$$

□

Proposition 1. *Let (A1), (A2), and (A5) hold. Given $h \in L^2(0,T;H)$, the Fréchet derivative $Dy(v)h$ is locally Lipschitz continuous on $L^2(0,T;H)$. Indeed, it is satisfied that*

$$\left\|Dy(v_1)h - Dy(v_2)h\right\|_{\mathcal{S}(0,T)} \le C\left\|v_1 - v_2\right\|_{L^2(0,T;H)}\|h\|_{L^2(0,T;H)}. \tag{80}$$

Proof. Let $z_i = Dy(v_i)h\,(i = 1,2)$ be the weak solutions of Equation (62), corresponding to $v_i\,(i = 1,2)$. The author sets $z = z_1 - z_2$. Then, it is deduced that z satisfies

$$\begin{cases} M_\gamma z'' + Az + \left(1 + \|y(v_1)\|_{V_1}^2\right)A^{(1/2)}z + 2(y(v_1), z)_1 A^{(1/2)}y(v_1) \\ +g'(y(v_1))z = v_1 z' + \sum_{i=1}^{5} I_i \text{ in } (0,T), \\ z(0) = 0, \\ z'(0) = 0. \end{cases} \quad (81)$$

In the weak sense,

$$\begin{aligned} I_1 &= -((y(v_1) - y(v_2), y(v_1) + y(v_2)))_1 A^{(1/2)} z_2, \\ I_2 &= -2((y(v_1) - y(v_2), z_2))_1 A^{(1/2)} y(v_1), \\ I_3 &= -2((y(v_2), z_2))_1 \left(A^{(1/2)} y(v_1) - A^{(1/2)} y(v_2)\right), \\ I_4 &= -(g'(y(v_1)) - g'(y(v_2)))z_2, \\ I_5 &= (v_1 - v_2)z_2' + h(y'(v_1)) - y'(v_2). \end{aligned} \quad (82)$$

By analogy with (70), it is deduced that the weak solution z of Equation (81) can be estimated as

$$\|z\|_{\mathcal{S}(0,T)} \leq C\left\|\sum_{i=1}^{5} I_i\right\|_{L^2(0,T;V_1')}. \quad (83)$$

From (65), the following equation can be obtained:

$$\|z_i\|_{\mathcal{S}(0,T)} \leq C\|h\|_{L^2(0,T;H)} \, (i = 1, 2). \quad (84)$$

Hence, by Theorem 2 and (84), I_1 can be estimated as

$$\|I_1\|_{L^2(0,T;H)} \leq C\|v_1 - v_2\|_{L^2(0,T;H)}\|h\|_{L^2(0,T;H)}. \quad (85)$$

Similarly, $I_i \, (i = 2, 3)$ can be estimated as follows:

$$\|I_i\|_{L^2(0,T;H)} \leq C\|v_1 - v_2\|_{L^2(0,T;H)}\|h\|_{L^2(0,T;H)} \, (i = 2, 3). \quad (86)$$

Thanks to (A5) and (84) with $\mathcal{S}(0,T) \hookrightarrow C(\overline{Q})$, I_4 can be estimated by

$$\|I_4\|_{L^2(0,T;H)} \leq \|g'(y(v_1)) - g'(y(v_2))\|_{L^2(0,T;H)}\|z_2\|_{C([0,T];H)} \leq C\|v_1 - v_2\|_{L^2(0,T;H)}\|h\|_{L^2(0,T;H)}. \quad (87)$$

By analogy with (71) and (72), the following equation can be obtained:

$$\begin{aligned} \|I_5\|_{L^2(0,T;V_1')} &\leq \|(v_1 - v_2)z_2'\|_{L^2(0,T;V_1')} + \|h(y'(v_1) - y'(v_2))\|_{L^2(0,T;V_1')} \\ &\leq C\|v_1 - v_2\|_{L^2(0,T;H)}\|z_2\|_{\mathcal{S}(0,T)} + C\|y(v_1) - y(v_2)\|_{\mathcal{S}(0,T)}\|h\|_{L^2(0,T;H)} \\ &\leq C\|v_1 - v_2\|_{L^2(0,T;H)}\|h\|_{L^2(0,T;H)}. \end{aligned} \quad (88)$$

Finally, by (83)–(88), the following equation can be obtained:

$$\|z\|_{\mathcal{S}(0,T)} \leq C\|v_1 - v_2\|_{L^2(0,T;H)}\|h\|_{L^2(0,T;H)}. \quad (89)$$

Hence, the proof is concluded. □

5. Proof of the Existence

The goal of this section is to study the existence of an optimal pair (c^*, d^*) satisfying (5) under some reasonable conditions for the constants α and β in the quadratic cost. First, the following result is needed.

Proposition 2. *The mapping $(c, d) \mapsto y(c, d)$ is sequentially continuous from $\mathcal{W}_{ad}$, endowed with weak-$L^2(0,T;H)$ topology, to $W(0,T)$.*

Proof. Let $(\overline{c}, \overline{d}) = (\overline{q}) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, and let $(c_n, d_n)(= q_n) \in \mathcal{W}_{ad}(= \mathcal{U}_{ad} \times \mathcal{V}_{ad})$ be a sequence such that

$$(c_n, d_n) \rightharpoonup (\overline{c}, \overline{d}) \text{ weakly in } \left[L^2(0,T;H)\right]^2, \quad (90)$$

respectively, as $n \longrightarrow \infty$. From now on, $y(c_n, d_n)$ is denoted by y_n, which is the solution of Equation (49), in which c and d are replaced by c_n and d_n, respectively. From Theorem 2, the following equation can be obtained:

$$\left\|y_n\right\|_{\mathcal{S}(0,T)} \leq C\left\|\left(y_0, y_1, f, c_n+d_n\right)\right\|_{\mathcal{P}}. \tag{91}$$

Hence, by Rellich's extraction theorem, a subsequence can be extracted from $\{y_n\}$, say again $\{y_n\}$, and $\overline{y} \in W(0,T)$ can be found such that

$$y_n \rightharpoonup \overline{y} \text{ weakly in } W(0,T) \quad \text{as } n \longrightarrow \infty, \tag{92}$$

$$y_n \rightharpoonup \overline{y} \text{ weakly } * \text{ in } L^{\infty}\left(0,T;V_2\right) \quad \text{as } n \longrightarrow \infty, \tag{93}$$

$$y_n' \rightharpoonup \overline{y}' \text{ weakly } * \text{ in } L^{\infty}\left(0,T;V_1\right) \quad \text{as } n \longrightarrow \infty. \tag{94}$$

From the fact that embedding $V_2 \hookrightarrow (V_1 \text{ or } C(\overline{\Omega}))$ is compact, we can apply Simon's compact embedding theorem [24] to (93) and (94) to verify that

$$\{y_n\} \text{is pre}-\text{compact in } C\left([0,T];V_1\right) \cap C(\overline{Q}). \tag{95}$$

Hence, if necessary, a subsequence $\left\{y_{n_k}\right\} \subset \{y_n\}$ can be found such that

$$y_{n_k} \longrightarrow \overline{y} \text{ strongly in } C\left([0,T];V_1\right) \cap C(\overline{Q}). \tag{96}$$

Therefore, from (21), (92), and (96),

$$\begin{aligned} &\left\|y_{n_k}\right\|_{V_1}^2 A y_{n_k} \rightharpoonup \|\overline{y}\|_{V_1}^2 A\overline{y} \text{ weakly in } L^2(0,T;H) \quad \text{as } k \longrightarrow \infty, \\ &g\left(y_{n_k}\right) \longrightarrow g(\overline{y}) \text{ strongly in } L^2(0,T;H) \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{97}$$

Since embedding $V_1 \hookrightarrow H$ is compact, we can apply again Simon's compact embedding theorem [24] to (92) and (94) to verify that

$$\{y_n'\} \text{ is pre}-\text{compact in } C([0,T];H). \tag{98}$$

Hence, a subsequence $\left\{y_{n_k}'\right\} \subset \{y_n'\}$ can also be found if necessary such that

$$y_{n_k}' \longrightarrow \overline{y}' \text{ strongly in } C([0,T];H) \text{as } k \longrightarrow \infty. \tag{99}$$

For all $\varphi \in L^2(0,T;V_1)$, the following equation can be obtained:

$$\begin{aligned} &\left|\int_0^T \left(\left(c_{n_k}+d_{n_k}\right)y_{n_k}' - (\overline{c}+\overline{d})\overline{y}', \varphi\right)_2 \mathrm{d}t\right| \\ &\leq \left|\int_0^T \left(\left(c_{n_k}+d_{n_k}\right)\left(y_{n_k}'-\overline{y}'\right), \varphi\right)_2 \mathrm{d}t\right| \\ &+ \left|\int_0^T \left(\left(c_{n_k}+d_{n_k}-\overline{c}-\overline{d}\right)\overline{y}', \varphi\right)_2 \mathrm{d}t\right| \\ &\leq C\int_0^T \left\|y_{n_k}'-\overline{y}'\right\|_2 \|\varphi\|_2 \mathrm{d}t \\ &+ \left|\int_0^T \left(\left(c_{n_k}+d_{n_k}-\overline{c}-\overline{d}\right)\overline{y}', \varphi\right)_2 \mathrm{d}t\right|. \end{aligned} \tag{100}$$

By (99) and (90) with $\overline{y}'\varphi \in L^2(0,T;H)$, it is evident that (100) converges to 0 as $k \longrightarrow \infty$. Thus,

$$\left(c_{n_k}+d_{n_k}\right) = y_{n_k}' \longrightarrow (\overline{c}+\overline{d})\overline{y}' \text{ weakly in } L^2(0,T;V_1'), \tag{101}$$

as$k \longrightarrow \infty$. Finally, y in (28) is replaced by y_{n_k}, $\mathbf{U}$ in (28) is replaced by $c_{n_k}+d_{n_k}$, and $k \longrightarrow \infty$is taken. Then, the limit $\overline{y}$ satisfies

$$\begin{cases} M_\gamma \overline{y}'' + A\overline{y} + \left(1+\|\overline{y}\|_{V_1}^2\right)A^{1/2}\overline{y} + g(\overline{y}) = (\overline{c}+\overline{d})\overline{y} + f \text{ in } (0,T), \\ \overline{y}(0) = y_0, \overline{y}'(0) = y_1. \end{cases} \tag{102}$$

From Theorem 2, it is finally deduced that the unique solution $\overline{y}$ of Equation (102) equals to $y(\overline{q})$ in $W(0,T)$. Thus,

$$y\left(q_{n_k}\right) \longrightarrow y(\overline{q}) \text{weakly in } W(0,T) \text{as } k \longrightarrow \infty. \tag{103}$$

Hence, the proof is concluded. □

Remark 3. Since we are considering observation spaces Min which $W(0,T)$ is compactly embedded, the author infers from (103) that

$$\mathscr{C}y\left(q_{n_k}\right) \longrightarrow \mathscr{C}y(\overline{q}) \text{strongly in } M \text{ as } k \longrightarrow \infty, \tag{104}$$

where $\mathscr{C} \in \mathscr{L}((W(0,T)),M)$ is the observation operator in (50).

From Theorem 5, it is known that the Fréchet derivative of $y(q)$ at $q=\overline{q}$ in the direction $w=(l,h) \in [L^2(0,T;H)]^2$, which is denoted by $z = Dy(\overline{q})w$, is given as the weak solution of

$$\begin{cases} M_\gamma z'' + Az + \left(1+\|y(\overline{q})\|_{V_1}^2\right)A^{(1/2)}z + 2(y(\overline{q}),z)_1 A^{(1/2)}y(\overline{q}) + g'(y(\overline{q}))z \\ = (\overline{c}+\overline{d})z' + (l+h)y'(\overline{q}) \text{in } (0,T), \\ z(0) = 0, \\ z'(0) = 0. \end{cases} \tag{105}$$

Proposition 3. *Let (A1)– (A5) be satisfied. For sufficiently large α and β in (50),*

$$\begin{aligned}&\left(D_cJ(c_1,d)-D_cJ(c_2,d)\right)(c_1-c_2)\geq 0, \quad \forall c_1,c_2\in\mathscr{U}_{ad},\\&\left(D_dJ(c,d_1)-D_dJ(c,d_2)\right)(d_1-d_2)\leq 0, \quad \forall d_1,d_2\in\mathscr{V}_{ad},\end{aligned}\tag{106}$$

where $D_cJ(c_i,d)(c_1-c_2), D_dJ(c,d_i)(i=1,2)$ are the Gâteaux derivatives of J at $c_i,d_i(i=1,2)$ in the directions of c_1-c_2 and d_1-d_2, respectively.

Indeed, the maps $c\longrightarrow J(c,d)$ and $d\longrightarrow J(c,d)$ are convex for all $d\in\mathscr{V}_{ad}$ and concave for all $c\in\mathscr{U}_{ad}$, respectively.

Proof. From the Fréchet differentiability of the solution map $c\longrightarrow y(c,d)$ where d is fixed, (106) can be written again by

$$\begin{aligned}&\left(\mathscr{C}y(c_1,d)-Y_d,\mathscr{C}D_cy(c_1,d)(c_1-c_2)\right)_M\\&+\alpha\int_0^T(c_1,c_1-c_2)_2\mathrm{d}t\\&-\left(\mathscr{C}y(c_2,d)-Y_d,\mathscr{C}D_cy(c_2,d)(c_1-c_2)\right)_M\\&-\alpha\int_0^T(c_2,c_1-c_2)_2\mathrm{d}t\geq 0, \quad \forall c_1,c_2\in\mathscr{U}_{ad},\end{aligned}\tag{107}$$

where $D_cy(c_i,d)(c_1-c_2)(i=1,2)$ is the weak solution of Equation (105), in which $(\overline{c}+\overline{d})z'+(l+h)y'(\overline{q})$ is replaced by $(c_i+d)z'+(c_1-c_2)y'(c_i,d)(i=1,2)$. It can easily be verified that (107) equals to

$$\begin{aligned}&\left(\mathscr{C}y(c_1,d)-\mathscr{C}y(c_2,d),\mathscr{C}D_cy(c_1,d)(c_1-c_2)\right)_M\\&+\left(\mathscr{C}y(c_2,d)-Y_d,\mathscr{C}D_cy(c_1,d)(c_1-c_2)-\mathscr{C}D_cy(c_2,d)(c_1-c_2)\right)_M\\&+\alpha\left\|c_1-c_2\right\|^2_{L^2(0,T;H)}\geq 0\forall c_1,c_2\in\mathscr{U}_{ad}.\end{aligned}\tag{108}$$

By Theorem 2, (65), and Proposition 1, the left hand side of (108) can be estimated as follows:

$$\begin{aligned}&\left|\left(\mathscr{C}y(c_1,d)-\mathscr{C}y(c_2,d),\mathscr{C}D_cy(c_1,d)(c_1-c_2)\right)_M\right|\\&\leq\left\|\mathscr{C}y(c_1,d)-\mathscr{C}y(c_2,d)\right\|_M\left\|\mathscr{C}D_cy(c_1,d)(c_1,c_2)\right\|_M\\&\leq\|\mathscr{C}\|^2_{\mathscr{L}(W(0,T),M)}\left\|y(c_1,d)-y(c_2,d)\right\|_{W(0,T)}\left\|D_cy(c_1,d)(c_1,c_2)\right\|_{W(0,T)}\\&\leq C\left\|y(c_1,d)-y(c_2,d)\right\|_{\mathscr{S}(0,T)}\left\|D_cy(c_1,d)(c_1-c_2)\right\|_{\mathscr{S}(0,T)}\\&\leq C\left\|c_1-c_2\right\|^2_{L^2(0,T;H)},\end{aligned}\tag{109}$$

$$\begin{aligned}&\left|\left(\mathscr{C}y(c_2,d)-Y_d,\mathscr{C}D_cy(c_1,d)(c_1-c_2)-\mathscr{C}D_cy(c_2,d)(c_1-c_2)\right)_M\right|\\&\leq\left\|\mathscr{C}y(c_2,d)-Y_d\right\|_M\left\|\mathscr{C}D_cy(c_1,d)(c_1-c_2)-\mathscr{C}D_cy(c_2,d)(c_1-c_2)\right\|_M\\&\leq C\|\mathscr{C}\|_{\mathscr{L}(W(0,T),M)}\left\|D_cy(c_1,d)(c_1-c_2)-D_cy(c_2,d)(c_1-c_2)\right\|_{W(0,T)}\\&\leq C\left\|D_cy(c_1,d)(c_1-c_2)-D_cy(c_2,d)(c_1-c_2)\right\|_{\mathscr{S}(0,T)}\\&\leq C\left\|c_1+d-(c_2+d)\right\|_{L^2(0,T;H)}\left\|c_1-c_2\right\|_{L^2(0,T;H)}\\&=C\left\|c_1-c_2\right\|^2_{L^2(0,T;H)}.\end{aligned}\tag{110}$$

From (108)–(110), sufficiently large α_l can be found, depending on $\mathscr{P},\mathscr{W}_{ad},Y_d$, and $\mathscr{C}$, so that (106) is satisfied for any $\alpha>\alpha_l$. Thus, we know the map $c\longrightarrow J(c,d)$, where d is fixed is convex for such α.

Similarly, sufficiently large β_l can also be found, depending on $\mathscr{P},\mathscr{W}_{ad},Y_d$, and $\mathscr{C}$, so that (106) is satisfied for any $\beta>\beta_l$. This implies the concavity of the map $d\longrightarrow J(c,d)$, where $c\in\mathscr{U}_{ad}$ is fixed.

Hence, the proof is concluded.

Finally, the following existence theorem of optimal pairs is presented (cf. [17] and [20]). □

Theorem 6. *Assume that (A1)–(A5) is fulfilled. For sufficiently large α and β in (50), there exists $(\overline{c},\overline{d})\in\mathscr{W}_{ad}$ such that $(\overline{c},\overline{d})$ satisfies (3.3).*

Proof. By Proposition 3, for sufficiently large α and β, we know that the maps $c\longrightarrow J(c,d)$ with fixed d and $d\longrightarrow J(\cdot,d)$ with fixed c are convex and concave, respectively. Together with (104), this theorem can be proved by referring to Theorem 5 of [16].

Hence, the proof is concluded. □

6. Proofs of Necessary Optimality Conditions

This section is dedicated to deriving the optimal conditions required for each of the optimal pairs of minimax optimal control problems along with the costs from Theorem 3 and Theorem 4.

6.1. Proof for the Case of $M = L^2(0,T;H)$ and $\mathscr{C}y(q) = y'(q)$. For this purpose, adjoint Equation (53)is introduced. The well posedness of Equation (53) is clarified by the following proposition.

Proposition 4. *Equation (53) admits a unique weak solution $p \in \mathcal{S}(0,T)$.*

Proof. By reversing the direction of time $t \longrightarrow T-t$ in Equation (53), it can be seen that Equation (53) is changed as follows:

$$\begin{cases} M_\gamma p'' + Ap - \int_0^t (\mathscr{G}(y(\overline{q}), p') + g'(y(\overline{q}))p')\mathrm{d}s \\ = (\overline{c} + \overline{d})p' - y'(\overline{q}) - Y_d \text{ in } (0,T), \\ p(0) = 0, \\ p'(0) = 0, \end{cases} \tag{111}$$

where $\mathscr{G}(y(\overline{q}), p')$ is defined in (54) and $Y_d \in M = L^2(0,T;H)$.

Estimating the weak solution p of Equation (111) through energy equality, as given in the proof of Theorem 2, the following equation is obtained:

$$\begin{aligned} \|p'(t)\|_2^2 + \|p'(t)\|_{V_1}^2 + \|p(t)\|_{V_2}^2 \leq C\Bigg(&\left|\int_0^t\int_0^s \langle \mathscr{G}(y(\overline{q};\sigma), p'(\sigma)) + g'(y(\overline{q};\sigma))p'(\sigma), p'(s)\rangle_{V_1',V_1} \mathrm{d}\sigma\, \mathrm{d}s\right| \\ &+ \|y'(\overline{q}) + Y_d\|_{L^2(0,T;H)}^2\Bigg). \end{aligned} \tag{112}$$

We note

$$\mathscr{G}(y(\overline{q}), \cdot) \in \mathscr{L}(V_1, V_1'). \tag{113}$$

By (21) with (19), it is deduced that

$$\begin{aligned} &\left|\int_0^t\int_0^s \langle \mathscr{G}(y(\overline{q};\sigma), p'(\sigma)) + g'(y(\overline{q};\sigma))p'(\sigma), p'(s)\rangle_{V_1',V_1} \mathrm{d}\sigma \mathrm{d}s\right| \\ &\leq C\left(\int_0^t \|p'(s)\|_{V_1}\int_0^s \|p'(\sigma)\|_{V_1}\mathrm{d}\sigma\, \mathrm{d}s + \int_0^t \|p'(s)\|_2 \int_0^s \|p'(\sigma)\|_2 \mathrm{d}\sigma\, \mathrm{d}s\right) \\ &\leq C\int_0^t \|p'(s)\|_{V_1}\int_0^s \|p'(\sigma)\|_{V_1}\mathrm{d}\sigma\, \mathrm{d}s \leq \mathrm{CT}\|p'\|_{L^2(0,t;V_1)}^2. \end{aligned} \tag{114}$$

Therefore, if we combine (112) and (114) and apply Gronwall's lemma to the combined inequality, we obtain

$$\|p\|_{\mathcal{S}(0,T)} \leq C. \tag{115}$$

Based on this, using Faedo–Galerkin's approximation procedure, it can be verified that Equation (111) has a unique weak solution $p \in \mathcal{S}(0,T)$.

Hence, the proof is concluded.

Before discussing the first-order optimality condition for the minimax optimal control problem (51) for the observation case of Theorem 3, the following remark is noted. □

Remark 4. In deriving the optimality condition for the observation case of Theorem 3, the weak form of Equation (53) needs to be multiplied by z'. However, as is often encountered in hyperbolic problems, it is the only formal procedure because $z' \in C([0,T];V_1)$ and $M_\gamma p'', Ap \in L^2(0,T;V_2')$. In [25], this difficulty can be eliminated by employing the regularization method proposed by Lions ([13], pp. 286–288).

Proof of Theorem 1. Thanks to the assumptions of Theorem 3, it is evident from Theorem 6 that there exists an optimal pair in (51) for the observation case of Theorem 3.

Let $\overline{q} = (\overline{c}, \overline{d}) \in \mathscr{W}_{ad}$ be an optimal pair in (51) with the cost (50) through the observation case of Theorem 3 and $y(\overline{q})$ be the corresponding weak solution of Equation (49).

The Gâteaux derivative of the cost (50) through the observation case of Theorem 3 at $q = \overline{q}$ in the direction $w = (l, h) \in [L^2(0,T;H)]^2$, with $\overline{q} + \varepsilon w \in \mathscr{W}_{ad}$ for sufficiently small $\varepsilon > 0$, yields

$$\begin{aligned} \mathrm{DJ}(\overline{c}, \overline{d})(l,h) &= \lim_{\varepsilon \longrightarrow 0^+} \frac{J(\overline{c} + \varepsilon l, \overline{d} + \varepsilon h) - J(\overline{c}, \overline{d})}{\varepsilon} \\ &= (y'(\overline{q}) - Y_d, z')_{L^2(0,T;H)} + \alpha(\overline{c}, l)_{L^2(0,T;H)} \\ &\quad - \beta(\overline{d}, h)_{L^2(0,T;H)}, \end{aligned} \tag{116}$$

where $z = Dy(\overline{q})w$ is a solution of Equation (105).

To eliminate the difficulty caused by the regularity of weak solutions, Lions' regularization method is employed [13] (cf. [25]). For that, the author extends the time domains of Equation (105) and Equation (53) to $\mathbb{R}_t$ as follows:

$$\begin{cases} M_\gamma \phi'' + A\phi + \mathscr{G}(y(\overline{q}), \phi) + g'(y(\overline{q}))\phi - (\overline{c} + \overline{d})\phi' \\ = \begin{cases} (l+h)y'(\overline{q}), & \text{in}\,(0,T), \\ 0, & \text{otherwise}, \end{cases} \\ \phi(0) = 0, \\ \phi'(0) = 0. \end{cases} \tag{117}$$

And

$$\begin{cases} M_\gamma \xi'' + A\xi - \int_t^T (\mathscr{G}(y(\overline{q}), \xi') + g'(y(\overline{q}))\xi')\mathrm{d}s + (\overline{c} + \overline{d})\xi' \\ = \begin{cases} y'(\overline{q}) - Y_d, & \text{in}\,(0,T), \\ 0, & \text{otherwise}, \end{cases} \\ \xi(T) = 0, \\ \xi'(T) = 0. \end{cases} \tag{118}$$

In fact,

$$\begin{aligned} \psi &= z \,\text{in}\,(0,T), \\ \xi &= p \,\text{in}\,(0,T), \end{aligned} \tag{119}$$

and

$$(y'(\overline{q}) - Y_d, z')_{L^2(0,T;H)} = \int_{\mathbb{R}_t} \left(M_\gamma \xi'' + A\xi - \int_t^t (\mathscr{G}y((\overline{q}), \xi') + g'(y(\overline{q}))\xi')\mathrm{d}s(\overline{c} + \overline{d}) + \xi', \phi' \right)_2 \mathrm{d}t. \tag{120}$$

For convenience of representation, the scalar product on H or antiduality $V_i, V_i'(i = 1,2)$ is denoted as $[\cdot,\cdot]$. Let $\{\rho_n\}$ be a mollifying sequence on $\mathbb{R}_t$. Then, the right hand side of (120) equals

$$\begin{aligned} &\lim_{n \longrightarrow \infty} \int_{\mathbb{R}_t} \left[\left(\left(M_\gamma \xi'' + A\xi - \int_t^T (\mathscr{G}(y(\overline{q}), \xi') + g'(y(\overline{q}))\xi')\mathrm{d}s + (\overline{c} + \overline{d})\xi' \right) * \rho_n, \phi' * \rho_n \right) \right] \mathrm{d}t \\ &= \lim_{n \longrightarrow \infty} X_n. \end{aligned} \tag{121}$$

X_n may be integrated by parts to obtain

$$\begin{aligned} X_n = \int_{\mathbb{R}_t} \Big(&[-\xi' * \rho_n, M_\gamma \phi'' * \rho_n] + [(A\xi) * \rho_n, \phi' * \rho_n] \\ &- [(\mathscr{G}(y(\overline{q}), \xi') + g'(y(\overline{q}))\xi') * \rho_n, \phi * \rho_n] + \big([(\overline{c} + \overline{d})\xi]' * \rho_n, \phi' * \rho_n \big] \Big) \mathrm{d}t. \end{aligned} \tag{122}$$

Using

$$M_\gamma \phi'' = -A\phi - \mathscr{G}(y(\overline{q}), \phi) - g'(y(\overline{q}))\phi + (\overline{c} + \overline{d})\phi' + (l+h)y'(\overline{q}). \tag{123}$$

(122) can be given again by

$$X_n = \int_{\mathbb{R}_t} [-\xi' * \rho_n, ((l+h)y'(\overline{q})) * \rho_n]\mathrm{d}t + \sum_{i=1}^{4} Y_n^i, \tag{124}$$

where

$$\begin{aligned} Y_n^1 &= \int_{\mathbb{R}_t} ([\xi' * \rho_n, (A\phi) * \rho_n] + [(A\xi) * \rho_n, \phi' * \rho_n])\mathrm{d}t, \\ Y_n^2 &= \int_{\mathbb{R}_t} ([\xi' * \rho_n, (\mathscr{G}(y(\overline{q}), \phi)) * \rho_n] - [(\mathscr{G}(y(\overline{q}), \xi')) * \rho_n, \phi * \rho_n])\mathrm{d}t, \\ Y_n^3 &= \int_{\mathbb{R}_t} ([\xi' * \rho_n, (g'(y(\overline{q}))\phi) * \rho_n] - [(g'(y(\overline{q})\xi')) * \rho_n, \phi * \rho_n])\mathrm{d}t, \\ Y_n^4 &= \int_{\mathbb{R}_t} \left(\left[\left((\overline{c} + \overline{d})\xi'\right) * \rho_n, \phi' * \rho_n\right] - \left[\xi' * \rho_n, \left((\overline{c} + \overline{d})\phi'\right) * \rho_n\right]\right)\mathrm{d}t. \end{aligned} \tag{125}$$

We immediately know $Y_n^1 = 0$ by integration by parts. Since

$$\begin{aligned} \int_{\mathbb{R}_t} [\xi' * \rho_n, (\mathscr{G}(y(\overline{q}), \phi)) * \rho_n]\mathrm{d}t &\longrightarrow \int_{\mathbb{R}_t} \xi', \mathscr{G}[(y(\overline{q}), \phi)]\mathrm{d}t \\ &= \int_{\mathbb{R}_t} [\mathscr{G}(y(\overline{q}), \xi'), \phi]\mathrm{d}t, \end{aligned} \tag{126}$$

as $n \longrightarrow \infty$. It can be deduced that $Y_n^2 \longrightarrow 0$ as $n \longrightarrow \infty$. In a similar manner, $Y_n^3 \longrightarrow 0$ and $Y_n^4 \longrightarrow 0$ as $n \longrightarrow \infty$. This justifies

$$\begin{aligned} \lim_{n \longrightarrow \infty} X_n &= \int_{\mathrm{R}_t} [-\xi', (l+h)y'(\overline{q})]\mathrm{d}t \\ &= \left(-p', (l+h)y'(\overline{q})\right)_{L^2(0,T;H)}. \end{aligned} \tag{127}$$

Since $\overline{q} = (\overline{c}, \overline{d}) \in \mathscr{W}_{ad}$ is an optimal pair in (51), the following equation is obtained:

$$\begin{aligned} &D_c J(\overline{c}, \overline{d})(l) \geq 0, \\ &D_d J(\overline{c}, \overline{d})(h) \leq 0, \\ &(l,h) \in \left[L^2(0,T;H)\right]^2. \end{aligned} \tag{128}$$

From (116), (120), and (127), (128) implies

$$\begin{aligned} &(\alpha\overline{c} - y'(\overline{q})p', l)_{L^2(0,T;H)} \geq 0, \\ &\left(-\beta\overline{d} - y'(\overline{q})p', h\right)_{L^2(0,T;H)} \leq 0, \end{aligned} \tag{129}$$

where $(l,h) \in [L^2(0,T;H)]^2$. By referring to the proof of ([16], Theorem 4.2), it can be deduced from (129) that the optimal pair $\overline{q} = (\overline{c}, \overline{d})$ can be given by

$$\begin{aligned} \overline{c} &= \max \left\{c_1, \min\left\{\frac{y'(\overline{q})p'}{\alpha}, c_2\right\}\right\}, \\ \overline{d} &= \max \left\{d_1, \min\left\{-\frac{y'(\overline{q})p'}{\beta}, d_2\right\}\right\}. \end{aligned} \tag{130}$$

Hence, the proof is concluded. □

6.2. *Proof for the Case of* $M = L^2(0,T;H) \times H$ *and* $\mathscr{C}y(q) = (y(q), y(q;T))$.

Before proceeding the proof of Theorem 4, the following remark is noted.

Remark 5. In the case of observation in Theorem 4, adjoint equations cannot be constructed directly, because they must involve the term $-((\overline{c} + \overline{d})p)'$, as shown in (58), where p is an adjoint state. However, the regularity of the admissible set is insufficient to guarantee the existence of adjoint equations. This difficulty is overcome by employing the modified transposition method.

The transposition method is explained. For any given $y(\overline{q}) = y(\overline{c}, \overline{d}) \in \mathscr{S}(0,T)$, the following equation is considered:

$$\begin{cases} M_\gamma \phi'' + A\phi + \mathcal{G}(y(\bar{q}),\phi) + g'(y(\bar{q}))\phi = (\bar{c}+\bar{d})\phi' + f \text{ in } (0,T), \\ \phi(0) = 0, \phi'(0) = 0, \end{cases} \tag{131}$$

where $f \in L^2(0,T;V_1')$, $(\bar{c},\bar{d}) \in [L^2(0,T;H)]^2$, and

$$\mathcal{G}(y(\bar{q}),\phi) \coloneqq 2((y(\bar{q}),(\phi))_1 A^{(1/2)} y(\bar{q}) + \left(1+\|y(\bar{q})\|_{V_1}^2\right) A^{(1/2)}\phi. \tag{132}$$

From Theorem 2, it is evident that Equation (131) has a unique weak solution $\phi \in \mathcal{S}(0,T)$ satisfying

$$\|\phi\|_{\mathcal{S}(0,T)} \le C\|f\|_{L^2(0,T;V_1')}. \tag{133}$$

Thus, the space $\mathcal{X} = \{\phi | \phi \text{ satisfies Equation (131) as } f \text{ ranges over } L^2(0,T;V_1')\}$.

Then,

$$\mathcal{X} \subset \mathcal{S}(0,T). \tag{134}$$

Endowed with the norm $\|z\|_{\mathcal{X}} = \|f\|_{L^2(0,T;V_1')}$, $\mathcal{X}$ is the Hilbert space, and the map L defined by

$$L(\phi) = M_\gamma \phi'' + A\phi + \mathcal{G}(y(\bar{q}),\phi) + g'(y(\bar{q}))\phi - (\bar{c}+\bar{d})\phi', \tag{135}$$

is an isomorphism from $\mathcal{X}$ onto $L^2(0,T;V_1')$. Therefore, for any continuous linear functional l on $\mathcal{X}$, there exists unique $\eta \in L^2(0,T;V_1)$ such that

$$\int_0^T \langle \eta, L(\phi)\rangle V_1, V_1' dt = l(\phi)\forall, \quad \phi \in \mathcal{X}. \tag{136}$$

We call such a η to be a solution by the transposition method associated with the bounded linear functional l on $\mathcal{X}$ for the transposed equation with respect to L.

To discuss the existence of a solution to (formal) adjoint system Equation (58), l in (136) is taken as follows:

$$l(\phi) = (y(\bar{q}) - Y_d, \phi)_{L^2(0,T;H)} + \left(y(\bar{q};T) - Y_d^T, \phi(T)\right)_2 \forall\phi \in \mathcal{X}. \tag{137}$$

It is easily verified that

$$\begin{aligned} \left|(y(\bar{q}) - Y_d, \phi)_{L^2(0,T;H)}\right| &\le \left\|y(\bar{q}) - Y_d\right\|_{L^2(0,T;H)} \|\phi\|_{L^2(0,T;H)} \\ &\le C\left\|y(\bar{q}) - Y_d\right\|_{L^2(0,T;H)} \|\phi\|_{\mathcal{S}(0,T)} \\ &\le C\left\|y(\bar{q}) - Y_d\right\|_{L^2(0,T;H)} \|L(\phi)\|_{L^2(0,T;V_1')} \\ &\equiv C\left\|y(\bar{q}) - Y_d\right\|_{L^2(0,T;H)} \|\phi\|_{\mathcal{X}}, \\ \left|\left(y(\bar{q};T) - Y_d^T, \phi(T)\right)_2\right| &\le \left\|y(\bar{q};T) - Y_d^T\right\|_2 \|\phi(T)\|_2 \\ &\le \left\|y(\bar{q};T) - Y_d^T\right\|_2 \|\phi\|_{C([0,T];H)} \\ &\le C\left\|y(\bar{q};T) - Y_d^T\right\|_2 \|\phi\|_{\mathcal{S}(0,T)} \\ &\le C\left\|y(\bar{q};T) - Y_d^T\right\|_2 \|L(\phi)\|_{L^2(0,T;V_1')} \\ &\equiv C\left\|y(\bar{q};T) - Y_d^T\right\|_2 \|\phi\|_{\mathcal{X}}. \end{aligned} \tag{138}$$

Hence, from (137)–(139), the following equation is obtained:

$$|l(\phi)| \le C\left(\left\|y(\bar{q}) - Y_d\right\|_{L^2(0,T;H)} + \left\|y(\bar{q};T) - Y_d^T\right\|_2\right)\|\phi\|_{\mathcal{X}} \forall z \in \mathcal{X}. \tag{139}$$

As a consequence, the functional l is linear and bounded on $\mathcal{X}$. Also, it is deduced that the map $(y(\bar{q}) - Y_d, y(q^*;T) - Y_d^T) \longrightarrow p$ is continuous from $L^2(0,T;H) \times H$ to $L^2(0,T;V_1)$. Hence, the following proposition is obtained.

Proposition 5. *For given $y(\bar{q}) - Y_d \in L^2(0,T;H)$ and $y(\bar{q};T) - Y_d^T \in H$, there exists a unique solution $p \in L^2(0,T;V_1)$ of Equation (58) such that*

$$\int_0^T \langle p, L(\phi)\rangle_{V_1,V_1'} dt = l(\phi)\forall\phi \in \mathcal{X}, \tag{140}$$

where L is defined in (135) and $l(\phi)$ is defined in (137). Moreover, the solution p of (141) satisfies

$$\|p\|_{L^2(0,T;V_1)} \le C\left(\left\|y(\bar{q}) - Y_d\right\|_{L^2(0,T;H)} + \left\|y(\bar{q};T) - Y_d^T\right\|_2\right). \tag{141}$$

Proof of Theorem 2. Let $\bar{q} = (\bar{c},\bar{d}) \in \mathcal{W}_{ad}$ be an optimal pair in (51) with the cost (50) through the observation case of Theorem 4 and $y(\bar{q})$ be the corresponding weak solution of (49).

The Gâteaux derivative of the cost (50) through the observation case of Theorem 4 at $q = \bar{q}$ in the direction $w =$

$(l,h) \in [L^2(0,T;H)]^2$, with $\overline{q} + \varepsilon w \in \mathscr{W}_{ad}$ for sufficiently small $\varepsilon > 0$, yields

$$\begin{aligned}
\mathrm{DJ}(\overline{c},\overline{d})(l,h) &= \lim_{\varepsilon \longrightarrow 0^+} \frac{J(\overline{c}+\varepsilon l, \overline{d}+\varepsilon h) - J(\overline{c},\overline{d})}{\varepsilon} \\
&= \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{2} \int_0^T \left(y(\overline{q}+\varepsilon w) + y(\overline{q}) - 2Y_d, \frac{y(\overline{q}+\varepsilon w) - y(\overline{q})}{\varepsilon} \right)_2 \mathrm{d}t \\
&\quad + \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{2} \left(y(\overline{q}+\varepsilon w;T) + y(\overline{q};T) - 2Y_d^T, \frac{y(\overline{q}+\varepsilon w;T) - y(\overline{q};T)}{\varepsilon} \right)_2 \\
&\quad + \lim_{\varepsilon \longrightarrow 0^+} \left[\frac{\alpha}{2} \int_0^T \left(2(\overline{c},l)_2 + \varepsilon \|l\|_2^2 \right) \mathrm{d}t - \frac{\beta}{2} \int_0^T \left(2(\overline{d},h)_2 + \varepsilon \|h\|_2^2 \right) \mathrm{d}t \right] \\
&= \left(y(\overline{q}) - Y_d, z \right)_{L^2(0,T;H)} + \left(y(\overline{q};T) - Y_d^T, z(T) \right)_2 + \alpha(\overline{c},l)_{L^2(0,T;H)} - \beta(\overline{d},h)_{L^2(0,T;H)},
\end{aligned} \tag{142}$$

where $z = Dy(\overline{q})w$ is a weak solution of Equation (105). Hence, if the author takes $\phi = z$ in (141), then it is inferred from (142) that

$$\begin{aligned}
\mathrm{DJ}(\overline{c},\overline{d})(l,h) = &\left(p, (l+h)y'(\overline{q}) \right)_{L^2(0,T;H)} \\
&+ \alpha(\overline{c},l)_{L^2(0,T;H)} - \beta(\overline{d},h)_{L^2(0,T;H)}.
\end{aligned} \tag{143}$$

Since $\overline{q} = (\overline{c},\overline{d}) \in \mathscr{W}_{ad}$ is an optimal pair in (50), the following equation is obtained:

$$\begin{aligned}
&D_c J(\overline{c},\overline{d})(l) \geq 0, \\
&D_d J(\overline{c},\overline{d})(h) \leq 0, \\
&(l,h) \in \left[L^2(0,T;H) \right]^2.
\end{aligned} \tag{144}$$

Thus, from (144) and (145), the following equation is obtained:

$$\begin{aligned}
&\left(\alpha\overline{c} + y'(\overline{q})p, l \right)_{L^2(0,T;H)} \geq 0, \\
&\left(-\beta\overline{d} + y'(\overline{q})p, h \right)_{L^2(0,T;H)} \leq 0,
\end{aligned} \tag{145}$$

where $(l,h) \in [L^2(0,T;H)]^2$. By analogy with the case of (130), it can be inferred that from (145), the optimal pair $\overline{q} = (\overline{c},\overline{d})$ can be given as (possibly not unique)

$$\begin{aligned}
\overline{c} &= \max \left\{ c_1, \min \left\{ -\frac{y'(\overline{q})p}{\alpha}, c_2 \right\} \right\}, \\
\overline{d} &= \max \left\{ d_1, \min \left\{ \frac{y'(\overline{q})p}{\beta}, d_2 \right\} \right\}.
\end{aligned} \tag{146}$$

Hence, the proof is concluded. □

7. Conclusion

In this study, the author investigated the bilinear minimax optimal control problems for an extensible beam equation with rotational inertia effects. The well posedness of the solution to the Cauchy problem and the improved regularity theorem were given for the underlying equation. The author formulated the bilinear minimax optimal control problem for the state equation by adding the bilinear control and noise inputs of the velocity term to the state equation. Using and analyzing the properties of the Fréchet derivative of the nonlinear solution map from the bilinear control and noise inputs to the solution of the equation, the author proved the existence of an optimal pair and found its necessary optimality conditions corresponding to the following physically meaningful observation cases. The main novelties of this study can be summarized as follows: In the case of velocity distribution observation, the double regularization method was used to derive the optimality condition for the optimal pair through the adjoint equation. For the distributive and terminal value observations, instead of assuming regularity for the data and control sets required to construct an explicit adjoint equation, the transposition method was used to derive the necessary optimality condition for the optimal pair.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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