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Research Article

System Level Extropy of the Past Life of a Coherent System

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In this paper, we take a new approach to uncertainty in a coherent system, where components are assumed to be all inactive in a given time. In particular, the signature-based method is used to quantify the extropy of the past lifetime of the system, which serves as a valuable indicator of its predictability. The results provide several key findings, including some bounds and stochastic ordering aspects for this measure. We also introduce a new formula to select the system that is preferable based on its relative extropy in the past. The results of this work can provide insights for designing systems to improve their reliability and resilience.

1. Introduction

An important research topic in many fields, including biology, survival analysis, reliability engineering, econometrics, statistics, and demography, is the analysis of distribution functions based on partial knowledge. The model selection, estimation, hypothesis testing, evaluation of inequality and poverty, and portfolio analysis are all relevant tasks in this area. The entropy of a probability distribution, which has numerous applications in information science, physics, probability, statistics, communication theory, and economics, is one of the most commonly used measurements in this field. The Shannon entropy of a nonnegative random variable (rv) with probability density function (pdf) f, introduced by Shannon in his seminal work [1], is defined as $H(X) = H(f) = -E[\log f(X)]$, provided the expectation exists.

Extropy, a complementary dual uncertainty measure developed by Frank et al. [2], has garnered more attention in recent years. The extropy measure is applicable to absolutely continuous, nonnegative rvs X supported on the interval $[0, \infty)$. The expression of it involves the survival function (sf) S given by $S(x) = P(X > x)$ and pdf f. The extropy of X is designated as follows:

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)\mathrm{d}x. \tag{1}$$

It is crucial for engineers to appropriately measure uncertainty across a system's lifetime. Systems with a longer lifespan and a lower uncertainty are generally thought to be superior since dependability declines as uncertainty rises (see, e.g., Ebrahimi and Pellerey [3]). Although $J(X)$ is a useful indicator of the lifetime (given by X) uncertainty of a new system, it might not be acceptable in circumstances when understanding of operators of the system's current age is limited. For instance, the $J(X)$ measure is no longer applicable if it is known that the system is operational at time t and want to quantify the uncertainty related to its previous lifetime, as signified by $Xt = X - t|X > t$. A novel metric known as residual extropy has been proposed to get around this restriction. Residual extropy is measured via the following formula:

$$J(X_t) = -\frac{1}{2}\int_t^\infty \left(\frac{f(x)}{S(t)}\right)^2 \mathrm{d}x, \tag{2}$$

for all $t > 0$. Extropy has been the subject of extensive investigation by several researchers, including Frank et al. [2], Qiu [4], Qiu and Jia [5], and others cited therein. In a recent study, Qiu et al. [6] examined an assertion for the extropy of the random lifetime due to a coherent system.

In real systems, there is often a pervasive element of uncertainty that affects both the past and the present. Against this background, a complementary concept of entropy has been established, which is distinct from residual entropy, which describes the uncertainty of future events and captures the uncertainty of past events. As can be seen from publications such as those by Di Crescenzo and Longobardi [7], Unnikrishnan Nair and Sunoj [8], and Kayid and Shrahili [9], the literature has paid considerable attention to the topic of past entropy and its statistical applications.

The research conducted by Gupta et al. [10] on the properties of historical entropy in the context of ordered random variables has led to a substantial advancement of the area. Gupta et al. [10] have studied, in particular, the past entropy and, moreover, the residual entropy of ordered random variables and have dealt with stochastic order aspects arising from these random variables that have provided new insight into past entropy's fundamental concepts and its use in statistical analysis.

This study includes a thorough analysis of extropy applied to the distribution of past lifetimes as well as a generalized version of equation (2). By introducing parameter 2, which allows for a variety of weightings of the conditional probabilities, our proposed measure allows for a nuanced comparison of the shapes of different distributions of past lifetimes. Our results demonstrate the immense potential of this measure to provide new insights into the underlying mechanics of these distributions, with applications beyond the scope of our current work.

Consider a coherent system with n components that are all inactive at time t in order to further explore the applicability of the measure we suggest in this study. We use the system signature approach to calculate the extropy of the coherent system's past-life distribution. With potential applications in network science, reliability engineering, and industrial systems, the findings have major ramifications for comprehending and simulating complex systems.

2. Results on the Past Extropy

We assume that the rv X stands as the lifetime of a system. Note that the pdf of $X_t = [X|X<t]$ is $f_t(x) = f(x)/F(t)$, so that $x \in (0,t)$. The past extropy of X at time t is derived by the following formula (see Krishnan et al. [11]):

$$\begin{aligned}\overline{J}(X_t) &= -\frac{1}{2}\int_0^{+\infty} f_t^2(x)\mathrm{d}x \\ &= -\frac{1}{2}\int_0^t \left(\frac{f(x)}{F(t)}\right)^2 \mathrm{d}x.\end{aligned} \tag{3}$$

It is to be mentioned here that the extropy $\overline{J}(X_t)$, of the past life, can take values from negative infinity to zero. The captivating research papers by Krishnan et al. [11], Kamari and Buono [12], and Toomaj et al. [13] delve into a plethora of fascinating topics concerning the investigation of past extropy. These works rigorously analyze expressions, bounds, characterizations, ageing properties, stochastic orders, and other valuable properties, providing a comprehensive understanding of this complex subject.

When a system fails, $\overline{J}(X_t)$ offers a measure of uncertainty regarding the system's previous lifetime, assuming it failed at time t. This metric is particularly useful for comparing random lives because it allows us to spot subtle differences in the morphologies of diverse distributions of past lifetimes. To illustrate the importance of past extropy in comparing random lifetimes, consider the next example.

Example 1. Let us contemplate two system components which have life lengths X and Y, with respective pdfs:

$$\begin{aligned}f(x) &= 2x, \quad x \in (0,1), \\ g(x) &= 2(1-x), \quad x \in (0,1).\end{aligned} \tag{4}$$

The extropy of X and, further, that of Y are appealingly recorded by knowing that $\overline{J}(X) = \overline{J}(Y) = -2/3$. This discovery has significant ramifications for anticipating the extropy measurements on rvs X and Y.

In particular, our results suggest that the expected uncertainty associated with X and Y is the same when pdfs f and g are assigned for these rvs. Let us assume that both components were found to be defective during an inspection at a time t that lies between 0 and 1. In such a case, we can use the notion of historical extropy to quantify the uncertainty associated with the relative failure times. Clearly, equation (3) can be taken into account in order to evaluate the past extropy by the identities:

$$\begin{aligned}\overline{J}(X_t) &= -\frac{2}{3t}, \\ \overline{J}(Y_t) &= \frac{2t^2 - 6t + 6}{3t - 6},\end{aligned} \tag{5}$$

for all $t \in (0,1)$. The results of our analysis, presented in Figure 1, reveal a compelling trend. Specifically, we demonstrate that the extropy of X_t is in command of Y_t for $t \in (0,1)$, despite the fact that $\overline{J}(X) = \overline{J}(Y)$.

It is understood from Equation (3) that the extropy of $[t-X|X\le t]$ (the idle time) is a striking finding. This alternate identification offers new insight into the system's underlying dynamics and works for a useful criterion for describing the extropy of the system's temporal behavior. In addition to this alternative identification, equation (3) also gives another formula for the past extropy, as follows:

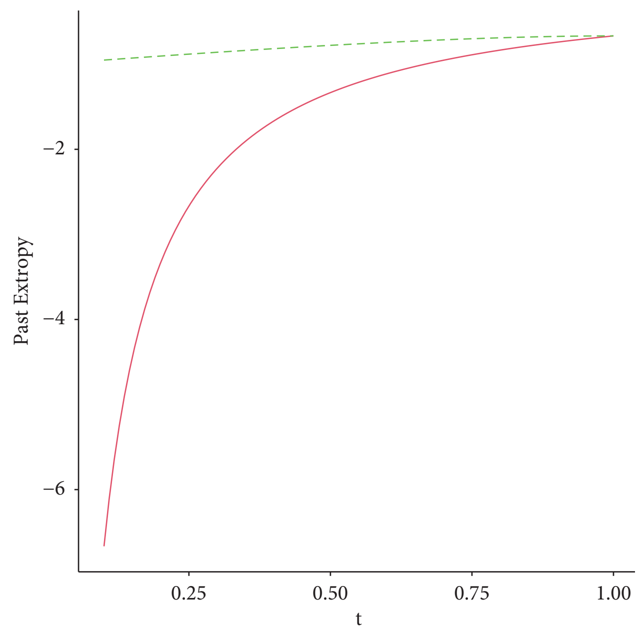

FIGURE 1: The extropy of $\overline{J}(X_t)$ (solid line) and $\overline{J}(Y_t)$ (dashed line) in Example 1.

$$\begin{aligned}\overline{J}(X_t) &= -\frac{1}{2}\int_0^t \frac{f^2(x)}{F^2(t)}\mathrm{d}x \\ &= -\frac{1}{2}\int_0^t \frac{f(x)}{F(x)}\frac{f(x)}{F(t)}\frac{F(x)}{F(t)}\mathrm{d}x \\ &= -\frac{1}{2}\int_0^{+\infty} \widetilde{r}(x) f_t(x) F_t(x)\mathrm{d}x \\ &= -\frac{1}{4}\int_0^{+\infty} \widetilde{r}(x) f_{2,t}(x)\mathrm{d}x \\ &= -\frac{1}{4}E\left[\widetilde{r}\left(X_{2,t}\right)\right],\end{aligned} \tag{6}$$

in which $\widetilde{r}(x) = d/dx \ln(F(x))$ stands as the reversed hazard rate of X and $X_{2,t}$ that follows the pdf

$$f_{2,t}(x) = 2 f_t(x) F_t(x), \tag{7}$$

such that $F_t(x) = F(x)/F(t)$ for all $x \in (0, t)$. Monotonicity of past extropy is a fundamental property of rvs that exhibit the decreasing reverse hazard rate (DRHR) property. This property, defined by the decreasing property of $\widetilde{r}(x)$ for all $x > 0$, is commonly observed in a number of real-world applications. Here, we establish a theorem that provides fresh insight into how historical extropy behaves when the DRHR property is present. In particular, we show that the past extropy of a DRHR rv increases monotonically over time and provides a formal proof of this result.

Theorem 1. *If X has a distribution with DRHR property, then $\overline{J}(X_t)$ increases in $t > 0$.*

Proof. Let us differentiate (3) in terms of t to obtain

$$\begin{aligned}-2\overline{J}'(X_t) &= \widetilde{r}^2(t) - 2\widetilde{r}(t)\int_0^t \frac{f^2(x)}{F^2(t)}\mathrm{d}x \\ &= \widetilde{r}^2(t) - \widetilde{r}(t)\int_0^t \widetilde{r}^2(x) f_{2,t}(x)\mathrm{d}x,\end{aligned} \tag{8}$$

where $f_{2,t}(x)$ is acquired from (7). If it is known that the distribution of X induces the DRHR property, then $\widetilde{r}(x)$ satisfies the inequality $\widetilde{r}(x) \geq \widetilde{r}(t)$ for all $x \leq t$. Now, appealing to (8), we obtain

$$\widetilde{r}^2(t) - \widetilde{r}(t)\int_0^t \widetilde{r}(x) f_{2,t}(x)\mathrm{d}x \leq 0, \tag{9}$$

which can be rearranged as follows:

$$-2\overline{J}'(X_t) \leq 0, \tag{10}$$

where $\overline{J}'(X_t)$ indicates the derivative of the past extropy in terms of t. It follows that $\overline{J}'(X_t)$ increases t which completes the proof. □

Below, we provide an upper bound for $\overline{J}(X_t)$ on the basis of $\widetilde{r}$.

Theorem 2. *If $\widetilde{r}(x) < +\infty$ and distribution of X induces DRHR property, then it holds that*

$$\overline{J}(X_t) \leq -\frac{\widetilde{r}(t)}{4}, \quad t > 0. \tag{11}$$

Proof. The DRHR property yields $-\widetilde{r}$ as an increasing function. Thus, for all $x \in (0, t)$, one has $-\widetilde{r}(x) \leq -\widetilde{r}(t)$, and thus in spirit of (4), one obtains

$$\begin{aligned}\overline{J}(X_t) &= \frac{1}{4}\int_0^t (-\widetilde{r}(x)) f_{2,t}(x)\mathrm{d}x \\ &\leq \frac{-\widetilde{r}(t)}{4}\int_0^t f_{2,t}(x)\mathrm{d}x \\ &= -\frac{\widetilde{r}(t)}{4},\end{aligned} \tag{12}$$

where the last equality follows from the identity $\int_0^t f_{2,t}(x)\mathrm{d}x = 1$. □

3. Extropy of the Past Lifetime of the Coherent System

Assuming that every component has failed at a given time, we show how to use of the signature-based approach to determine the past lifespan extropy of a coherent system of any structure. A coherent system satisfies the conditions of being devoid of superfluous parts and having a monotonic structure function. This system is recognized by the signature vector $\mathbf{p} = (p_1, \ldots, p_n)$, in which the i th element $p_i = P(T = X_{i:n})$ serves as the probability that the i th component in the system is the last failed component (see [14]).

Contemplate a coherent system in which component lifetimes are denoted by $X_1, \ldots, X_n$ and they are independent and identically distributed (i.i.d.). Let us assume that the signature vector $\mathbf{p} = (p_1, \ldots, p_n)$ is due to the considering system. Let $T_t = [t - T \mid X_{n:n} \leq t]$ stands as the past lifetime of the coherent system, considering that components of the system are all inactive at time t. From Khaledi and Kochar [15], the sf of T_t is derivable as follows:

$$P(T_t > x) = \sum_{i=1}^{n} p_i P\left(T_t^i > x\right), \tag{13}$$

where

$$P\left(T_t^i > x\right) = P\left(t - X_{i:n} > x \middle| X_{n:n} \leq t\right) = \sum_{k=i}^{n} \binom{n}{k} \left(\frac{F(t-x)}{F(t)}\right)^k \left(1 - \frac{F(t-x)}{F(t)}\right)^{n-k}, \quad 0 < x < t, \tag{14}$$

signifies the sf of the past lifetime due to an i-out-of-n system seeing that the components are all idle at time t. From (13), we obtain

$$f_{T_t}(x) = \sum_{i=1}^{n} p_i f_{T_t^i}(x), \tag{15}$$

where

$$f_{T_t^i}(x) = d_i \left(\frac{F(t-x)}{F(t)}\right)^{i-1} \left(1 - \frac{F(t-x)}{F(t)}\right)^{n-i} \frac{f(t-x)}{F(t)}, \quad x \in (0, t), \tag{16}$$

where $d_i = \Gamma(n+1)/\Gamma(i)\Gamma(n-i+1)$ in which $\Gamma(\cdot)$ is the well-known gamma function and $T_t^i = [t - X_{i:n} \mid X_{n:n} \leq t], i = 1, 2, \ldots, n$, is the time elapsed since failure of the component in the system which has a lifetime $X_{i:n}$ provided that the system has become inactive prior to time t. It is noteworthy from (13) that T_t^i signifies the i th order statistics arisen from of n i.i.d. components lifetimes with the common cumulative distribution function (cdf) $F(t-x)/F(t), x \in (0, t)$. We now give a formula for the extropy of T_t. Now, we set $F_t(x) = F(x)/F(t), 0 < x < t$. The transformation $V = F_t(T_t)$ plays a necessary role in our context. It is evidently seen that $U_{i:n} = F_t(T_t^i)$ has a beta distribution with parameters i and $n - i + 1$. Next, we give a formula for the extropy of T_t.

Theorem 3. *Let T_t be the past life of the system given that, at time t, all components in the system have become inactive. Then, the extropy of T_t is*

$$\bar{J}(T_t) = -\frac{1}{2} \int_0^1 g_V^2(u) f_t\left(F_t^{-1}(u)\right) \mathrm{d}u, \tag{17}$$

for all $t > 0$.

Proof. Let us assume that $g_V(u) = \sum_{i=1}^{n} p_i g_i(u)$, in which g_i is the pdf of a beta distribution with parameters i and $n - i + 1$. We denoted by g_V the density of the rv V which denotes the lifetime of a system having a same structure function as the underlying system, in which components which have i.i.d. lifetimes according to the uniform distribution. By combining (1) and (8) and making the change $z = t - x$, one obtains

$$\begin{aligned}
\bar{J}(T_t) &= -\frac{1}{2} \int_0^t \left(f_{T_t}(x)\right)^2 \mathrm{d}x \\
&= -\frac{1}{2} \int_0^t \left(\sum_{i=1}^{n} p_i f_{T_t^i}(x)\right)^2 \mathrm{d}x \\
&= -\frac{1}{2} \int_0^t \left(\sum_{i=1}^{n} p_i d_i \left(\frac{F(t-x)}{F(t)}\right)^{i-1} \left(1 - \frac{F(t-x)}{F(t)}\right)^{n-i} \frac{f(t-x)}{F(t)}\right)^2 \mathrm{d}x \\
&= -\frac{1}{2} \int_0^t \left(\sum_{i=1}^{n} p_i d_i \left(F_t(z)\right)^{i-1} \left(1 - F_t(z)\right)^{n-i} f_t(z)\right)^2 \mathrm{d}z \\
&= -\frac{1}{2} \int_0^1 g_V^2(u) f_t\left(F_t^{-1}(u)\right) \mathrm{d}u.
\end{aligned} \tag{18}$$

The last identity results from the change of the variable z into $u = F_t(z)$. □

Theorem 3 presents a statement for the extropy of a system's lifespan provided that component failures at a certain moment make a significant advance to our understanding of system dependability. This expression is particularly helpful in circumstances when component failures happen at predetermined times for assessing the strength of uncertainty behind the randomness in the lifetime of a system.

If we assume that all components are idle at time t, then $\overline{J}(T_t)$ is capable to measure the uncertainty which is expected to be induced via the conditional pdf of $t - T$ under the condition that $X_{n:n} \leq t$. Let us consider an i-out-of-n system with signature $\mathbf{p} = (0, \ldots, 0, 1_i, 0, \ldots, 0), i = 1, 2, \ldots, n$, then equation (18) brings to

$$\overline{J}(T_t) = -\frac{1}{2}\int_0^1 g_i^2(u) f_t\left(F_t^{-1}(u)\right)\mathrm{d}u, \quad t > 0. \tag{19}$$

The following result is an immediate consequence of Theorem 3.

Theorem 4. *If X has a distribution with the DRHR property, then $\overline{J}(T_t)$ increases in t.*

Proof. By noting that $f_t(F_t^{-1}(x)) = x\tilde{r}_t(F_t^{-1}(x))$, equation (19) can be rewritten as

$$-2\overline{J}(T_t) = \int_0^1 g_V^2(u)u\tilde{r}_t\left(F_t^{-1}(u)\right)\mathrm{d}u, \tag{20}$$

for all $t > 0$. We plainly observe that $F_t^{-1}(u) = F^{-1}(uF(t))$, whenever $u \in (0, 1)$, and as a result:

$$\tilde{r}_t\left(F_t^{-1}(u)\right) = \tilde{r}\left(F^{-1}(uF(t))\right), \quad 0 < u < 1. \tag{21}$$

If $t_1 \leq t_2$, then $F^{-1}(uF(t_1)) \leq F^{-1}(uF(t_2))$. Therefore, as X has a distribution with DRHR property, thus

$$\begin{aligned}\int_0^1 g_V^2(u)u\left(\tilde{r}_{t_1}\left(F_{t_1}^{-1}(u)\right)\right)\mathrm{d}u &= \int_0^1 g_V^2(u)u\left(\tilde{r}\left(F^{-1}(uF(t_1))\right)\right)\mathrm{d}u \\ &\geq \int_0^1 g_V^2(u)u\left(\tilde{r}\left(F^{-1}(uF(t_2))\right)\right)\mathrm{d}u \\ &= \int_0^1 g_V^2(u)u\left(\tilde{r}_{t_2}\left(F_{t_2}^{-1}(u)\right)\right)\mathrm{d}u,\end{aligned} \tag{22}$$

for all $t_1 \leq t_2$. Using (18), we obtain

$$-2\overline{J}\left(T_{t_1}\right) \geq -2\overline{J}\left(T_{t_2}\right). \tag{23}$$

This implies that $\overline{J}(T_{t_1}) \leq \overline{J}(T_{t_2})$. □

The below example is given to apply Theorems 3 and 4.

Example 2. We consider a coherent system of order 4, depicted in Figure 2, in which the component lifetimes are independently and identically distributed with a common cdf $F(x) = e^{-x^{-k}}$, where $k > 0$. The signature vector for this coherent system can be evaluated as $\mathbf{p} = (0, 1/6, 7/12, 1/4)$.

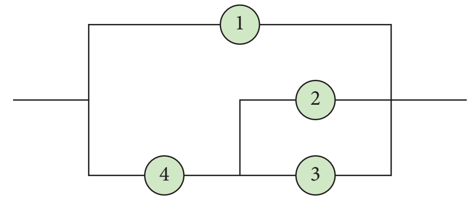

Figure 2: A coherent system with signature $\mathbf{p} = (0, 1/6, 7/12, 1/4)$.

To compute the exact value of the expected lifetime $\overline{J}(T_t)$, we use relation (18) to obtain the following expression:

$$\overline{J}(T_t) = -\frac{k}{2}\int_0^1 \left(t^{-k} - \log u\right)^{1/k+1} u g_V^2(u)\mathrm{d}u, \quad t > 0. \tag{24}$$

Although it is not that easy to derive a perfectly clear statement for (19), we can obtain meaningful results through numerical calculations. Specifically, we consider some values of $k > 0$ and calculate the extropy of T_t as a function of time t using numerical methods. Figure 3 shows the resulting extropy values for T_t versus time. Remarkably, the underlying cdf F induces DRHR property, for all $k > 0$, as stated in Theorem 4. Consistent with this theorem, we observe that $\overline{J}(T_t)$ increases with t for $k > 0$, as shown in Figure 3.

The description given earlier sheds important insights on the intricate connection between a rv's extropy and time and highlights the importance of considering the DRHR property while scrutinizing such systems. The findings imply that the DRHR feature of X is key in determining how the extropy of T_t behaves over time, which has significant implications for a variety of applications. In particular, the results may lead to a substantial understanding of complex systems in which the DRHR property of component lifetimes is common. By characterizing the temporal behavior of the extropy of coherent systems with DRHR components, we can gain deeper insights into the performance and reliability of such systems. Overall, the given example highlights the power of information-theoretic methods for analyzing complex systems and underscores the importance of considering the underlying distributional properties of component lifetimes in such analyzes.

In order to substantial reduction of the complicated computations involved in identifying the signature vector, it is useful to think of a system as having dualities. Kochar et al. [16] presented a duality link between a system's signature and that of its dual, and it can be utilized to simplify the derivation of the past extropy for coherent systems. In more detail, if $\mathbf{p} = (p_1, \ldots, p_n)$ signifies the signature of a coherent system with lifetime T, then $\mathbf{p}^D = (p_n, \ldots, p_1)$ gives the signature of its dual system with lifetime T^D. For coherent systems, the calculation of the past extropy can be made easier using this duality characteristic. The duality principle is used in the following theorem to produce a condensed expression for the past extropy of coherent systems. The following lemma will be useful in our end.

Lemma 5. *Let ϕ be a continuous function on (0,1) so that $\int_0^1 x^n\phi(x)\mathrm{d}x = 0$ for all $n \geq 0$. Then, it holds that $\phi(x) = 0$ for every $x \in [0, 1]$.*

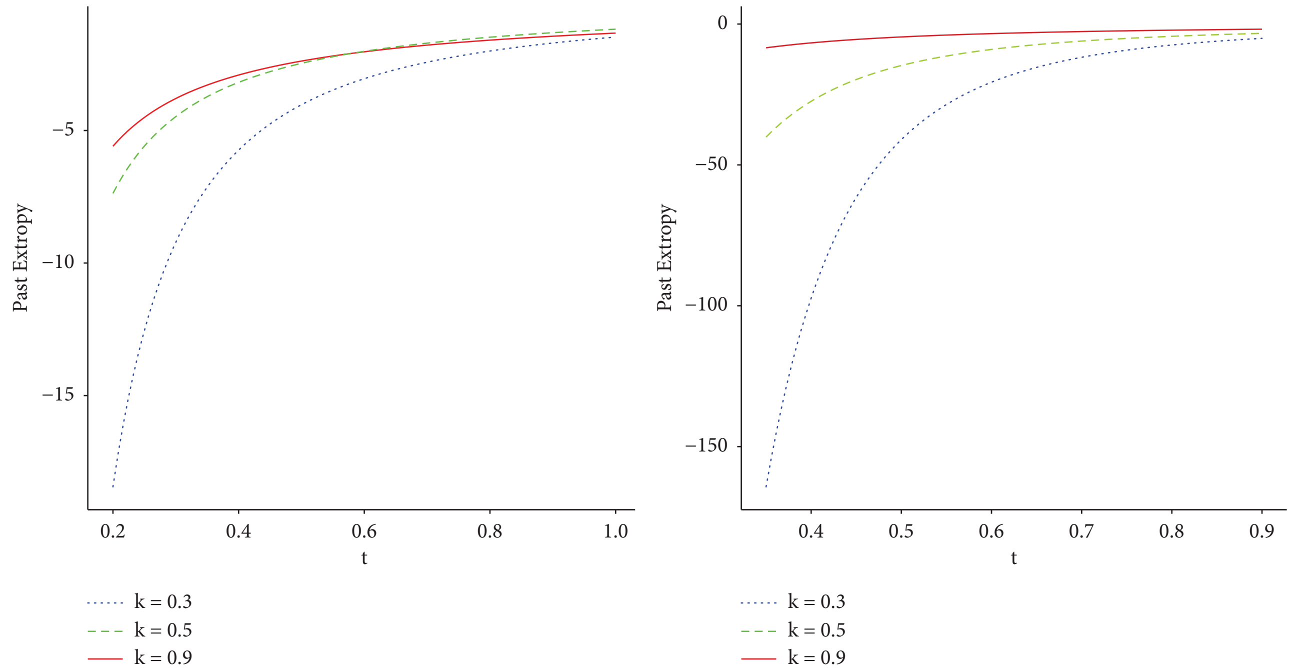

Figure 3: Exact value of $\overline{J}(T_t)$ in Example 1 for various values of k.

Theorem 6. *Let T_t denote the life length a coherent system with signature* **p** *has, assuming that it has n components with i.i.d. lifetimes. If $f_t(F_t^{-1}(u)) = f_t(F_t^{-1}(1-u))$ satisfies for all $u \in (0,1)$ and for all $t > 0$, then $\overline{J}(T_t) = \overline{J}(T_t^D)$ for all* **p** *and every n.*

Proof. We first prove the sufficient part. To this end, let $f_t(F_t^{-1}(u)) = f_t(F_t^{-1}(1-u))$ for all $u \in (0,1)$. Remark that $g_i(1-u) = g_{n-i+1}(u)$ for all $i = 1, \ldots, n$ and $u \in (0,1)$. Therefore, in spirit of (11), one can derive

$$\begin{aligned}
\int_0^1 g_{V^D}^2(u) f_t\left(F_t^{-1}(u)\right)\mathrm{d}u &= -\int_0^1 \left(\sum_{i=1}^n p_{n-i+1} g_i(u)\right)^2 f_t\left(F_t^{-1}(u)\right)\mathrm{d}u \\
&= \int_0^1 \left(\sum_{r=1}^n p_r g_{n-r+1}(u)\right)^2 f_t\left(F_t^{-1}(u)\right)\mathrm{d}u \\
&= \int_0^1 \left(\sum_{r=1}^n p_r g_r(1-u)\right)^2 f_t\left(F_t^{-1}(u)\right)\mathrm{d}u \\
&= \int_0^1 \left(\sum_{r=1}^n p_r g_r(u)\right)^2 f_t\left(F_t^{-1}(u)\right)\mathrm{d}u \\
&= \int_0^1 g_V^2(u) f_t\left(F_t^{-1}(u)\right)\mathrm{d}u.
\end{aligned} \tag{25}$$

Now, appealing to equation (18) will complete the proof. □

For the i-out-of-n systems, an immediate consequence of the previous theorem is given below.

Corollary 7. *Suppose that T_t^i is the life length of an i-out-of-n system having n i.i.d. components. Let $f_t(F_t^{-1}(u)) = f_t(F_t^{-1}(1-u))$ holds true for all $u \in (0,1)$ and for every $t > 0$. Then, $\overline{J}(T_t^i) = \overline{J}(T_t^{n-i+1})$ for all n and $i = 1, 2, \ldots, n/2$ if n is even and, moreover, $i = 1, 2, \ldots, (n-1)/2$ if n is odd.*

4. Bounds for Extropy of the Past Lifetime

In complex systems, where the number of components is large, calculating the expected lifetime $\overline{J}(T_t)$ is a challenging task. This situation occurs frequently in practice, and it might be challenging to find reliable estimates of the system lifetime. Utilizing a prior extropy bound to estimate the lifetime of the coherent system is one way to get around this problem. Past extropy bounds have been shown to be effective in estimating the uncertainty in the lifetime of complex systems, as demonstrated in recent research studies [6, 17]. In the next result, we present bounds on the past extropy of the system with respect to the past extropy of the parent distribution $\overline{J}(X_t)$. These constraints are a helpful tool for determining how long a coherent system will last in complicated systems with many of constituent parts.

Theorem 8. *Let $T_t = [t - T \mid X_{n:n} \le t]$ signify the past lifetime of the system. Components are assumed to have a common sf S and the signature of the system is $\mathbf{p} = (p_1, \ldots, p_n)$. Suppose that $\overline{J}(T_t) < \infty$ for all t. It holds that*

$$\overline{J}(T_t) \ge (B_n(\mathbf{p}))^2 \overline{J}(X_t), \tag{26}$$

where $B_n(\mathbf{p}) = p_1 g_1(m_1) + \cdots + p_n g_n(m_n)$ and $m_i = i - 1/n - 1$.

Proof. It is plain to observe that the mode of the beta distribution with parameters i and $n - i + 1$ is $m_i = i - 1/n - 1$. Therefore, we obtain

$$g_V(v) \le p_1 g_1(m_1) + \cdots + p_n g_n(m_n) = B_n(\mathbf{p}), v \in (0,1). \tag{27}$$

Thus, one has

$$\begin{aligned} -2\overline{J}(T_t) &= \int_0^1 g_V^2(v) f_t\left(F_t^{-1}(v)\right) dv \\ &\le (B_n(\mathbf{p}))^2 \int_0^1 f_t\left(F_t^{-1}(v)\right) dv \\ &= -2(B_n(\mathbf{p}))^2 \overline{J}(X_t). \end{aligned} \tag{28}$$

The last equality is obtained by noting that

$$\overline{J}(X_t) = \int_0^1 f_t\left(F_t^{-1}(v)\right) dv, \tag{29}$$

by which the final result is validated. □

When a system has a complex structure or a large number of components, the supplied bound in (24) is quite useful. Now, using extropy measure, we derive a public lower bound.

Theorem 9. *Underneath the requirements of Theorem 8, we have*

$$\overline{J}(T_t) \ge \sum_{i=1}^n p_i \overline{J}\left(T_t^i\right), \tag{30}$$

for all t.

Proof. From Jensen's inequality when applying to the concave function $k(t) = -t^2/2$, we obtain

$$-\frac{1}{2}\left(\sum_{i=1}^n p_i f_{T_t^i}(x)\right)^2 \ge -\frac{1}{2}\sum_{i=1}^n p_i f_{T_t^i}^2(x), \quad x, t > 0, \tag{31}$$

and hence

$$\begin{aligned} \overline{J}(T_t) &= -\frac{1}{2}\left(\int_0^t f_{T_t}^2(x) dx\right) \\ &\ge -\frac{1}{2}\left(\sum_{i=1}^n p_i \int_0^t f_{T_t^i}^2(x) dx\right) \\ &= \sum_{i=1}^n p_i \overline{J}\left(T_t^i\right). \end{aligned} \tag{32}$$

Here, this completes the proof. □

Remark that equality in (28) is satisfied for i-out-of-n systems as $p_j = 0$, for $j \ne i$, and $p_j = 1$, for $j = i$. That is, $\overline{J}(T_t) = \overline{J}(T_t^i)$. When the lower bounds in Theorems 8 and 9 are computable, the maximum value can be considered as a sharper lower bound.

Example 3. Suppose that $T_t = [t - T | X_{5:5} \le t]$ indicates the past lifetime of a system with signature $\mathbf{p} = (0, 3/10, 5/10, 2/10, 0)$ consisting of $n = 5$ i.i.d. component lifetimes having a common cdf $F(x) = x^2, x \in (0,1)$. It is easy to verify that $B_5(\mathbf{p}) = 2$. Thus, by Theorem 8, the extropy of T_t is dominated as follows:

$$\overline{J}(T_t) \ge \frac{2}{t}, \quad 0 < t < 1. \tag{33}$$

Furthermore, the lower bound achieved in (30) is derived as follows:

$$\overline{J}(T_t) \ge \frac{[\Gamma(n+1)]^2}{2t\Gamma(2n)} \sum_{i=1}^n p_i \frac{\Gamma(2i-1)\Gamma(2n-2i+1)}{[\Gamma(i)\Gamma(n-i+1)]^2}, \tag{34}$$

for all $t \in (0,1)$. Suppose that the component lifetimes are distributed uniformly. We computed the bounds in (33) (dotted line), as well as the exact value of $\overline{J}(T_t)$ acquired from (17), and further the bounds in (34) (dashed line). The results are put on show in Figure 4. In this example, the lower bound in (34) is preferable than the lower bound acquired as (33).

5. Preferable System

Pairwise comparisons are a common task in engineering and scientific research, but the physical type of particular structures often precludes the use of traditional stochastic ordering methods. Indeed, there are many pairs of systems that simply cannot be compared using standard stochastic indices. To address this limitation, we propose a novel approach to comparing system performance in this context. In the continuing part, we illustrate a novel method to make comparison of information measures when no traditional stochastic ordering is available. In particular, we focus on the important technical criterion of system longevity, which is widely regarded as a key indicator of performance. To ensure a fair comparison, we assume that the competing systems have similar characteristics. Under this assumption, our analysis shows that the parallel system design is more proper than other systems because it induces a greater production and a better expected lifetime.

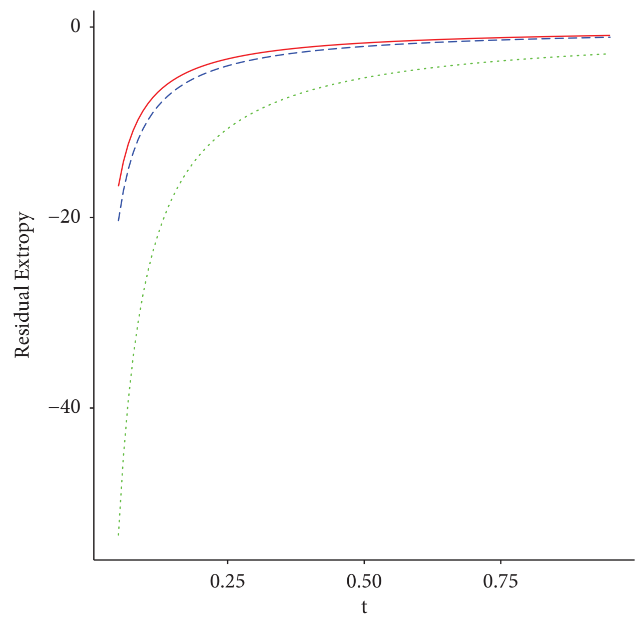

Figure 4: Exact value of $\overline{J}(T_t)$ (solid line) and the corresponding lower bounds (33) (dotted line) and (34) (dashed line) for the standard uniform distribution at inspection time t.

To quantify the reliability of the proposed approach, we use the well-established concept of survival analysis. In particular, we derive a key property from (13) that allows us to rigorously evaluate the reliability of our approach and demonstrate its superiority over existing methods. Clearly,

$$P\left(T_t^1 > x\right) \le P\left(T_t > x\right) \le P\left(T_t^n > x\right), \quad x > 0, \tag{35}$$

for all $t > 0$. Instead of relying only on pairwise comparisons, another approach is to choose a system whose structure or distribution is more similar to that of the parallel system. To achieve this, we ask the following question: which of these systems has a configuration more similar (or closer) to the parallel system and more dissimilar (or more distant) to the serial system? We propose to use the idea of the relative extropy distinction to answer this question. More specifically, we use relative extropy, a powerful information-theoretic metric that has proven useful in describing the differences between probability distributions. The past relative extropy induced by the pdf $f_t(x) = f(x)/F(t)$ with respect to $g_t(x) = g(x)/G(t)$ is defined as

$$\overline{D}\left(X_t\colon Y_t\right) = \frac{1}{2}\int_0^t \left[f_t(x) - (x)g_t(x)\right]^2 \mathrm{d}x \ge 0, \tag{36}$$

under the condition that the integral exists. The equality holds if and only if $f_t(x) = g_t(x)$ almost everywhere. We have

$$\overline{D}\left(T_t\colon T_t^1\right) = \frac{1}{2}\int_0^t \left[f_{T_t}(x) - f_{T_t^1}(x)\right]^2 \mathrm{d}x, \tag{37}$$

$$\overline{D}\left(T_t\colon T_t^n\right) = \frac{1}{2}\int_0^t \left[f_{T_t}(x) - f_{T_t^n}(x)\right]^2 \mathrm{d}x. \tag{38}$$

High (low) values of $\overline{D}(T_t\colon T_t^1)$ and $\overline{D}(T_t\colon T_t^n)$ indicate that T_t is significantly different from that of the parallel and serial systems, respectively. These differences may have a negative impact on the performance of the coherent system, making it less preferable to the parallel or series system, depending on which system has a more similar past-life distribution. To formalize this notion of preference, we provide a tool for a preferred system on the basis of the relative extropy difference measure. Specifically, we define a system as preferable if its past lifetime distribution is more similar to that of the parallel system than to that of the serial system, as measured by the relative past extropy discrimination. In other words, we define

$$\overline{\mathcal{J}}\left(T_t\right) = \frac{\overline{D}\left(T_t\colon T_t^1\right) - \overline{D}\left(T_t\colon T_t^n\right)}{\overline{D}\left(T_t\colon T_t^1\right) + \overline{D}\left(T_t\colon T_t^n\right)}, \tag{39}$$

for all $t > 0$. The range of past relative extropy divergence $\overline{\mathcal{J}}(T_t)$ is bounded by -1 and 1 for all $t > 0$. More precisely, $\overline{\mathcal{J}}(T_t) = 1$ only if $T_t = T_t^n$, and $\overline{\mathcal{J}}(T_t) = -1$ only if $T_t = T_t^1$. In other words, when $\overline{\mathcal{J}}(T_t)$ is closer to 1, the past-life distribution of T_t more closely resembles that of the parallel system, and when $\overline{\mathcal{J}}(T_t)$ is closer to -1, the past-life distribution of T_t more closely resembles that of the series system.

It is worth noting that the past relative extropia divergence given by the equation DKL relies on the signature of the system and further the parent distribution. This observation highlights the importance of carefully selecting the appropriate divergence measure for a given problem. With this in mind, we propose the following definition for selecting a preferred system.

Definition 10. Let $T_{1,t}$ and $T_{2,t}$ be past lifetimes of two coherent systems with n i.i.d. component lifetimes and signatures $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. We say that $T_{2,t}$ is more preferable than $T_{1,t}$ in terms of the past relative extropy (PRE) at time t, denoted by $T_{1,t} \le_{\mathrm{PRE}} T_{2,t}$, if and only if $\overline{\mathcal{J}}(T_{1,t}) \le \overline{\mathcal{J}}(T_{2,t})$ for all $t > 0$.

By using the change of $u = F_t(x)$, equations (35) and (36) can be rewritten as follows:

$$\begin{aligned}
\overline{D}\left(T_t\colon T_t^1\right) &= \frac{1}{2}\int_0^1 \left[g_V(v) - g_1(v)\right]^2 f_t\left(F_t(v)\right)\mathrm{d}v,\\
\overline{D}\left(T_t\colon T_t^n\right) &= \frac{1}{2}\int_0^1 \left[g_V(v) - g_n(v)\right]^2 f_t\left(F_t(v)\right)\mathrm{d}v.
\end{aligned} \tag{40}$$

An example is given as follows to illustrate the measure proposed.

Example 4. Consider two signatures $\mathbf{p}_1 = (0, 2/3, 1/3, 0)$ and $\mathbf{p}_2 = (1/4, 1/4, 1/2, 0)$, which are associated with the rvs $T_{1,t}$ and $T_{2,t}$, respectively. We assume that the component lifetimes are i.i.d. according to the standard uniform distribution in $(0, 1)$, and therefore we have $f_t(F_t^{-1}(u)) = 1/t, 00$. It is obvious that they are not comparable with the usual stochastic ordering methods. We can use the method PRE to compare the past lifetime distributions of the systems. In this case, we obtain $\overline{\mathcal{J}}(T_{1,t}) =$

0.7230 and $\overline{\mathcal{J}}(T_{2,t}) = 0.7324$, indicating that $T_{2,t}$ is more similar to the past lifetime distribution of the parallel system than $T_{1,t}$. Based on the preferred system criterion proposed earlier, we can conclude that $T_{2,t}$ is the preferred system because its past-life distribution is more similar to that of the parallel system than that of the serial system. In other words, it is more likely that $T_{2,t}$ has better performance compared to $T_{1,t}$.

The abovementioned example illustrates the potential advantages of the PRE method for comparing complex systems that cannot be compared using traditional stochastic ordering methods. By using this powerful information-theoretic tool, we can make more sophisticated and informed design decisions in engineering and scientific research.

6. Conclusion

Quantifying the uncertainty linked to the lifetime of engineering systems has garnered more attention in recent years. Predictability is a crucial component of system reliability, and quantifying a system's lifespan predictability can help with crucial design decisions. In such circumstances, extropy, a Shannon entropy extension, has shown to be a useful tool by enabling us to quantify the uncertainty behind the randomness of the lifetime of a system. In the current study, we have obtained a formula for the extropy of a system's lifetime under the assumption that all of its constituent parts have failed at time t. We look at this measure's numerous characteristics and derive bounds for its value with significant design choices in engineering and science. We give various examples and show their outcomes to demonstrate the usefulness of the suggested measures and partial orderings. The criteria for choosing a preferable system was then developed, and it was based on the relative extropy between the past lifetime distributions of competing systems. This criterion can help with crucial design choices in engineering and scientific research since it offers a simple and straightforward framework for comparing the performance of complicated systems. In general, this approach supports ongoing attempts to provide more complicated tools for describing and contrasting the reliability of complex systems. We can obtain deeper insights into how these systems behave and improve design choices by utilizing the rich theories of information theory and probability theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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