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Research Article

Bifurcations in a Plant-Pollinator Model with Multiple Delays

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The plant-pollinator model is a common model widely researched by scholars in population dynamics. In fact, its complex dynamical behaviors are universally and simply expressed as a class of delay differential-difference equations. In this paper, based on several early plant-pollinator models, we consider a plant-pollinator model with two combined delays to further describe the mutual constraints between the two populations under different time delays and qualitatively analyze its stability and Hopf bifurcation. Specifically, by selecting different combinations of two delays as branch parameters and analyzing in detail the distribution of roots of the corresponding characteristic transcendental equation, we investigate the local stability of the positive equilibrium point of equations, derive the sufficient conditions of asymptotic stability, and demonstrate the Hopf bifurcation for the system. Under the condition that two delays are not equal, some explicit formulas for determining the direction of Hopf bifurcation and some conditions for the stability of periodic solutions of bifurcation are obtained for delay differential equations by using the theory of norm form and the theorem of center manifold. In the end, some examples are presented and corresponding computer numerical simulations are taken to demonstrate and support effectiveness of our theoretical predictions.

1. Introduction

It is estimated that there are about 3,50,000 plant species in the nature world, which are classified as seed plants, bryophytes, ferns, and algae. Until 2004, almost certainly over 2,87,655 species had been identified, including 2,58,650 flowering plants, 16,000 bryophytes, 11,000 ferns, and 8000 green algae. Obviously, the flowering plants, accounting for about 90%, make up the majority of the identified plants. It indicated that most of flowering plants rely on some certain medium to transmit pollen, and 90% of medium are animals, especially insects, except for a few by wind and water. Therefore, the research on the interaction between plants and pollinators has important application value in biodiversity conservation and agriculture.

Pollinators are an important part of the ecosystem; their species composition, quantity change, and pollination objects directly or indirectly reflect the ecological environment and its development trend. In addition, they also provide important ecological services for the ecosystem, which plays an important role in maintaining the dynamic balance and relative stability of the ecosystem. The plant pollinator population system, as an important branch of population ecology, has been a hot issue in the field of biomathematics for half a century. Discussing about the dynamic characteristics between plants and pollinators has real significance in biodiversity conservation, species origin and formation mechanism, and agricultural production. It suggested that the species categories and ecological evolution processes of organisms are diverse. There are complex interactive relationships between plants and pollinators, including reciprocity, hostility, and defense that are generally expressed by differential equation models.

It is a hot topic on the plant-pollinator population dynamics for a half century. As early as 1976, considering about saturation effects of the benefits that the plant derives from the pollinator, May proposed a model of an obligate relationship between a plant and its pollinator and discussed the curvilinear isoclines with stabilizing effect [1, 2]. In 1981, Soberon et al. presented a mathematical model describing the dynamics of the plant-pollinator dynamical interaction with function response. They mainly considered the effects of two parameters on the stability of their system, including the nectar reward provided by plants to pollinators and the specificity of pollinators to plants [3]. In 1955, Lundberg and Ingvarsson [4] generalized an earlier model of the interactions between plants and pollinators and analyzed the equilibrium points of the system. It showed that there is a threshold standard in the system, which would not last long when it is lower than the standard, and the impact of the existence of the threshold standard on the persistence of plants was further discussed.

After then, some dynamic models of three population interactions appeared, such as the herbivore-plant-pollinator interactions proposed by Jang [5] and the plant-pollinator-robber system raised by Wang et al. [6, 7]. In [5], by analyzing the parameter energy reward and the specificity of pollinators to plants, Jang et al. considered the effect of herbivores on the pollinator's flower visiting rate and further discussed the possible mechanism of herbivores accompanying pollinators to increase the pollinator's flower visiting rate. The results showed that this mechanism could promote the persistence of the interaction between the three populations. The literature [6, 7] focused on the factors that led to the widespread occurrence and stability of interaction by analyzing the mathematical model of plant-pollinator-robber coexistence and dynamic properties by using the relevant theory of the dynamic system. Fishman and Hadany formulated and discussed a multigeneration population dynamics model for plants' interaction with central place pollinators [8]. In [9], Wang et al. investigated a plant-pollinator model with diffusion and analyzed the uniqueness and stability of positive steady state solution by using the regular perturbation theorem and the monotone dynamical system theory. However, in these existing mathematical models, most researchers did not consider the time delay factor. Due to its inevitability and importance, the influence of time delay on dynamic behavior has been adequately considered in some models, such as the predator-prey model [10–14], the competition and cooperation model of two enterprises [15–19], the neuron network model [20–22], the competition model of internet [23, 24], the chemical reaction model [25, 26], and the epidemic model [27–31]. The introduction of the delay factor can more accurately reflect the objective facts and development laws of things. Therefore, most of the scholars tend to be more interested in analyzing the delay differential system when analyzing the differential equation model.

In [7], Wang et al. derived a classical plant-pollinator model.

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 + \dfrac{\alpha_{12} N_1 N_2}{1 + aN_1 + bN_2} - \beta_1 N_1 N_2 - d_1 {N_1}^2, \\ \\ \dfrac{dN_2}{dt} = \dfrac{\alpha_{21} N_1 N_2}{1 + aN_1 + bN_2} - d_2 N_2, \end{cases} \tag{1}$$

where a, b, r_1, β_1, d_1, d_2, α_{12}, and α_{21} are the positive constants. $N_1(t)$ denotes the population densities of plants, $N_2(t)$ denotes the population densities of pollinators, r_1 represents the intrinsic growth rate of plants, d_1 is the self-incompatible degree, a is the effective equilibrium constant for plant-pollinator interactions, b is the intensity of exploitation competition among pollinators, α_{12} represents the plants efficiency in translation plant-pollinator interactions into fitness, α_{21} is the corresponding value for the pollinators, β_1 denotes the per-capita negative effect of pollinators on plants, and d_2 is the per-capita mortality rate of pollinators.

Recently, Huang et al. [32] considered the following plant-pollinator model with a diffusion term and a time delay:

$$\begin{cases} \dfrac{\partial N_1}{\partial t} = N_1 \left[r_1 + \dfrac{\alpha_{12} N_2}{1 + aN_1 + bN_2} - \beta_1 N_2 - d_1 N_1 \right], \\ \\ \dfrac{\partial N_2}{\partial t} = D_2 \Delta N_2 + N_2 \left[\dfrac{\alpha_{21} N_1 (t-\tau, x)}{1 + aN_1 (t-\tau, x) + bN_2 (t-\tau, x)} - d_2 \right], \end{cases} \tag{2}$$

where $\tau > 0$ is a time delay, Δ is the Laplacian operator, and $D_2 > 0$ is a diffusion coefficient. By analyzing eigenvalues of the linearized equation, Huang et al. gave some conditions about the stability of the positive constant steady state and existence of spatially homogeneous and spatially inhomogeneous periodic solutions.

Based on (1) and (2), we incorporate two different delays into the model to reflect the dynamical behaviours depending on the histories. We shall consider the following system:

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 + \dfrac{\alpha_{12} N_1 N_2}{1 + aN_1 + bN_2} - \beta_1 N_1 N_2 - d_1 {N_1}^2, \\ \\ \dfrac{dN_2}{dt} = N_2 \left[\dfrac{\alpha_{21} N_1 (t-\tau_1)}{1 + aN_1 (t-\tau_1) + bN_2 (t-\tau_2)} - d_2 \right], \\ \\ N_1(t) = \phi(t), N_2(t) = \psi(t), t \in \left[-\max\limits_{i=1,2} \{\tau_i\}, 0 \right], \end{cases} \tag{3}$$

where $\tau_i \geq 0\, (i = 1, 2)$ denotes the delay effects in the process when the pollinators translate plant-pollinator interactions into the fitness.

The present article is organized as follows: In Section 2, by selecting two different varying delays $\tau_i\, (i = 1, 2)$ as the bifurcation parameters and considering the distribution of corresponding characteristic roots, we shall give the conditions on the stability of the positive equilibrium and the existence of Hopf bifurcation of system (3). In Section 3, based on the normal form method and the center manifold reduction used by Hassard et al. in [33], we shall derivate some formulas for deciding the stability and the directions of periodic solutions and Hopf bifurcation. In Section 4, some numerical simulations are carried out to illustrate the validity of the main results.

2. Model Description

In this section, we shall mainly analyze the local stability of the positive constant steady state and the existence of Hopf bifurcation of system (3) by using the methods in [32, 34, 35].

According to [32], for the existence and uniqueness of the positive equilibrium of (3), we have the following result:

Lemma 1. *Assume that one of the following conditions holds:*

(A1) $\alpha_{21} > ad_2, a_1 < 0, a_1^2 - 4a_0a_2 = 0$;

(A2) $\alpha_{21} > ad_2, 4a_0a_2 < 0$;

where $a_0 = b\beta_1/(\alpha_{21} - ad_2) + d_1d_2b^2/(\alpha_{21} - ad_2)^2$, $a_1 = (\beta_1 - br_1)/(\alpha_{21} - ad_2) + 2d_1d_2b/(\alpha_{21} - ad_2)^2 - \alpha_{12}/\alpha_{21}$, *and*

$$a_2 = -\frac{r_1}{\alpha_{21} - ad_2} + \frac{d_1d_2}{(\alpha_{21} - ad_2)^2}. \tag{4}$$

Then, (3) has two boundary equilibria, $E_0(0,0)$ and $E_1(r_1/d_1, 0)$, and a unique positive equilibrium, $E^*(N_1^*, N_2^*)$, where

$$\begin{aligned} N_1^* &= \frac{2a_0d_2 - a_1bd_2 + bd_2\sqrt{a_1^2 - 4a_0a_2}}{2a_0(\alpha_{21} - ad_2)}, \\ N_2^* &= \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0}. \end{aligned} \tag{5}$$

Let $u_1 = N_1 - N_1^*$ and $u_2 = N_2 - N_2^*$, then (3) can be rewritten as follows:

$$\begin{cases} \dfrac{du_1}{dt} = (u_1 + N_1^*)\left[r_1 + \dfrac{\alpha_{12}(u_2 + N_2^*)}{1 + a(u_1 + N_1^*) + b(u_2 + N_2^*)} - \beta_1(u_2 + N_2^*) - d_1(u_1 + N_1^*)\right], \\ \dfrac{du_2}{dt} = (u_2 + N_2^*)\left[\dfrac{\alpha_{21}(u_1(t-\tau_1) + N_1^*)}{1 + a(u_1(t-\tau_1) + N_1^*) + b(u_2(t-\tau_2) + N_2^*)} - d_2\right], \\ u_1(t) = \phi(t) - N_1^*, u_2(t) = \psi(t) - N_2^*, t \in \left[-\max\limits_{i=1,2}\{\tau_i\}, 0\right]. \end{cases} \tag{6}$$

Thus, the positive equilibrium point $E^*(N_1^*, N_2^*)$ of system (3) coincides with the zero equilibrium of system (6).

Let

$$\begin{aligned} f^{(1)}(u_1, u_2) &= (u_1 + N_1^*)\left[r_1 + \frac{\alpha_{12}(u_2 + N_2^*)}{1 + a(u_1 + N_1^*) + b(u_2 + N_2^*)} - \beta_1(u_2 + N_2^*) - d_1(u_1 + N_1^*)\right], \\ f^{(2)}(u_1, u_2, w) &= (w + N_2^*)\left[\frac{\alpha_{21}(u_1 + N_1^*)}{1 + a(u_1 + N_1^*) + b(u_2 + N_2^*)} - d_2\right]. \end{aligned} \tag{7}$$

For $i, j, l \in \mathrm{N}_0 = \{0, 1, 2, \ldots\}$, we define $f_{ij}^{(1)}(i + j \geq 1)$ and $f_{ijl}^{(2)}(i + j + l \geq 1)$ as follows:

$$f_{ij}^{(1)} = \frac{\partial^{i+j} f^{(1)}(0,0)}{\partial u_1^i \partial u_2^j}, f_{ijl}^{(2)} = \frac{\partial^{i+j+l} f^{(2)}(0,0,0)}{\partial u_1^i \partial u_2^j \partial w^l}, \tag{8}$$

in particular.

$$\begin{aligned} \alpha_1 &= f_{10}^{(1)} = -d_1N_1^* - \frac{\alpha_{12}aN_1^*N_2^*}{(1 + aN_1^* + bN_2^*)^2} < 0, \alpha_2 = f_{01}^{(1)} = \frac{\alpha_{12}N_1^*(1 + aN_1^*)}{(1 + aN_1^* + bN_2^*)^2} - \beta_1N_1^*, \\ \gamma_1 &= f_{100}^{(2)} = \frac{\alpha_{21}N_2^*(1 + bN_2^*)}{(1 + aN_1^* + bN_2^*)^2} > 0, \gamma_2 = f_{010}^{(2)} = -\frac{b\alpha_{21}N_1^*N_2^*}{(1 + aN_1^* + bN_2^*)^2} < 0. \end{aligned} \tag{9}$$

By Taylor expansion, (6) can become as follows:

$$\begin{cases} \dfrac{du_1}{dt} = \alpha_1 u_1(t) + \alpha_2 u_2(t) + \sum_{i+j\ge 2} \dfrac{1}{i!j!} f_{ij}^{(1)} u_1{}^i(t) u_2{}^j(t), \\ \dfrac{du_2}{dt} = \gamma_1 u_1(t-\tau_1) + \gamma_2 u_2(t-\tau_2) + \sum_{i+j+l\ge 2} \dfrac{1}{i!j!l!} f_{ijl}^{(2)} u_1{}^i(t-\tau_1) u_2{}^j(t-\tau_2) u_2{}^l(t), \end{cases} \tag{10}$$

and $u_1(t) = \phi(t) - N_1^*, u_2(t) = \psi(t) - N_2^*, t \in [-\max_{i=1,2}\{\tau_i\}, 0]$.

It is easy to see that system (10) about the equilibrium point $(0,0)$ yields the following linear system:

$$\begin{cases} \dfrac{du_1}{dt} = \alpha_1 u_1(t) + \alpha_2 u_2(t), \\ \dfrac{du_2}{dt} = \gamma_1 u_1(t-\tau_1) + \gamma_2 u_2(t-\tau_2). \end{cases} \tag{11}$$

The corresponding characteristic equation of the system (11) is as follows:

$$\lambda^2 - \alpha_1\lambda - (\gamma_2\lambda - \gamma_2\alpha_1)e^{-\lambda\tau_2} - \gamma_1\alpha_2 e^{-\lambda\tau_1} = 0. \tag{12}$$

We use the following Lemma in [35] to investigate the distribution of roots of the transcendental equation (12).

Lemma 2. *For the transcendental equation,*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} \\ &\quad + \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} \\ &= 0. \end{aligned} \tag{13}$$

As $(\tau_1, \tau_2, \tau_3, \dots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ in the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Now, we shall consider the following different cases:

Case 1. $\tau_1 = \tau_2 = 0$. Then, (12) becomes

$$\lambda^2 - (\alpha_1 + \gamma_2)\lambda + \gamma_2\alpha_1 - \gamma_1\alpha_2 = 0. \tag{14}$$

Since $\alpha_1 < 0, \gamma_2 < 0, -(\alpha_1 + \gamma_2) > 0$ holds. It is obvious that when $\tau_1 = \tau_2 = 0$, the condition that all roots of (14) have negative real parts is given in the following form:

$$(\mathrm{H1}){:}\ \gamma_2\alpha_1 - \gamma_1\alpha_2 > 0 \tag{15}$$

Thus, we have the following result:

Lemma 3. *Assume that (H1) holds. Then, for $\tau_1 = \tau_2 = 0$, the positive equilibrium point E^* of (3) is asymptotically stable.*

Case 2. $\tau_1 > 0, \tau_2 = 0$. Then, (12) becomes

$$\lambda^2 - (\alpha_1 + \gamma_2)\lambda + \gamma_2\alpha_1 - \gamma_1\alpha_2 e^{-\lambda\tau_1} = 0. \tag{16}$$

Suppose that $\tau_1 > 0, \tau_2 = 0$ and $\lambda = i\omega_1\ (\omega_1 > 0)$ are roots of (16). Then, we have

$$\begin{aligned} &-\omega_1{}^2 - i\omega_1(\alpha_1 + \gamma_2) + \gamma_2\alpha_1 \\ &\quad - \gamma_1\alpha_2(\cos\omega_1\tau_1 - i\sin\omega_1\tau_1) = 0. \end{aligned} \tag{17}$$

Separating the real and imaginary parts of (17), we can obtain

$$\begin{cases} \omega_1^2 - \gamma_2\alpha_1 = -\gamma_1\alpha_2\cos\omega_1\tau_1, \\ \omega_1(\alpha_1 + \gamma_2) = \gamma_1\alpha_2\sin\omega_1\tau_1. \end{cases} \tag{18}$$

By simple calculation, we can have

$$\omega_1^4 + \left(\alpha_1{}^2 + \gamma_2{}^2\right)\omega_1^2 + (\gamma_2\alpha_1)^2 - (\gamma_1\alpha_2)^2 = 0. \tag{19}$$

Let $z = \omega_1^2$, then (19) becomes

$$z^2 + \left(\alpha_1{}^2 + \gamma_2{}^2\right)z + (\gamma_2\alpha_1)^2 - (\gamma_1\alpha_2)^2 = 0. \tag{20}$$

Notice that $\alpha_1{}^2 + \gamma_2{}^2 > 0$. If the following condition (H2): $(\gamma_2\alpha_1)^2 - (\gamma_1\alpha_2)^2 > 0$ holds, then (20) has no positive solution. Thus, all solutions of (16) have negative real parts when $\tau_1 > 0$ under (H2). So, we have the following result:

Theorem 1. *Let $\tau_2 = 0$ and (A1) or (A2) holds. When (H2) holds, the positive equilibrium point E^* of (3) is asymptotically stable for all $\tau_1 > 0$.*

However, if the condition (H3): $(\gamma_2\alpha_1)^2 - (\gamma_1\alpha_2)^2 < 0$ holds, then (20) has a unique positive root.

$$z_0 = \omega_{10}^2 = \frac{1}{2}\left[-\left(\alpha_1{}^2 + \gamma_2{}^2\right) + \sqrt{\left(\alpha_1^2 - \gamma_2^2\right)^2 + 4\gamma_1{}^2\alpha_2{}^2}\right]. \tag{21}$$

Thus, $\omega_{10} = \{1/2[-(\alpha_1{}^2 + \gamma_2{}^2) + \sqrt{(\alpha_1^2 - \gamma_2^2)^2 + 4\gamma_1{}^2\alpha_2{}^2}]\}^{1/2}$, and (16) has a pair of purely imaginary roots $\pm i\omega_{10}$. Substituting ω_{10} into (18), we can get

$$\tau_{1_j} = \frac{1}{\omega_{10}}\arccos\left\{\frac{\omega_{10}{}^2 - \gamma_2\alpha_1}{-\gamma_1\alpha_2}\right\} + \frac{2j\pi}{\omega_{10}}, \ (j \in \mathbb{N}_0). \tag{22}$$

Let

$$F(\lambda, \tau_1) = \lambda^2 - (\alpha_1 + \gamma_2)\lambda + \gamma_2\alpha_1 + \gamma_1\alpha_2 e^{-\lambda\tau_1}. \tag{23}$$

Then, $F(\pm i\omega_{10}, \tau_{1j}) = 0$. Suppose $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ is a root of (16) near $\tau_1 = \tau_{1j}$ and $\alpha(\tau_{1j}) = 0$, $\omega(\tau_{1j}) = \omega_{10}$. From the function differential equation (FDE) theory, for every τ_{1j} $(j = 0, 1, 2, \ldots)$ there exists $\varepsilon > 0$, such that $\lambda(\tau_1)$ is continuously differentiable at τ_1 for $|\tau_1 - \tau_{1j}| < \varepsilon$. Substituting $\lambda(\tau_1)$ into the left-hand side of (16) and differentiating with respect to τ_1, we can have

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{[2\lambda - (\alpha_1 + \gamma_2)]e^{\lambda\tau_1}}{-\lambda\gamma_1\alpha_2} - \frac{\tau_1}{\lambda}. \tag{24}$$

Hence,

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\Bigg|_{\tau_1=\tau_{1j}} = \frac{[2i\omega_{10} - (\alpha_1 + \gamma_2)]\left(\cos\omega_{10}\tau_{1j} + i\sin\omega_{10}\tau_{1j}\right)}{-i\omega_{10}\gamma_1\alpha_2} - \frac{\tau_{1j}}{i\omega_{10}}. \tag{25}$$

Noting that,

$$\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau_1}\right\}^{-1}_{\tau_1=\tau_{1j}} = \frac{2\omega_{10}{}^2 + \alpha_1{}^2 + \gamma_2{}^2}{(\gamma_1\alpha_2)^2} > 0. \tag{26}$$

From the above discussions, we can get $d(\mathrm{Re}\lambda)/d\tau_1|_{\tau_1=\tau_{1j}} > 0$ and derive the following theorem:

Theorem 2. *For $\tau_2 = 0$, we assume that (A1) or (A2) holds. Assume further that (H3) holds. The following conclusions are true:*

(i) If $\tau_1 \in [0, \tau_{10})$, then the positive equilibrium E^* of (3) is asymptotically stable

(ii) If $\tau_1 > \tau_{10}$, then the positive equilibrium E^* of (3) is unstable

(iii) If $\tau_1 = \tau_{10}$, then system (3) undergoes a Hopf bifurcation at the positive equilibrium E^*

Case 3. $\tau_1 = 0, \tau_2 > 0$. Then, (12) becomes

$$\lambda^2 - \alpha_1\lambda - \gamma_1\alpha_2 - (\gamma_2\lambda - \gamma_2\alpha_1)e^{-\lambda\tau_2} = 0. \tag{27}$$

Similar to the second case, we suppose that $\tau_1 = 0, \tau_2 > 0$ and $\lambda = i\omega_2$ $(\omega_2 > 0)$ is a root of (27). Through some calculations, we can get

$$z^2 + \left(\alpha_1{}^2 + 2\gamma_1\alpha_2 - \gamma_2{}^2\right)z + (\gamma_1\alpha_2)^2 - (\gamma_2\alpha_1)^2 = 0, \tag{28}$$

where $z = \omega_2{}^2$. It is easy to see that if the condition (H4): $\alpha_1{}^2 + 2\gamma_1\alpha_2 - \gamma_2{}^2 > 0$, $(\gamma_1\alpha_2)^2 - (\gamma_2\alpha_1)^2 > 0$ holds, (28) has no positive solution. Thus, all solutions of (27) have negative real parts when $\tau_2 > 0$ under (H4).

Theorem 3. *Let $\tau_1 = 0$ and (A1) or (A2) holds. When (H4) holds, then the positive equilibrium E^* of system (3) is asymptotically stable for all $\tau_2 > 0$.*

However, if the condition

(H5): $\alpha_1{}^2 + 2\gamma_1\alpha_2 - \gamma_2{}^2 > 0$, $(\gamma_1\alpha_2)^2 - (\gamma_2\alpha_1)^2 < 0$

holds, then (28) has a unique positive solution $z_0 = \omega_{20}^2$ and (27) has a pair of purely imaginary roots $\pm i\omega_{20}$, where

$$\omega_{20} = \left\{\frac{1}{2}\left[-\left(\alpha_1{}^2 + 2\gamma_1\alpha_2 - \gamma_2{}^2\right) + \sqrt{\left(\alpha_1{}^2 + 2\gamma_1\alpha_2 - \gamma_2{}^2\right)^2 - 4\left(\gamma_1{}^2\alpha_2{}^2 - \gamma_2{}^2\alpha_1{}^2\right)}\right]\right\}^{1/2}. \tag{29}$$

At this time, we have

$$\tau_{2j} = \frac{1}{\omega_{20}}\arccos\left\{\frac{\gamma_1\alpha_1\alpha_2}{\gamma_2\left(\omega_{20}^2 + \alpha_1{}^2\right)}\right\} + \frac{2j\pi}{\omega_{20}}, j \in \mathbb{N}_0. \tag{30}$$

Let $F(\lambda, \tau_2) = \lambda^2 - \alpha_1\lambda + \gamma_1\alpha_2 + (-\gamma_2\lambda + \gamma_2\alpha_1)e^{-\lambda\tau_2}$, then $F(\pm i\omega_{20}, \tau_{2j}) = 0$.

Suppose $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ is a root of (27) near $\tau_2 = \tau_{2j}$ and $\alpha(\tau_{2j}) = 0$, then $\omega(\tau_{2j}) = \omega_{20}$. Substituting $\lambda(\tau_2)$ into (27) and differentiating with respect to τ_2, we can obtain

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{(2\lambda-\alpha_1)e^{\lambda\tau_2}-\gamma_2}{(\gamma_2\alpha_1-\gamma_2\lambda)\lambda} - \frac{\tau_2}{\lambda},$$

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{2j}} = \frac{(2i\omega_{20}-\alpha_1)\left(\cos\omega_{20}\tau_{2j}+i\sin\omega_{20}\tau_{2j}\right)-\gamma_2}{(\gamma_2\alpha_1-i\omega_{20}\gamma_2)i\omega_{20}} - \frac{\tau_{2j}}{i\omega_{20}}, \left\{\frac{d(\mathrm{Re}\lambda)}{d\tau_2}\right\}^{-1}_{\tau_2=\tau_{2j}} = \frac{2\gamma_1\alpha_2-\gamma_2{}^2+\alpha_1^2+2\omega_0^2}{\gamma_2{}^2\left(\omega_0^2+\alpha_1^2\right)}>0. \tag{31}$$

Therefore, we can easily get $d(\mathrm{Re}\lambda)/d\tau_2|_{\tau_2=\tau_{2j}}>0$. Based on the above analysis, we have the following theorem:

Theorem 4. *For $\tau_1=0$, assume that (A1) or (A2) holds. Assume further that (H5) holds. The following results are true:*

(i) If $\tau_2\in[0,\tau_{20})$, then the positive equilibrium E^* of (3) is asymptotically stable

(ii) If $\tau_2>\tau_{20}$, then the positive equilibrium E^* of (3) is unstable

(iii) If $\tau_2=\tau_{20}$, then system (3) undergoes a Hopf bifurcation at the positive equilibrium E^*

Case 4. $\tau_1>0, \tau_2\in[0,\tau_{20}]$. We consider (12) with τ_2 in its stable interval. Regarding τ_1 as a parameter, without loss of generality, we consider system (3) under assumption (A1) or (A2) and Case 3. Let $i\omega\,(\omega>0)$ be a root of (12), by calculating, we can obtain

$$\omega^4+k_1\omega^3+k_2\omega^2+k_3\omega+k_4=0, \tag{32}$$

where $k_1=2\gamma_2\sin\omega\tau_2, k_2=\gamma_2{}^2+\alpha_1{}^2, k_3=2\gamma_2\alpha_1{}^2\sin\omega\tau_2, k_4=(\gamma_2\alpha_1)^2-(\gamma_1\alpha_2)^2$.

Denoting $H(\omega)=\omega^4+k_1\omega^3+k_2\omega^2+k_3\omega+k_4$, we make the assumption that

(H6): equation (32) has finite positive roots $\omega_1,\omega_2,\ldots,\omega_n$, and for every fixed $\omega_i, i=1,2,\ldots,k$, there exists a sequence $\left\{\tau_{3i}^j \mid j=0,1,2,\ldots\right\}$, such that (32) holds, where

$$\tau_{3i}^j=\frac{1}{\omega_3^{(i)}}\arccos\frac{-\left(\omega_3^{(i)}\right)^2+\gamma_2\alpha_1\cos\omega_3^{(i)}\tau_2-\gamma_2\omega_3^{(i)}\sin\omega_3^{(i)}\tau_2}{\alpha_2\gamma_1}+\frac{2\pi j}{\omega_3^{(i)}}. \tag{33}$$

$$i=1,2,\ldots,k;\ j=0,1,2,\ldots$$

Let

$$\tau_{30}=\min\left\{\tau_{3i}^j \,\middle|\, i=1,2,\ldots,k;\ j=0,1,2,\ldots\right\}. \tag{34}$$

When $\tau_1=\tau_{30}$, (12) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau_2\in[0,\tau_{20}]$. In the following, we further assume that (H7): $[d(\mathrm{Re}\lambda)/d\tau_1]_{\tau_1=\tau_{30}}\neq 0$. By the general Hopf bifurcation theorem for FDE, we have a result on the stability and Hopf bifurcation for system (3).

Theorem 5. *For system (3), we assume that (A1) or (A2) holds and assume further that (H5), (H6), and (H7) are satisfied, and $\tau_2\in[0,\tau_{20})$. Then, the positive equilibrium E^* is asymptotically stable for $\tau_1\in(0,\tau_{30})$ and is unstable for $\tau_1\in(\tau_{30},+\infty)$. System (3) undergoes a Hopf bifurcation at the E^* for $\tau_1=\tau_{30}$.*

Case 5. $\tau_2>0, \tau_1\in[0,\tau_{10}]$. We consider (12) with τ_1 in its stable interval. Regarding τ_2 as a parameter, without loss of generality, we consider system (3) under assumption (A1) or (A2) and Case 2. Let $i\omega\,(\omega>0)$ be a root of (12), by calculating, we can obtain

$$\omega^4+k_1\omega^2+k_2\omega+k_3=0, \tag{35}$$

where $k_1=\alpha_1{}^2-\gamma_2{}^2+2\gamma_1\alpha_2\cos\omega\tau_1, k_2=-2\gamma_1\alpha_1\alpha_2\sin\omega\tau_1, k_3=(\gamma_1\alpha_2)^2-(\gamma_2\alpha_1)^2$. Denoting $H(\omega)=\omega^4+k_1\omega^2+k_2\omega+k_3$, we make the assumption that (H8): equation (35) has finite positive roots $\omega_1,\omega_2,\ldots,\omega_n$, and for every fixed $\omega_i, i=1,2,\ldots,k$, there exists a sequence $\left\{\tau_{4i}^j \mid j=0,1,2,\ldots\right\}$ such that (35) holds, where

$$\tau_{4i}^{j} = \frac{1}{\omega_4^{(i)}} \arccos \frac{-\left(\omega_4^{(i)}\right)^2 + \gamma_2\alpha_1 \cos\omega_4^{(i)}\tau_2 - \gamma_2\omega_4^{(i)} \sin\omega_4^{(i)}\tau_2}{\alpha_2\gamma_1} + \frac{2\pi j}{\omega_4^{(i)}}. \tag{36}$$

$$i = 1,2,\ldots,k;\ j = 0,1,2,\ldots$$

Let

$$\tau_{40} = \min\left\{\tau_{4i}^{j} \,\middle|\, i = 1,2,\ldots,k;\ j = 0,1,2,\ldots\right\}. \tag{37}$$

When $\tau_2 = \tau_{40}$, (12) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau_1 \in [0, \tau_{10}]$. In the following, we further assume that (H9): $[d(\mathrm{Re}\lambda)/d\tau_2]_{\tau_2=\tau_{40}} \neq 0$ Similarly, we have a result on the stability and Hopf bifurcation for system (3).

Theorem 6. *For system (3), assume that (A1) or (A2) holds and further assume that (H8) and (H9) are satisfied, and $\tau_1 \in [0, \tau_{10})$. Then, the positive equilibrium E^* is asymptotically stable for $\tau_2 \in (0, \tau_{40})$ and is unstable for $\tau_2 \in (\tau_{40}, +\infty)$. System (3) undergoes a Hopf bifurcation at the E^* for $\tau_2 = \tau_{40}$.*

3. Direction of Hopf Bifurcation and Stability of Bifurcating Periodic Solution

In the previous section, we have obtained stability and existence of Hopf bifurcation of system (3) at the positive equilibrium E^* by taking delay τ_i $(i = 1,2)$ as the bifurcation parameter and applying the linearization method. In the present section, we will discuss the direction of Hopf bifurcation and the stability of bifurcation periodic solutions by employing the normal form method and the center manifold theorem by Hassard et al. [33]. We always assume that system (3) undergoes Hopf bifurcation at the positive equilibrium E^* for $\tau_2 = \tau_{20}$, and $\pm i\omega^*$ denotes the corresponding purely imaginary roots of the characteristic equation at E^*.

Without loss of generality, we assume $\tau_1^* < \tau_{40}$, where $\tau_1^* \in (0, \tau_{40})$ and τ_{40} is defined by (37). For convenience, let $\overline{u_i}(t) = u_i(\tau t)$, $u_i(t) = N_i(t) - N_i^*$, $(i = 1,2)$, $\tau_2 = \tau_{40} + \mu$, and $\mu \in \mathbb{R}$, then $\mu = 0$ is the Hopf bifurcation value of (3). Thus, (3) can be rewritten as an FDE in $C = C([-1,0], \mathbb{R}^2)$ as follows:

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{38}$$

where $u(t) = (u_1(t), u_2(t))^T \in C$ and $u_t(\theta) = u(t+\theta)$.

Define the linear operator L_μ: $C \longrightarrow \mathbb{R}$ by

$$L_\mu(\phi) = (\tau_{40} + \mu)\left[B\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix} + C\begin{pmatrix}\phi_1\left(-\dfrac{\tau_1^*}{\tau_{40}}\right)\\ \phi_2\left(-\dfrac{\tau_1^*}{\tau_{40}}\right)\end{pmatrix} + D\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix}\right], \tag{39}$$

and nonlinear operator $F(\mu,\cdot)$: $\mathbb{R} \times C \longrightarrow \mathbb{R}$ by

$$F(\mu, \phi) = (\tau_{40} + \mu)(f_1, f_2)^T, \tag{40}$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$, $B = \begin{pmatrix}\alpha_1 & \alpha_2\\ 0 & 0\end{pmatrix}$, $C = \begin{pmatrix}0 & 0\\ 0 & \gamma_2\end{pmatrix}$, $D = \begin{pmatrix}0 & 0\\ \gamma_1 & 0\end{pmatrix}$, and

$$\begin{aligned} f_1 &= \sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} {\phi_1}^i(0) {\phi_2}^j(0),\\ f_2 &= \sum_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} {\phi_1}^i(-1) {\phi_2}^j\left(-\frac{\tau_1^*}{\tau_{40}}\right) {\phi_2}^l(0). \end{aligned} \tag{41}$$

Based on the Riesz representation theorem, we know that there is a matrix function with bounded variation components $\eta(\theta, \mu)$ and $\theta \in [-1, 0]$, such that

$$L_{\mu}(\phi) = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \phi(\theta) \in C\left([-1,0],\mathbb{R}^2\right). \tag{42}$$

In fact, we can take

$$\eta(\theta,\mu) = \begin{cases} (\tau_{40}+\mu)(B+C+D), & \theta = 0 \\ (\tau_{40}+\mu)(C+D), & \theta \in \left[-\dfrac{\tau_1^*}{\tau_{40}}, 0\right] \\ (\tau_{40}+\mu)D, & \theta \in \left(-1, -\dfrac{\tau_1^*}{\tau_{40}}\right) \\ 0. & \theta = -1. \end{cases} \tag{43}$$

In the sequel, we define the operators $A(\mu)$ and $R(\mu)$ by

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, \theta \in [-1,0) \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(s,\mu)\phi(s), \theta = 0 \end{cases}, \ R(\mu)\phi = \begin{cases} 0, \theta \in [-1,0) \\ F(\mu,\phi), \theta = 0 \end{cases}. \tag{44}$$

Then, (38) can be further rewritten into the following equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{45}$$

where $u_t(\theta) = u(t+\theta), \theta \in [-1,0]$. For $\psi \in C([0,1], (\mathbb{R}^2)^*)$, we define

$$A^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{ds}, s \in (0,1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta^T(t,0)\psi(-t), s = 0. \end{cases} \tag{46}$$

For $\phi \in C([-1,0], (\mathbb{R}^2)^*)$ and $\psi \in C([0,1], (\mathbb{R}^2)^*)$, we define the bilinear inner product.

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{47}$$

where $\eta(\theta) = \eta(\theta,0)$. Let $A = A(0)$, then A and A^* are a pair of adjoint operators. Since $\pm i\omega^*\tau_{40}$ is a pair of eigenvalues of $A(0)$, it follows that they are also a pair of eigenvalues of A^*.

Suppose that $q(\theta) = (1,\alpha)^T e^{i\omega^*\tau_{40}\theta}$ is an eigenvector of the operator $A(0)$ corresponding to the eigenvalues $i\omega^*\tau_{40}$ and $q^*(s) = M(1,\alpha^*)e^{i\omega^*\tau_{40}s}$ is an eigenvector of $A^*(0)$ corresponding to $-i\omega^*\tau_{40}$, where $M = 1/K$. From the definitions of A, $L_{\mu}\phi$, and $\eta(\theta,\mu)$, we can obtain

$$\begin{pmatrix} i\omega^* - \alpha_1 & -\alpha_2 \\ -\gamma_1 e^{-\lambda\tau_1} & i\omega^* - \gamma_2 e^{-\lambda\tau_2} \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{48}$$

Based on discussions in the last section, we can see

$$\det\begin{pmatrix} i\omega^* - \alpha_1 & -\alpha_2 \\ -\gamma_1 e^{-\lambda\tau_1} & i\omega^* - \gamma_2 e^{-\lambda\tau_2} \end{pmatrix} = 0. \tag{49}$$

Therefore, we can take $\alpha = (i\omega^* - \alpha_1)/\alpha_2$ and $\alpha^* = (-)(i\omega^* + \alpha_1)/\gamma_1 e^{-i\omega^*\tau_{40}}$.

Let $K \in C$ and $q^*(s) = M(1,\alpha^*)e^{i\omega^*\tau_{40}s}$ such that $\langle q^*(s), q(\theta)\rangle = 1$. Then, we can see that $q^*(s)$ is still an eigenvector of the operator A^* corresponding to the eigenvalue $-i\omega^*\tau_{40}$.

From the bilinear inner product of (47), we can get

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \overline{q}^*(0)q(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi \\ &= \overline{M}(1,\overline{\alpha}^*)(1,\alpha)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{M}(1,\overline{\alpha}^*)e^{-i\omega^*\tau_{40}(\xi-\theta)}d\eta(\theta)(1,\alpha)^T e^{i\omega^*\tau_{40}\xi}d\xi \\ &= \overline{M}(1,\overline{\alpha}^*)(1,\alpha)^T - \overline{M}(1,\overline{\alpha}^*)\int_{-1}^{0}\int_{\xi=0}^{\theta} d\eta(\theta)(1,\alpha)^T e^{i\omega^*\tau_{40}\theta}d\xi \\ &= \overline{M}(1,\overline{\alpha}^*)(1,\alpha)^T - \overline{M}(1,\overline{\alpha}^*)\int_{-1}^{0} \theta e^{i\omega^*\tau_{40}\theta}d\eta(\theta)(1,\alpha)^T \\ &= \overline{M}(1,\overline{\alpha}^*)(1,\alpha)^T - \overline{M}(1,\overline{\alpha}^*)\left[\tau_{40}B\phi(0) + \tau_{40}C\phi\left(-\frac{\tau_1^*}{\tau_{40}}\right) + \tau_{40}D\phi(-1)\right](1,\alpha)^T \\ &= \overline{M}(1,\overline{\alpha}^*)(1,\alpha)^T - \overline{M}(1,\overline{\alpha}^*)\left[\tau_{40}\begin{pmatrix}0 & 0\\ 0 & \gamma_2\end{pmatrix}\left(-\frac{\tau_1^*}{\tau_{40}}\right)e^{-i\omega^*\tau_1^*} - \tau_{40}\begin{pmatrix}0 & 0\\ \gamma_1 & 0\end{pmatrix}e^{-i\omega^*\tau_{40}}\right](1,\alpha)^T \\ &= \overline{M}(1+\alpha\overline{\alpha}^*) + \overline{M}\tau_1^*\gamma_2\alpha\overline{\alpha}^*e^{-i\omega^*\tau_1^*} + \overline{M}\tau_{40}\gamma_1\overline{\alpha}^*e^{-i\omega^*\tau_{40}}.\end{aligned} \tag{50}$$

Thus, we can choose $\overline{K}$ as follows:

$$\begin{aligned}\overline{K} &= 1 + \alpha\overline{\alpha}^* + \tau_1^*\gamma_2\alpha\overline{\alpha}^*e^{-i\omega^*\tau_1^*} + \tau_{40}\gamma_1\overline{\alpha}^*e^{-i\omega^*\tau_{40}}, \\ K &= 1 + \overline{\alpha}\alpha^* + \gamma_2\overline{\alpha}\alpha^*\tau_1^*e^{i\omega^*\tau_1^*} + \gamma_1\tau_{40}\alpha^*e^{i\omega^*\tau_{40}}.\end{aligned} \tag{51}$$

In addition, from $\langle\psi, A\phi\rangle = \langle A^*\psi, \phi\rangle$ and $A\overline{q}(\theta) = -i\omega^*\overline{q}(\theta)$, we can get

$$\begin{aligned}-i\omega^*\langle q^*, \overline{q}\rangle &= \langle q^*, A\overline{q}\rangle = \langle A^*q^*, \overline{q}\rangle = \langle -i\omega^* q^*, \overline{q}\rangle \\ &= i\omega^*\langle q^*, \overline{q}\rangle.\end{aligned} \tag{52}$$

Hence, $\langle q^*(\theta), \overline{q}(\theta)\rangle = 0$.

Next, using the algorithms given in [4], we can calculate the projection system of (3) on the center manifold C_0 when $\mu = 0$. For the solution u_t of (38), let $z(t) = \langle q^*, u_t\rangle$, and then by (45) and (47), we can have

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{u}_t\rangle = \langle q^*, A(0)u_t + R(0)u_t\rangle = \langle q^*, A(0)u_t\rangle + \langle q^*, R(0)u_t\rangle \\ &= \langle A^*(0)q^*, u_t\rangle + \overline{q}^*(0)F(0, u_t) \\ &= i\omega^*\tau_{40}z + g(z, \overline{z}),\end{aligned} \tag{53}$$

where $g(z, \overline{z}) = \overline{q}^*(0)F(0, u_t) = g_{20}(\theta)z^2/2 + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\overline{z}^2/2 + g_{21}(\theta)z^2\overline{z}/2\cdots$.

Let

$$\begin{aligned}W(t, \theta) &= u_t(\theta) - z(t)q(\theta) - \overline{z}(t)\overline{q}(\theta) \\ &= u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}.\end{aligned} \tag{54}$$

Then, on the center manifold C_0, we can have

$$W(t, \theta) = W(z(t), \overline{z}(t), \theta), \tag{55}$$

where

$$\begin{aligned}W(z, \overline{z}, \theta) &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} \\ &\quad + W_{30}(\theta)\frac{z^3}{6} + \cdots,\end{aligned} \tag{56}$$

where z and $\overline{z}$ are local coordinates for the center manifold C_0 in the directions of q^* and $\overline{q}^*$. Noting that W is real if u_t is real. We consider only real solution. From (54), we can see

$$
\begin{aligned}
u_t(\theta) &= W(t,\theta) + 2\mathrm{Re}\{z(t)q(\theta)\} = W(t,\theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta) \\
&= (1,\alpha)^T e^{i\omega^*\tau_{40}\theta} z + (1,\overline{\alpha})^T e^{-i\omega^*\tau_{40}\theta}\overline{z} + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\overline{z} \\
&\quad + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots .
\end{aligned} \tag{57}
$$

Therefore, $u_{1t}(0)$, $u_{2t}(0)$, $u_{1t}(-1)$, $u_{2t}(-1)$, $u_{1t}(-\tau_1^*/\tau_{40})$, and $u_{2t}(-\tau_1^*/\tau_{40})$ can be obtained.

$$
\begin{aligned}
u_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
u_{2t}(0) &= z\alpha + \overline{z\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
u_{1t}(-1) &= z e^{-i\omega^*\tau_{40}} + \overline{z} e^{i\omega^*\tau_{40}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots, \\
u_{2t}(-1) &= z\alpha e^{-i\omega^*\tau_{40}} + \overline{z\alpha} e^{i\omega^*\tau_{40}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1) z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots, \\
u_{1t}\left(-\frac{\tau_1^*}{\tau_{40}}\right) &= z e^{-i\omega^*\tau_1^*} + \overline{z} e^{i\omega^*\tau_1^*} + W_{20}^{(1)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\frac{z^2}{2} + W_{11}^{(1)}\left(-\frac{\tau_1^*}{\tau_{40}}\right) z\overline{z} + W_{02}^{(1)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\frac{\overline{z}^2}{2} + \cdots, \\
u_{2t}\left(-\frac{\tau_1^*}{\tau_{40}}\right) &= z\alpha e^{-i\omega^*\tau_1^*} + \overline{z\alpha} e^{i\omega^*\tau_1^*} + W_{20}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right) z\overline{z} + W_{02}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\frac{\overline{z}^2}{2} + \cdots .
\end{aligned} \tag{58}
$$

Furthermore, we can get $g(z,\overline{z})$, g_{20}, g_{11}, g_{02}, and g_{21} as follows:

$$g(z,\overline{z}) = \overline{q}^*(0)F(0,u_t) = \overline{M}(1,\overline{\alpha}^*)F(0,u_t)$$

$$= \overline{M}(1,\overline{\alpha}^*)\tau_{40}\begin{pmatrix} \frac{1}{2}f_{20}^{(1)}u_{1t}^2(0) + \frac{1}{2}f_{02}^{(1)}u_{2t}^2(0) + f_{11}^{(1)}u_{1t}(0)u_{2t}(0) \\ \frac{1}{2}f_{200}^{(2)}u_{1t}^2(-1) + \frac{1}{2}f_{020}^{(2)}u_{2t}^2\left(-\frac{\tau_1^*}{\tau_{40}}\right) + \frac{1}{2}f_{002}^{(2)}u_{2t}^2(0) + f_{110}^{(2)}u_{1t}(-1) \\ u_{2t}\left(-\frac{\tau_1^*}{\tau_{40}}\right) + f_{101}^{(2)}u_{1t}(-1)u_{2t}(0) + f_{011}^{(2)}u_{2t}(0)u_{2t}\left(-\frac{\tau_1^*}{\tau_{40}}\right) \end{pmatrix}$$

$$\begin{aligned}
g_{20} &= 2\overline{M}\tau_{40}\left[\frac{1}{2}f_{20}^{(1)} + \frac{1}{2}f_{02}^{(1)}\alpha^2 + f_{11}^{(1)}\alpha + \overline{\alpha}^*\left(\frac{1}{2}f_{200}^{(2)}e^{-2i\omega^*\tau_{40}} + \frac{1}{2}f_{020}^{(2)}\alpha^2 e^{-2i\omega^*\tau_1^*} + \frac{1}{2}f_{002}^{(2)}\alpha^2\right.\right. \\
&\quad \left.\left. + f_{110}^{(2)}\alpha e^{-i\omega^*\tau_{40}}e^{-i\omega^*\tau_1^*} + f_{101}^{(2)}\alpha e^{-i\omega^*\tau_{40}} + f_{011}^{(2)}\alpha^2 e^{-i\omega^*\tau_1^*}\right)\right]; \\
g_{11} &= \overline{M}\tau_{40}\left[f_{20}^{(1)} + f_{02}^{(1)}\alpha\overline{\alpha} + f_{11}^{(1)}(\alpha+\overline{\alpha}) + \overline{\alpha}^*\left(f_{200}^{(2)} + f_{020}^{(2)}\alpha\overline{\alpha} + f_{002}^{(2)}\alpha\overline{\alpha} + f_{110}^{(2)}\left(\overline{\alpha}e^{-i\omega^*\tau_{40}}e^{i\omega^*\tau_1^*} + \alpha e^{i\omega^*\tau_{40}}e^{-i\omega^*\tau_1^*}\right)\right.\right. \\
&\quad \left.\left. + f_{101}^{(2)}\left(\overline{\alpha}e^{-i\omega^*\tau_{40}} + \alpha e^{i\omega^*\tau_{40}}\right) + f_{011}^{(2)}\left(\alpha\overline{\alpha}e^{-i\omega^*\tau_1^*} + \alpha\overline{\alpha}e^{i\omega^*\tau_1^*}\right)\right)\right]; \\
g_{02} &= 2\overline{M}\tau_{40}\left[\frac{1}{2}f_{20}^{(1)} + \frac{1}{2}f_{02}^{(1)}\overline{\alpha}^2 + f_{11}^{(1)}\overline{\alpha} + \overline{\alpha}^*\left(\frac{1}{2}f_{200}^{(2)}e^{2i\omega^*\tau_{40}} + \frac{1}{2}f_{020}^{(2)}\overline{\alpha}^2 e^{2i\omega^*\tau_1^*}\right.\right. \\
&\quad \left.\left. + \frac{1}{2}f_{002}^{(2)}\overline{\alpha}^2 + f_{110}^{(2)}\overline{\alpha}e^{i\omega^*\tau_{40}}e^{i\omega^*\tau_1^*} + f_{101}^{(2)}\overline{\alpha}e^{i\omega^*\tau_{40}} + f_{011}^{(2)}\overline{\alpha}^2 e^{i\omega^*\tau_1^*}\right)\right] \\
g_{21} &= 2\overline{M}\tau_{40}\left[f_{20}^{(1)}\left(\frac{1}{2}W_{20}^{(1)}(0) + W_{11}^{(1)}(0)\right) + f_{02}^{(1)}\left(\frac{1}{2}\overline{\alpha}W_{20}^{(2)}(0) + \alpha W_{11}^{(2)}(0)\right) + f_{11}^{(1)}\left(\frac{1}{2}W_{20}^{(1)}(0)\right.\right. \\
&\quad \left.\overline{\alpha} + \frac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(1)}(0)\alpha + W_{11}^{(2)}(0)\right) + \overline{\alpha}^*\left(f_{200}^{(2)}\left(\frac{1}{2}W_{20}^{(1)}(-1)e^{i\omega^*\tau_{40}} + W_{11}^{(1)}(-1)e^{-i\omega^*\tau_{40}}\right)\right. \\
&\quad + f_{020}^{(2)}\left(\frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\overline{\alpha}e^{i\omega^*\tau_1^*} + W_{11}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\alpha e^{-i\omega^*\tau_1^*}\right) + f_{002}^{(2)}\left(\frac{1}{2}\overline{\alpha}W_{20}^{(2)}(0) + \alpha W_{11}^{(2)}(0)\right) \\
&\quad + f_{110}^{(2)}\left(\frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)e^{i\omega^*\tau_{40}} + W_{11}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)e^{-i\omega^*\tau_{40}} + \frac{1}{2}W_{20}^{(1)}(-1)\overline{\alpha}e^{i\omega^*\tau_1^*} + \alpha W_{11}^{(1)}(-1)\right. \\
&\quad \left.\cdot e^{-i\omega^*\tau_1^*}\right) + f_{101}^{(2)}\left(W_{11}^{(2)}(0)e^{-i\omega^*\tau_{40}} + \frac{1}{2}W_{20}^{(2)}(0)e^{i\omega^*\tau_{40}} + \frac{1}{2}\overline{\alpha}W_{20}^{(1)}(-1) + \alpha W_{11}^{(1)}(-1)\right) + f_{011}^{(2)} \\
&\quad \left.\left.\cdot\left(\alpha W_{11}^{(2)}(0)e^{-i\omega^*\tau_1^*} + \frac{1}{2}\overline{\alpha}W_{20}^{(2)}(0)e^{i\omega^*\tau_1^*} + \frac{1}{2}\overline{\alpha}W_{20}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right) + \alpha W_{11}^{(2)}\left(-\frac{\tau_1^*}{\tau_{40}}\right)\right)\right)\right].
\end{aligned} \tag{59}$$

Next, we compute $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21} by using the method in [8, 9, 32]. From (38) and (53), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q} = \begin{cases} AW - 2\mathrm{Re}\{\overline{q}^*(0)F(0,u_t)q(\theta)\}, & \theta \in [-1,0) \\ AW - 2\mathrm{Re}\{\overline{q}^*(0)F(0,u_t)q(0)\} + F(0,u_t), & \theta = 0 \end{cases} \triangleq AW + H(z,\overline{z},\theta), \tag{60}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{61}$$

Thus, we have

$$AW(t,\theta) - \dot{W} = -H(z,\overline{z},\theta) = -H_{20}(\theta)\frac{z^2}{2} - H_{11}(\theta)z\overline{z} - H_{02}(\theta)\frac{\overline{z}^2}{2} - \cdots. \tag{62}$$

From (56), we can get

$$AW(t,\theta) = AW_{20}(\theta)\frac{z^2}{2} + AW_{11}(\theta)z\overline{z} + AW_{02}(\theta)\frac{\overline{z}^2}{2} + AW_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{63}$$

and

$$\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}} = W_{20}(\theta)z\dot{z} + W_{11}(\theta)(\dot{z}\overline{z} + z\dot{\overline{z}}) + \cdots = 2i\omega^*\tau_{40}W_{20}(\theta)\frac{z^2}{2} + \cdots. \tag{64}$$

Therefore, we can get

$$(A - 2i\omega^*\tau_{40})W_{20}(\theta) = -H_{20}(\theta), AW_{11}(\theta) = -H_{11}(\theta). \tag{65}$$

For $\theta \in [-1,0)$, we have

$$\begin{aligned} H(z,\overline{z},\theta) &= -\overline{q}^*(0)F(0,u_t)q(\theta) - q^*(0)\overline{F}(0,u_t)\overline{q}(\theta) \\ &= -g(z,\overline{z})q(\theta) - \overline{g}(z,\overline{z})\overline{q}(\theta). \end{aligned} \tag{66}$$

Comparing coefficients of (66) with (61), we can obtain

$$H_{20}(\theta) = -\left(g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta)\right), \tag{67}$$

$$H_{11}(\theta) = -\left(g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta)\right). \tag{68}$$

From (66) and (68), and the definition of A, we can get

$$\dot{W}_{20}(\theta) = 2i\omega^*\tau_{40}W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{69}$$

Noticing that $q(\theta) = q(0)e^{i\omega^*\tau_{40}\theta} = (1,\alpha)^T e^{i\omega^*\tau_{40}\theta}$.
By computing (69), we can get

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau_{40}}q(0)e^{i\omega^*\tau_{40}\theta} + \frac{i\overline{g}_{02}}{3\omega^*\tau_{40}}\overline{q}(0)e^{-i\omega^*\tau_{40}\theta} + E_1 e^{2i\omega^*\tau_{40}\theta}, \tag{70}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in \mathbb{R}^2$ is a constant vector.

Similarly, from (66) and (68) and the definition of A, we can get

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta), \tag{71}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\tau_{40}}q(0)e^{i\omega^*\tau_{40}\theta} + \frac{i\overline{g}_{11}}{\omega^*\tau_{40}}\overline{q}(0)e^{-i\omega^*\tau_{40}\theta} + E_2, \tag{72}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in \mathbb{R}^2$ is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (70) and (72), respectively. From the definition of A and (65), we can have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega^*\tau_{40}W_{20}(0) - H_{20}(0), \tag{73}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0). \tag{74}$$

Thus,

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_{40}(P_1,P_2)^T, \tag{75}$$

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \tau_{40}(Q_1,Q_2)^T, \tag{76}$$

where $P_1 = 1/2f_{20}^{(1)} + 1/2f_{02}^{(1)}\alpha^2 + f_{11}^{(1)}\alpha$ and

$$\begin{aligned}
P_2 &= \frac{1}{2} f_{200}^{(2)} e^{-2i\omega^* \tau_{40}} + \frac{1}{2} f_{020}^{(2)} \alpha^2 e^{-2i\omega^* \tau_1^*} + \frac{1}{2} f_{002}^{(2)} \alpha^2 + f_{110}^{(2)} \alpha e^{-i\omega^* \tau_{40}} e^{-i\omega^* \tau_1^*} \\
&\quad + f_{101}^{(2)} \alpha e^{-i\omega^* \tau_{40}} + f_{011}^{(2)} \alpha^2 e^{-i\omega^* \tau_1^*}, \\
Q_1 &= f_{20}^{(1)} + f_{02}^{(1)} \alpha\overline{\alpha} + f_{11}^{(1)} (\alpha + \overline{\alpha}), \\
Q_2 &= f_{200}^{(2)} + f_{020}^{(2)} \alpha\overline{\alpha} + f_{002}^{(2)} \alpha\overline{\alpha} + f_{110}^{(2)} \left(\overline{\alpha} e^{-i\omega^* \tau_{40}} e^{i\omega^* \tau_1^*} + \alpha e^{i\omega^* \tau_{40}} e^{-i\omega^* \tau_1^*}\right) \\
&\quad + f_{101}^{(2)} \left(\overline{\alpha} e^{-i\omega^* \tau_{40}} + \alpha e^{i\omega^* \tau_{40}}\right) + f_{011}^{(2)} \left(\alpha\overline{\alpha} e^{-i\omega^* \tau_1^*} + \alpha\overline{\alpha} e^{i\omega^* \tau_1^*}\right).
\end{aligned} \tag{77}$$

Noting that,

$$\left(i\omega^* \tau_{40} I - \int_{-1}^{0} e^{i\omega^* \tau_{40}\theta} d\eta(\theta)\right) q(0) = 0, \left(-i\omega^* \tau_{40} I - \int_{-1}^{0} e^{-i\omega^* \tau_{40}\theta} d\eta(\theta)\right) \overline{q}(0) = 0. \tag{78}$$

Substituting (71) and (75) into (73), we can have

$$\left(2i\omega^* \tau_{40} I - \int_{-1}^{0} e^{2i\omega^* \tau_{40}\theta} d\eta(\theta)\right) E_1 = 2\tau_{40} (P_1, P_2)^T, \tag{79}$$

i.e., $\begin{pmatrix} 2i\omega^* - \alpha_1 & -\alpha_2 \\ -\gamma_1 e^{-2i\omega^* \tau_1^*} & 2i\omega^* - \gamma_2 e^{-2i\omega^* \tau_{40}} \end{pmatrix} E_1 = 2 \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$.

It follows that,

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \tag{80}$$

where $\Delta_1 = \det\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, $\Delta_{11} = 2\det\begin{pmatrix} P_1 & m_2 \\ P_2 & m_4 \end{pmatrix}$, $\Delta_{12} = 2\det\begin{pmatrix} m_1 & P_1 \\ m_3 & P_2 \end{pmatrix}$, $m_1 = 2i\omega^* - \alpha_1$, $m_2 = -\alpha_2$, $m_3 = -\gamma_1 e^{-2i\omega^* \tau_1^*}$, and $m_4 = 2i\omega^* - \gamma_2 e^{-2i\omega^* \tau_{40}}$.

Similarly, substituting (72) and (74) into (76), we can have

$$\int_{-1}^{0} d\eta(\theta) E_2 = -\tau_{40} (Q_1, Q_2)^T, \tag{81}$$

i.e., $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} E_2 = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$.

It follows that,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \tag{82}$$

where $\Delta_2 = \det\begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}$, $\Delta_{21} = 2\det\begin{pmatrix} Q_1 & \alpha_2 \\ Q_2 & \gamma_2 \end{pmatrix}$, and $\Delta_{22} = 2\det\begin{pmatrix} \alpha_1 & Q_1 \\ \gamma_1 & Q_2 \end{pmatrix}$.

From (70), (72), (75), and (83), we can calculate g_{21} and derive the following values:

$$\begin{cases}
c_1(0) = \dfrac{i}{2\omega^* \tau_{40}} \left(g_{20} g_{11} - 2|g_{11}|^2 - \dfrac{1}{3}|g_{02}|^2\right) + \dfrac{1}{2} g_{21}, \\
\mu_2 = -\dfrac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\left\{\lambda'(\tau_{40})\right\}}, \\
\beta_2 = 2\operatorname{Re}\{c_1(0)\}, \\
T_2 = -\dfrac{1}{\omega^* \tau_{40}} \left(\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'(\tau_{40})\right\}\right).
\end{cases} \tag{83}$$

Now, the main result in this section is given.

Theorem 7. *For system (3), suppose that (A1) or (A2) holds, and conditions (H1), (H3), (H8), and (H9) hold. The periodic solution is supercritical (resp. subcritical) if $\mu_2 > 0$ (resp. $\mu_2 < 0$). The bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical phase (resp. unstable) if $\beta_2 < 0$ (resp. $\beta_2 > 0$). The period of the bifurcating periodic solutions increases (resp. decreases) if $T_2 > 0$ (resp. $T_2 < 0$).*

4. Numerical Simulations

In this section, we shall present some examples and corresponding numerical simulations to verify above mentioned theoretical results.

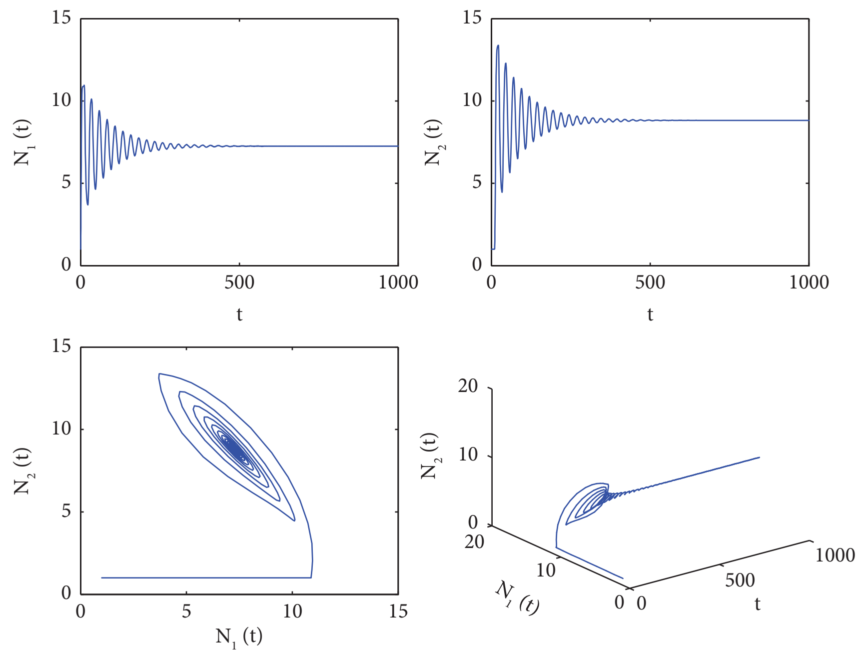

FIGURE 1: Behavior and phase portrait of the system (84) with $\tau_2 = 0, \tau_1 = 10$ ($\alpha_{21} = 6$).

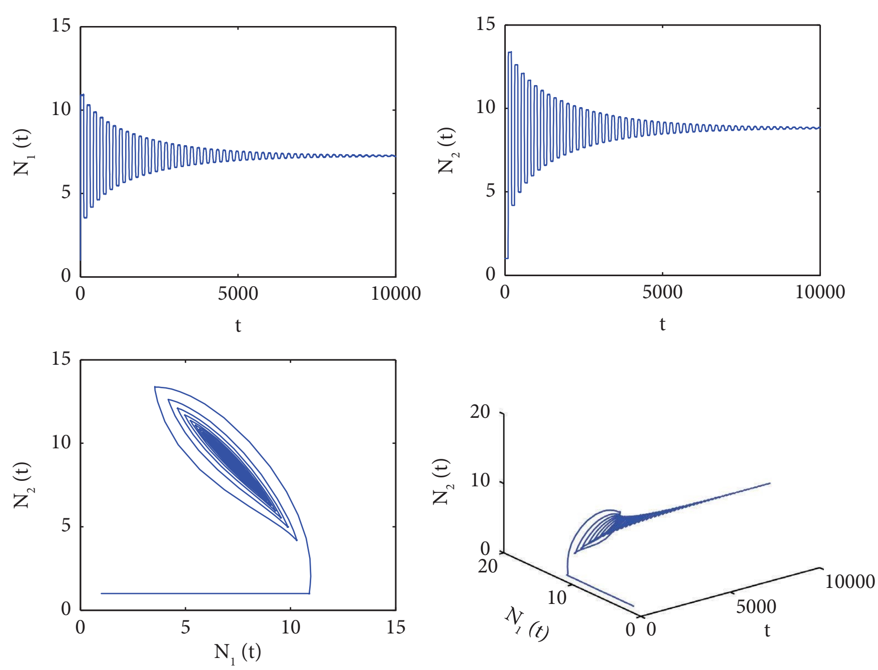

FIGURE 2: Behavior and phase portrait of the system (84) with $\tau_2 = 0, \tau_1 = 100$ ($\alpha_{21} = 6$).

Example 1. We consider system (3) under Case (2). First, we choose $a = 1$, $b = 4$, $r_1 = d_2 = 1$, $d_1 = \beta_1 = 0.1$, and $\alpha_{12} = 3$, and then the system (3) is

$$\begin{cases} \dfrac{dN_1}{dt} = N_1 + \dfrac{3N_1N_2}{1 + N_1 + 4N_2} - 0.1N_1N_2 - 0.1{N_1}^2, \\ \dfrac{dN_2}{dt} = \dfrac{\alpha_{21}N_1(t-\tau_1)N_2}{1 + N_1(t-\tau_1) + 4N_2(t-\tau_2)} - N_2, \end{cases} \quad (84)$$

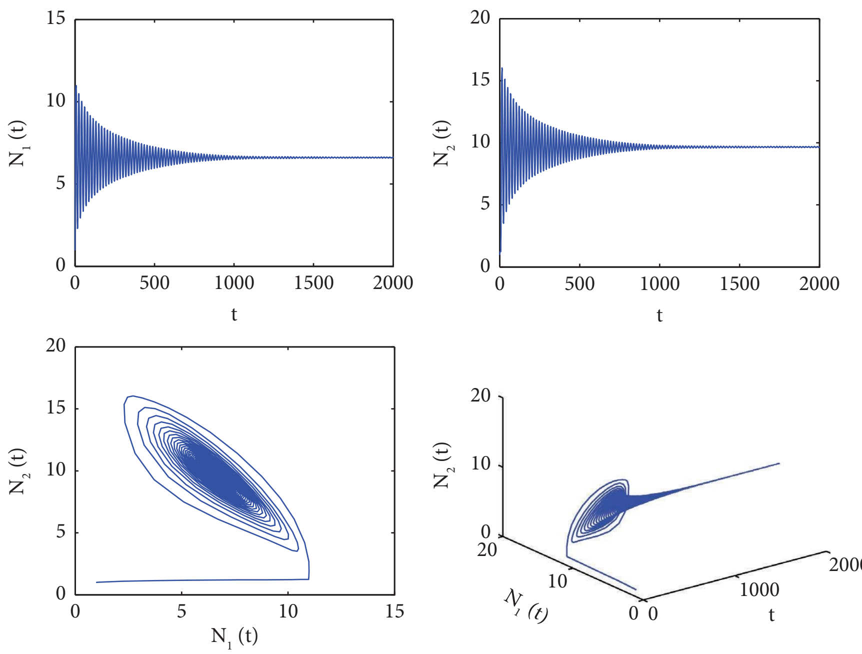

Figure 3: Behavior and phase portrait of the system (84) with $\tau_2 = 0, \tau_1 = 6.5$ ($\alpha_{21} = 7$).

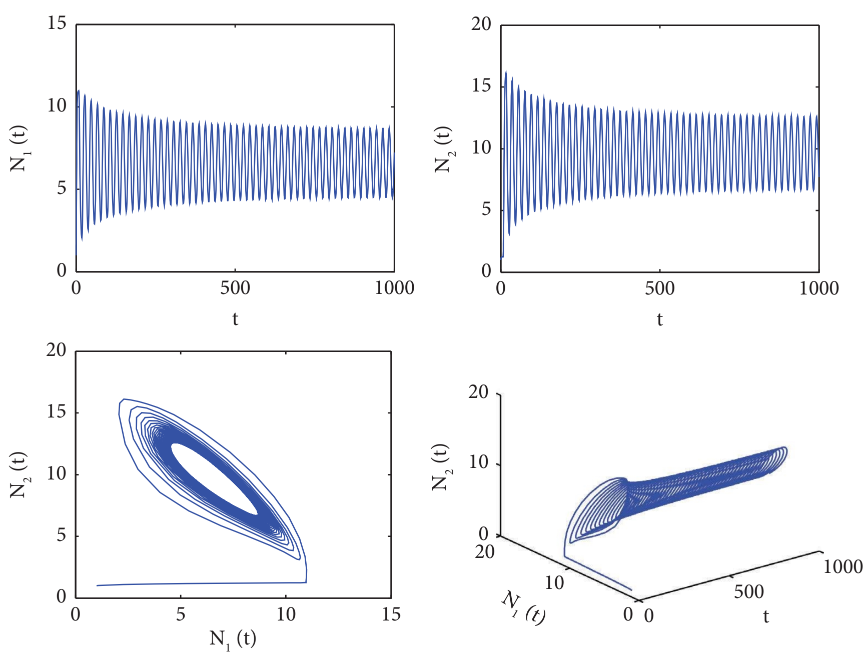

Figure 4: Behavior and phase portrait of the system (84) with $\tau_2 = 0, \tau_1 = 7.5$ ($\alpha_{21} = 7$).

where the initial value is $(1, 1)$. Let $\alpha_{21} = 6$, and by computations, we can obtain the unique positive equilibrium $E^* = (7.2568, 8.8210)$ of system (84). When $\tau_1 > 0$, $\tau_2 = 0$, (A1) or (A2) and (H2) hold, i.e. the conditions of Theorem 1 are satisfied, the positive equilibrium E^* of (84) is asymptotically stable for all $\tau_1 > 0$. In particular, we set $\tau_1 = 10$ and $\tau_1 = 100$, and the corresponding numerical simulation results are given in Figures 1 and 2.

Then, we consider system (84) with $\alpha_{21} = 7$ and keep other parameters unchanged when $\tau_1 > 0$, $\tau_2 = 0$, (A1) or (A2), and (H3) hold, i.e. the conditions of the Theorem 2 are satisfied. By computations, we can get the unique positive equilibrium $E^* = (6.6066, 9.6598)$, $\omega_{10} = 0.3351$, and $\tau_{10} = 6.9828$. If $\tau_1 \in [0, \tau_{10})$, then the positive equilibrium E^* of (84) is locally asymptotic stable under $\tau_2 = 0$, while it becomes unstable when τ_1 is gradually greater than this

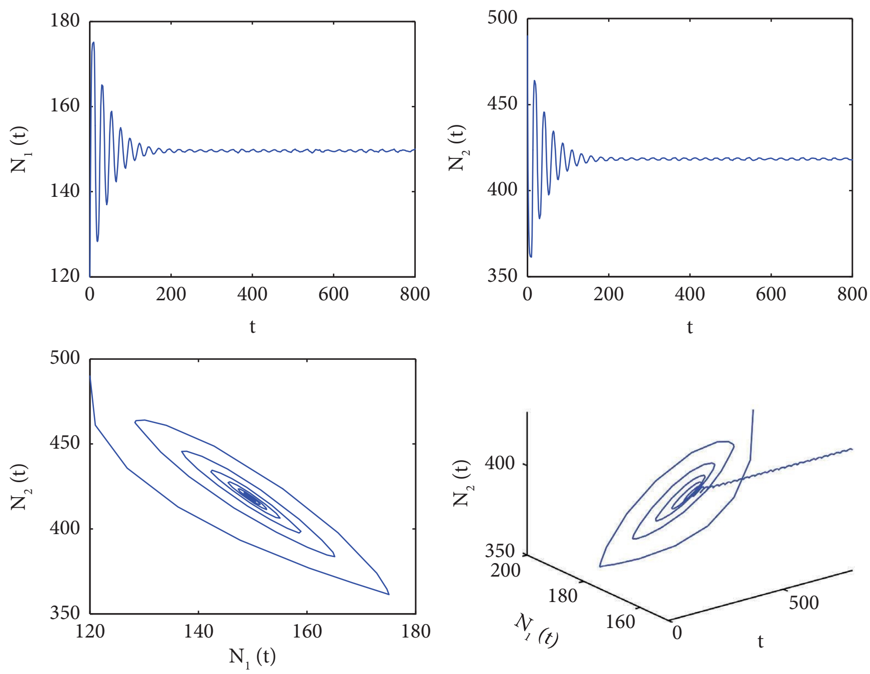

FIGURE 5: Behavior and phase portrait of the system (85) with $\tau_1 = 0, \tau_2 = 10$ ($\alpha_{21} = 3.8$).

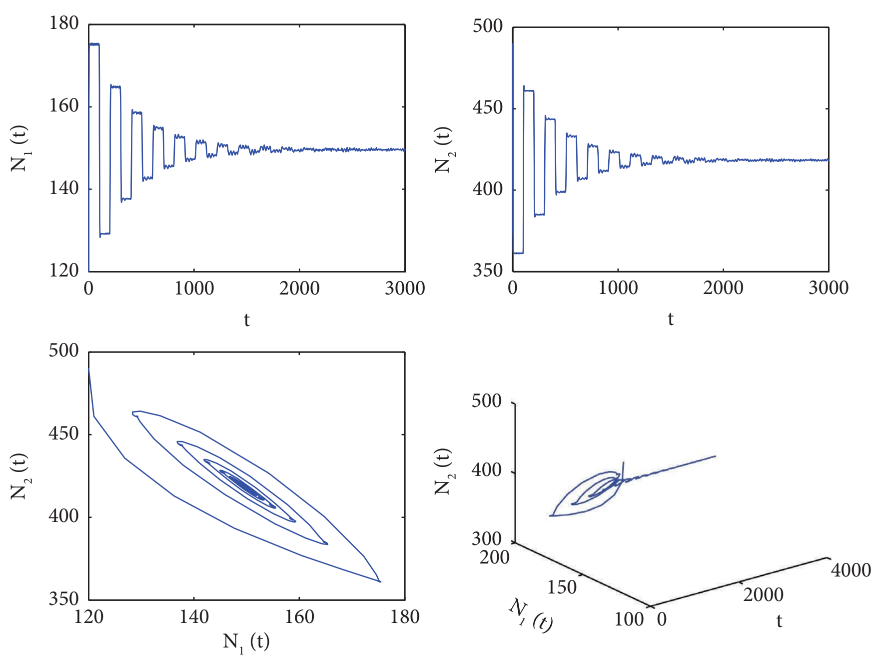

FIGURE 6: Behavior and phase portrait of the system (85) with $\tau_1 = 0, \tau_2 = 100$ ($\alpha_{21} = 3.8$).

critical value through Hopf bifurcation. The corresponding numerical simulation results are given in Figures 3 and 4.

Example 2. We consider system (3) under Case (3) and Case (4). First, we choose $a = b = 2$, $r_1 = 1$, $d_1 = \beta_1 = 0.005$, $d_2 = 0.5$, and $\alpha_{12} = 5$, and then the system (3) is

$$\begin{cases} \dfrac{dN_1}{dt} = N_1 + \dfrac{5N_1N_2}{1 + 2N_1 + 2N_2} - 0.005N_1N_2 - 0.005{N_1}^2, \\ \dfrac{dN_2}{dt} = \dfrac{\alpha_{21}N_1(t-\tau_1)N_2}{1 + 2N_1(t-\tau_1) + 2N_2(t-\tau_2)} - 0.5N_2, \end{cases} \tag{85}$$

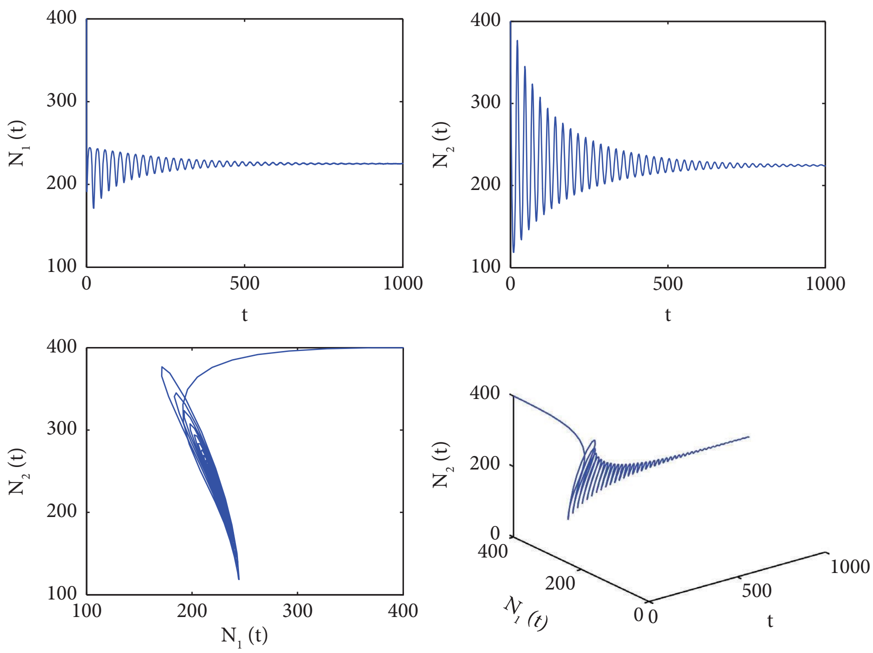

FIGURE 7: Behavior and phase portrait of the system (85) with $\tau_1 = 0, \tau_2 = 7$ ($\alpha_{21} = 2$).

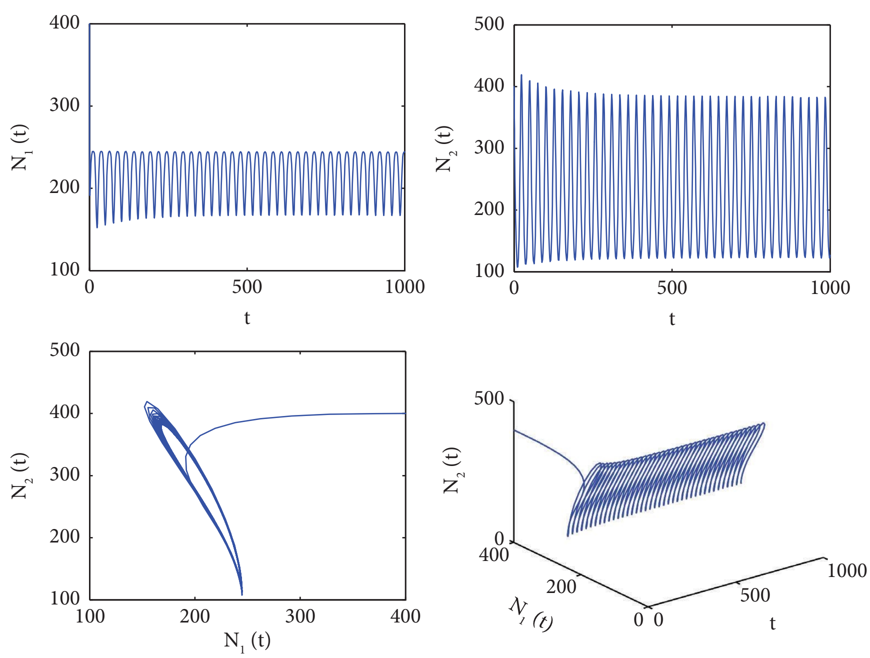

FIGURE 8: Behavior and phase portrait of the system (85) with $\tau_1 = 0, \tau_2 = 7.8$ ($\alpha_{21} = 2$).

where the initial value is $(120, 490)$. Let $\alpha_{21} = 3.8$, and by computations, we can obtain the unique positive equilibrium $E^* = (149.6003, 418.3809)$ of system (85). When $\tau_1 = 0, \tau_2 > 0$, (A1) or (A2), and (H4) hold, i.e. the conditions of the Theorem 3 are satisfied, the positive equilibrium E^* of (85) is asymptotically stable for all $\tau_2 > 0$. In particular, we set $\tau_2 = 10$ and $\tau_2 = 100$, and the corresponding numerical simulation results are given in Figures 5 and 6.

Then, we consider system (85) with $\alpha_{21} = 2$, whose initial value is $(400, 400)$ and keep other parameters unchanged. When $\tau_1 = 0$, $\tau_2 > 0$, (A1) or (A2), and (H5) hold, the conditions of the Theorem 4 are satisfied. By computations, we can get the unique positive equilibrium $E^* = (224.9722, 224.4722)$, $\omega_{20} = 0.2495$, and $\tau_{20} = 7.4321$. If $\tau_2 \in [0, \tau_{20})$, then the positive equilibrium E^* of (85) is locally asymptotic stable, while it becomes unstable when τ_2

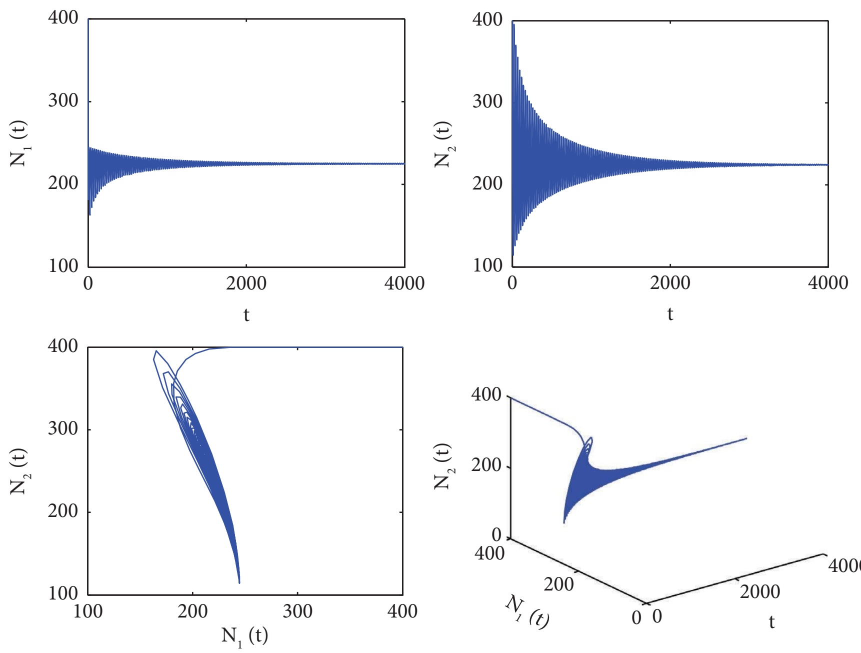

Figure 9: Behavior and phase portrait of the system (85) with $\tau_1 = 0.5 < \tau_{30}, \tau_2 = 7$ ($\alpha_{21} = 2$).

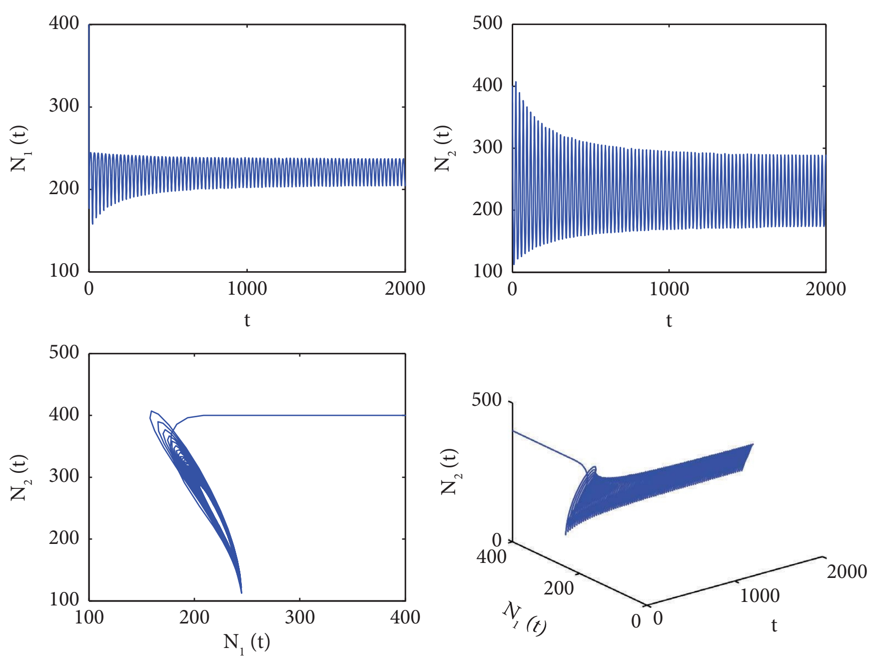

Figure 10: Behavior and phase portrait of the system (85) with $\tau_1 = 0.75 > \tau_{30}, \tau_2 = 7$ ($\alpha_{21} = 2$).

is gradually greater than this critical value through Hopf Bifurcation. The corresponding numerical simulation results are given in Figures 7 and 8.

Furthermore, we consider Case (4) of system (85) with $\alpha_{21} = 2$.We fix $\tau_2 = 7 < \tau_{20}$, and take τ_1 as a parameter. By computations, these parameters satisfied conditions (A1) or (A2), (H5), (H6), and (H7) of the Theorem 5, and we can get $\tau_{30} = 0.6768$. If $\tau_1 \in [0, \tau_{30})$, then the positive equilibrium E^* of (85) is locally asymptotic stable, while it becomes unstable when τ_1 is gradually greater than this critical value through Hopf bifurcation. The corresponding numerical simulation results are given in Figures 9 and 10.

Example 3. We consider system (3) under Case (5). Based on system (84) with $\alpha_{21} = 7$ in example 1, we further fix $\tau_1 = 6.5 < \tau_{10}$, and take τ_2 as a parameter. By computations,

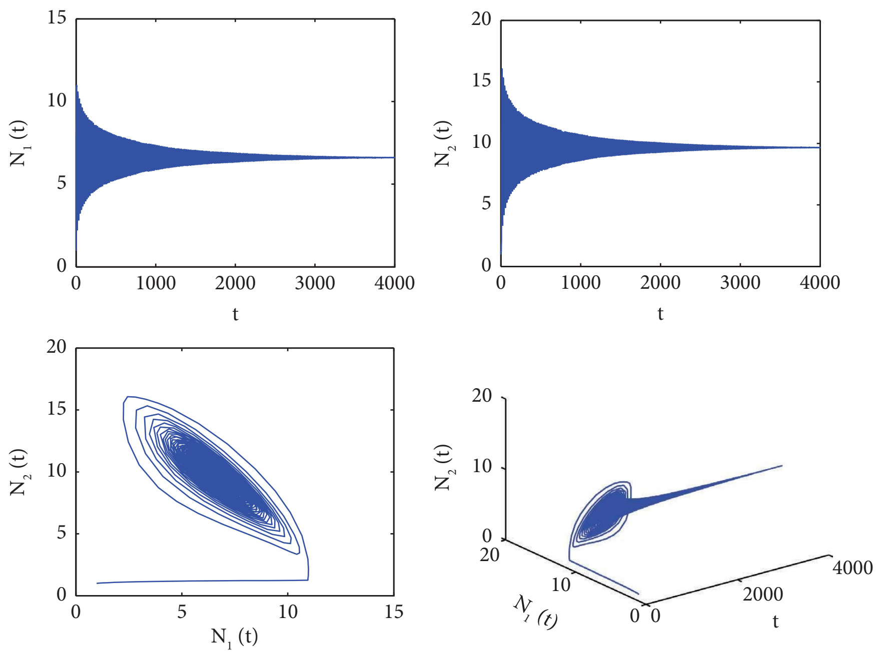

Figure 11: Behavior and phase portrait of the system (84) with $\tau_1 = 6.5, \tau_2 = 0.16 < \tau_{40}$ $(\alpha_{21} = 7)$.

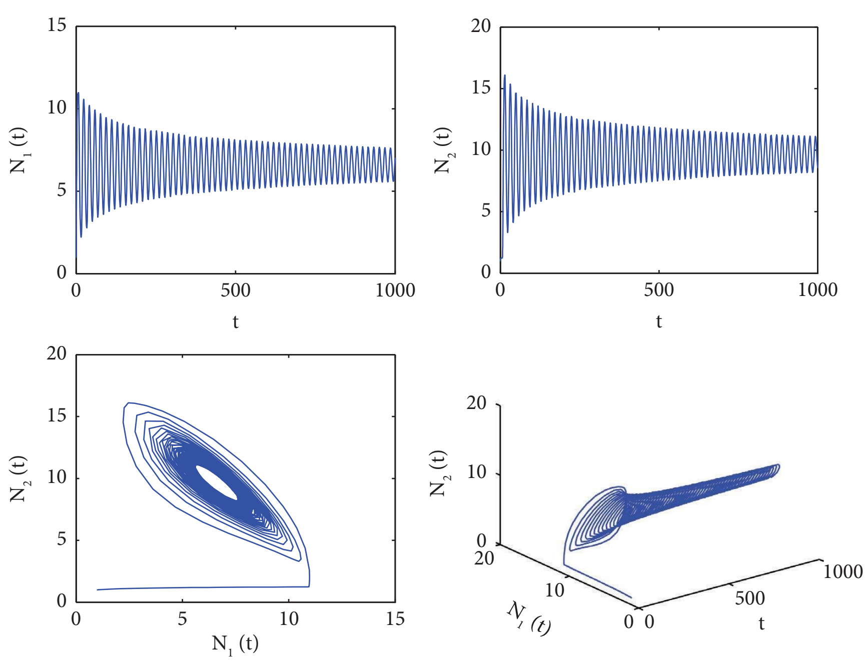

Figure 12: Behavior and phase portrait of the system (84) with $\tau_1 = 6.5, \tau_2 = 0.2 > \tau_{40}$ $(\alpha_{21} = 7)$.

these parameters satisfied conditions of Theorem 6, and we can get $\tau_{40} = 0.1740$. If $\tau_2 \in [0, \tau_{40})$, then the positive equilibrium E^* of (84) is locally asymptotic stable, while it becomes unstable when τ_2 is gradually greater than this critical value τ_{40} through Hopf bifurcation (see Figures 11 and 12). In addition, we can get $c_1(0) = -0.0431 - 1.9649i$, $\mu_2 = 12.9315 > 0$, and $\beta_2 = -0.0862 < 0$. Hence, when $\tau_1 > 0$ and $\tau_2 \in (0, \tau_{40})$, the Hopf bifurcation of system (84) is supercritical and the corresponding bifurcation periodic solutions are asymptotically stable.

Remark 1. In this paper, the dynamic properties of a plant-pollinator model with multiple delays are discussed by regarding different delays as bifurcating parameters; particularly, the theory of stability for its equilibrium point is analyzed in detail. The results of these aspects are comprehensive and satisfactory. We admit that only one case is implemented in the analysis of Hopf bifurcation for the reason that the remaining cases are similar to the one we discussed.

Remark 2. The main novel contributions are reflected as follows:

(1) The existing problems of bifurcations mainly focus on differential systems with unique delay or multiple identical delays owing to the fact that the stable regions with multiple different delays are difficult to determine. What merits further study in this work is that the dynamic properties of differential systems with multiple opposite delays are discussed in detail, which will stimulate us to further explore the bifurcating problems of the systems with multiple different delays.

(2) The bifurcating theories about the same delay in [32] are extended to multiple versions of different delays. The results are more accurate and less conservative.

(3) The exact bifurcating conditions caused by two different delays are derived by breaking through the difficulty of analyzing the characteristic equation.

Meanwhile, the challenges that accompany us in the future are how to design an appropriate controller to improve the stability for systems with multiple delays and how to extend the available system with integral order to the fractional order and continue to consider its dynamic properties, especially regarding the order as a bifurcating parameter, what may be a very meaningful topic.

5. Conclusions

In this paper, we considered the stability and Hopf bifurcation of a kind of a plant-pollination model with two delays. Taking the different combinations of the two delays τ_1 and τ_2 as the bifurcation parameters, we obtained the sufficient conditions for the local asymptotic stability and Hopf bifurcation of the positive equilibrium point E^* of the system model by using the Hopf bifurcation theorem. When $\tau_2 = 0$ and the parameters satisfy (H2) or $\tau_1 = 0$ and the parameters satisfy (H4), the positive equilibrium E^* of the model is asymptotically stable for any delay $\tau_1 > 0$ or $\tau_2 > 0$. It is found that when $\tau_2 = 0$ and the parameters satisfy (H3), there exists a critical value τ_{10} so that the positive equilibrium of the system is stable for $\tau_1 \in [0, \tau_{10})$, and it would be unstable for $\tau_1 \in (\tau_{10}, +\infty)$ and Hopf bifurcation at occurs E^* for $\tau_1 = \tau_{10}$. Similarly, when $\tau_1 = 0$ and the parameters satisfy (H5), there exists a critical value τ_{20} so that the positive equilibrium of the system is stable for $\tau_2 \in [0, \tau_{20})$, and it would be unstable for $\tau_2 \in (\tau_{20}, +\infty)$ and Hopf bifurcation occurs at E^* for $\tau_2 = \tau_{20}$. Furthermore, when $\tau_2 \in [0, \tau_{20})$ is fixed and τ_1 is taken as the bifurcation parameter, we obtain that there exists a critical value τ_{30} such that the positive equilibrium E^* of the system is stable for $\tau_1 \in [0, \tau_{30})$, and it would be unstable for $\tau_1 \in (\tau_{30}, +\infty)$ and undergoes Hopf bifurcation at E^* for $\tau_1 = \tau_{30}$. Analogously, when $\tau_1 \in [0, \tau_{10})$ is fixed and τ_2 is taken as the bifurcation parameter, we gain that there exists a critical value τ_{40} such that the positive equilibrium E^* of the system is stable for $\tau_2 \in [0, \tau_{40})$, and it would be unstable for $\tau_2 \in (\tau_{40}, +\infty)$ and undergoes Hopf bifurcation at E^* for $\tau_2 = \tau_{40}$. In a word, when τ crosses through a series of critical values including the above, the system can bifurcate a series of nontrivial periodic solutions from the positive equilibrium point. In addition, according to the normal form theory and the central manifold theorem of delay differential equations, some explicit formulas for determining Hopf bifurcation direction and stability of bifurcation periodic solutions are achieved. At last, some numerical simulations are conducted to demonstrate corresponding theoretical results. Our future work will focus on the following meaningful and promising aspects of researches:

(1) Extending the integral-order model to the fractional-order model. Fractional calculus has more advantages in describing some materials and processes with memory and genetic properties and can describe complex systems more concisely and clearly which has more potential to achieve some results that cannot be achieved by integral calculus.

(2) Further studying the dynamic properties of the fractional-order system with delays, which is a colorful project. As a bifurcation parameter, the bifurcating behavior caused by delay is worthy of our study, which is also a hot issue in recent years. So what results will be produced when order is used as a bifurcation parameter to study the bifurcation problem of the fractional-order model with delay? This is a topic worthy of our further discussion, and of course, it is also a challenging frontier issue.

Data Availability

The data used to support the findings of the current study are available from the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

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