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Research Article

Some Characterizations for Approximate Biflatness of Semigroup Algebras

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In this paper, we study an approximate biflatness of $l^1(S)$, where S is a Clifford semigroup. Indeed, we show that a Clifford semigroup algebra $l^1(S)$ is approximately biflat if and only if every maximal subgroup of S is amenable, $E(S)$ is locally finite, and $l^1(S)$ has an approximate identity in $c_{00}(S)$. Moreover, we prove that $l^1(S)$ is approximately biflat if and only if each maximal subgroup of S is amenable for an inverse semigroup S such that $E(S)$, the set of its idempotent elements, is totally ordered and locally finite.

1. Introduction and Preliminaries

The homological properties of a Banach algebra such as biflatness and biprojectivity were introduced by Helemskii in [1]. For every semilattice semigroup S, Choi in [2] proved that $l^1(S)$ is biflat if and only if S is uniformly locally finite. He also showed that $l^1(S)$ is biflat, where S is a Clifford semigroup if and only if $E(S)$ is uniformly locally finite and each maximal subgroup of S is amenable. This result was extended for every inverse semigroup S by Ramsden in [3]. Indeed, he proved that $l^1(S)$ is biflat if and only if S is a uniformly locally finite semigroup and the maximal subgroup of S at $p \in E(S)$ is amenable. Note that, in the presence of a bounded approximate identity, a Banach algebra A is biflat if and only if A is amenable, that is, there exists a bounded net (m_α) in $A\otimes_p A$ such that

$$\begin{aligned} &a\cdot m_\alpha - m_\alpha\cdot a \longrightarrow 0,\\ &\pi_A(m_\alpha)a \longrightarrow a\,(\forall a \in A), \end{aligned} \tag{1}$$

where $\pi_A\colon A\otimes_p A \longrightarrow A$ is the product morphism defined by $\pi_A(a\otimes b) = ab$, where $a\otimes b \in A\otimes_p A$. The projective tensor product $A\otimes_p A$ for a Banach algebra A is a Banach A-bimodule with the following actions:

$$\begin{aligned} &a\cdot(b\otimes c) = ab\otimes c,\\ &(b\otimes c)\cdot a = b\otimes ca\,(a,b,c \in A). \end{aligned} \tag{2}$$

Extending homological notions of Banach homology in the terms of "approximation" gave outstanding results in abstract harmonic analysis. Samei et al. introduced a concept of approximately biflat for Banach algebras. A Banach algebra A is approximately biflat, if there is a net of bounded A-bimodule morphisms $\theta_\lambda\colon (A\otimes_p A)^* \longrightarrow A^*$ such that

$$W^*OT - \lim_\lambda \theta_\lambda{}^\circ \pi_A^* = id_{A^*}, \tag{3}$$

where W^*OT is the weak* operator topology on $B(A^*)$ [4]. They proved that if an approximately biflat Banach algebra A has an approximate identity, then A is pseudoamenable. In fact A is pseudoamenable, if there exists a net $(m_\alpha)\subseteq A\otimes_p A$ such that $a\cdot m_\alpha - m_\alpha\cdot a \longrightarrow 0$ and $\pi_A(m_\alpha)a - a \longrightarrow 0$ for all $a \in A$ [5].

The module cohomological properties for Banach algebras, namely, module (approximate) biprojectivity and module (approximate) biflatness for Banach algebras which are generalization of the classical cases, were introduced in [6, 7]. In these papers, the authors found necessary and sufficient conditions for $l^1(S)$ to be module approximately biprojective and module approximately biflat, where S is an inverse semigroup.

In this paper, for a Clifford semigroup S, we study approximate biflatness of $l^1(S)$. Indeed, for a Clifford semigroup S, we obtain necessary and sufficient condition for $l^1(S)$ to be approximately biflat. We also show that for an inverse semigroup S with some mild assumptions, $l^1(S)$ is approximately biflat if and only if each maximal subgroup of S is amenable.

Some concepts in the semigroup theory are given here. For more details about semigroups, see [8]. Let S be a semigroup and let $E(S)$ be the collection of its idempotents. Then, a partial order on $E(S)$ is defined by

$$\begin{aligned} i \le j \Leftrightarrow i &= ij \\ &= ji\,(i, j \in E(S)). \end{aligned} \tag{4}$$

If $i = j$ whenever $i \le j$, then $i \in E(S)$ is called maximal. A commutative semigroup S is called semilattice if $E(S) = S$. The natural numbers $\mathbb{N}$ with the semigroup operations $m*_1 n = \min\{m, n\}$ or $m*_2 n = \max\{m, n\}$ becomes a semilattice.

Suppose that S is a discrete semigroup. If for each $s \in S$ there is a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$, then S is called an inverse semigroup. For an inverse semigroup S, there is a partial order on S defined by $s_1 \le s_2 \Leftrightarrow s_1 = s_1 s_1{}^* s_2$, where $s_1, s_2 \in S$. We set $(s] = \{t \in S\colon t \le s\}$, where $s \in S$. We remind that S is called locally finite, if $|(s]|$ is finite, for all $s \in S$.

Let S be an inverse semigroup. Then, $G_p = \{s \in S\colon ss^* = s^*s = p\}$ is called the maximal subgroup of S at p. An inverse semigroup S is called a Clifford semigroup if we have $ss^* = s^*s$ for every $s \in S$.

2. Approximately Biflat Property of Some Inverse Semigroup Algebras

In this section, we study approximate biflatness of some semigroup algebras.

Definition 1. Let A be a Banach algebra and let B be a closed subalgebra of A. Then, B is a retract of A if there is a continuous homomorphism $T\colon A \longrightarrow B$ such that its restriction to B is the identity map on B.

Lemma 2. *Suppose that A is a Banach algebra and B is a retract of A. Then, B is approximately biflat if A is approximately biflat*

Proof. Since A is approximately biflat, there is a net of bounded A-bimodule morphism $\theta_\lambda\colon (A \otimes_p A)^* \longrightarrow A^*$ such that $W^*OT - \lim_\lambda \theta_\lambda{}^\circ \pi_A^* = id_{A^*}$. By assumption, B is a retract of A. So, there is a continuous homomorphism $\eta\colon A \longrightarrow B$ such that its restriction to B is identity map on B. Now, we define

$$\widetilde{\theta_\lambda}\colon \left(B \otimes_p B\right)^* \longrightarrow B^*, \tag{5}$$

by $\widetilde{\theta_\lambda} = (i_B)^{*\circ} \theta_\lambda \circ (\eta \otimes \eta)^*$, where i_B is the inclusion map from B into A. Obviously, $\widetilde{\theta_\lambda}$ is a net of bounded B-bimodule morphisms. We show that $W^*OT - \lim_\lambda \widetilde{\theta_\lambda}{}^\circ \pi_B^* = id_{B^*}$. Let $\psi \in B^*$ and $b \in B$. Then, we have

$$\lim_\lambda \langle b, \widetilde{\theta_\lambda}{}^\circ \pi_B^*(\psi)\rangle = \lim_\lambda \langle b, (i_B)^{*\circ} \theta_\lambda \circ (\eta \otimes \eta)^* \circ \pi_B^*(\psi)\rangle. \tag{6}$$

Since $(\eta \otimes \eta)^{*\circ} \pi_B^* = \pi_A^*$, we have

$$\begin{aligned} \lim_\lambda \langle b, (i_B)^{*\circ} \theta_\lambda \circ (\eta \otimes \eta)^* \circ \pi_B^*(\psi)\rangle &= \lim_\lambda \langle b, (i_B)^{*\circ} \theta_\lambda \circ \pi_A^*(\psi)\rangle \\ &= \langle b, (i_B)^{*\circ} id_{A^*}(\psi)\rangle \\ &= \langle b, \psi\rangle. \end{aligned} \tag{7}$$

Hence, $W^*OT - \lim_\lambda \widetilde{\theta_\lambda}{}^\circ \pi_B^* = id_{B^*}$, and so, B is approximately biflat. □

Lemma 3. *Let A be an approximately biflat Banach algebra and let K be a closed ideal of A. Then, K is approximately biflat if K has an identity.*

Proof. Since A is approximately biflat, there exists a net $\theta_\lambda\colon (A \otimes_p A)^* \longrightarrow A^*$ such that $W^*OT - \lim_\lambda \theta_\lambda{}^\circ \pi_A^* = id_{A^*}$. For all $\lambda \in I$, we define $\sigma_\lambda = e \cdot \theta_\lambda \cdot e$, where e denotes the identity of K. Certainly, $(\sigma_\lambda)_{\lambda \in I}$ satisfies the definition of approximate biflatness of K. □

Theorem 4. *Suppose that $S = \cup_{p \in E(S)} G_p$ is a Clifford semigroup and $E(S)$ is locally finite. If $l^1(S)$ is approximately biflat, then G_p is amenable for all $p \in E(S)$. The converse is true, whenever $l^1(S)$ has an approximate identity in $c_{00}(S)$.*

Proof. Let $l^1(S)$ be approximately biflat. We regard the l^1-graded Banach algebra $\mathscr{B}_p = l^1 - \oplus_{q \in (p]} l^1(G_q)$ for all $p \in E(S)$ [9]. Obviously, $\mathscr{B}_p$ is a closed ideal of $l^1(S)$ and by [9, Proposition 2.1] $\mathscr{B}_p$ is unital and so Lemma 3 implies that $\mathscr{B}_p$ is approximately biflat. Since $E(S)$ is locally finite and so $(p]$ is finite, we can imply that $l^1(G_q)$ is approximately biflat for every $q \in (p]$. Thus, by [4, Theorem 4] follows that $l^1(G_q)$ is pseudoamenable for every $q \in (p]$ and therefore by [5, Proposition 4.1], we can deduce that G_q is amenable for every $q \in (p]$. In particular, G_p is amenable.

Conversely, we let $(e_\alpha)_{\alpha \in I}$ be an approximate identity for$l^1(S)$ in $c_{00}(S)$. Then, there is a finite subset $\gamma_\alpha \subseteq E(S)$ for every $\alpha \in I$ such that $e_\alpha \in \mathscr{B}_{\gamma_\alpha} = l^1 - \oplus_{q \in \langle \gamma_\alpha \rangle} l^1(G_q)$, where $\langle \gamma_\alpha \rangle = \cup_{p \in \gamma_\alpha}(p]$. By assumption, $E(S)$ is locally finite and so by [9, Proposition 2.5] follows that $\mathscr{B}_{\gamma_\alpha}$ is amenable. Suppose that M_{γ_α} is a vertual diagonal of $\mathscr{B}_{\gamma_\alpha}$. Thus, [9, Proposition 2.1] implies that $\mathscr{B}_{\gamma_\alpha}$ has an identity and we denote this identity by e_{γ_α}. Hence, we have $\pi^{**}(M_{\gamma_\alpha}) = e_{\gamma_\alpha}$. We put $M_\alpha = e_\alpha \cdot M_{\gamma_\alpha}$ for all $\alpha \in I$. Certainly, we have $M_\alpha \in (\mathscr{B}_{\gamma_\alpha} \otimes_p \mathscr{B}_{\gamma_\alpha})^{**}$. On the other hand, $\mathscr{B}_{\gamma_\alpha}$ is a complemented ideal of $l^1(S)$. Thus, we can consider $M_\alpha \in (l^1(S) \otimes_p l^1(S))^{**}$. Now, we define $\rho_\alpha: l^1(S) \longrightarrow (l^1(S) \otimes_p l^1(S))^{**}$ by $\rho_\alpha(a) = a \cdot M_\alpha$. It is easy to see that $(\rho_\alpha)_{\alpha \in I}$ is a net of bounded $l^1(S)$-bimodule morphism such that for every $a \in l^1(S)$, we have

$$\begin{aligned} \pi^{**}_{\ell^1(S)}(a \cdot M_\alpha) &= a\pi^{**}_{\ell^1(S)}(M_\alpha) \\ &= ae_\alpha e_{\gamma_\alpha} \\ &= ae_\alpha \longrightarrow a. \end{aligned} \tag{8}$$

It follows that $l^1(S)$ is approximately biflat. □

Remark 5. Suppose that Γ is a totally ordered set and every nonempty subset of Γ has a least element. Then, Γ is called well-ordered.

Proposition 6. *Let S be an inverse semigroup and E(S) be totally ordered and locally finite. Then, $l^1(S)$ is approximately biflat if and only if every maximal subgroup of S is amenable.*

Proof. Let $l^1(S)$ be approximately biflat. By assumption, $E(S)$ is totally ordered semilattice. On the other hand, $E(S)$ is locally finite. It follows that $E(S)$ is well-ordered [9, Remark 2.10]. Now, [8, Theorem 5.5.1 and Proposition 5.5.2] imply that every inverse semigroup with well-ordered idempotents set is a Clifford semigroup. Thus, the previous theorem gives that every maximal subgroup of S is amenable.

Conversely, let $E(S)$ be locally finite and every maximal subgroup of S be amenable. Then, $E(S)$ is well-ordered by [9, Remark 2.10]. Since $E(S)$ is totally ordered, [10, Theorem 16] implies that $l^1(S)$ has a bounded approximate identity. Now, by [9, Remark 2.7 (iii)], this bounded approximate identity could be chosen to be in $c_{00}(E(S))$. By [8, Theorem 5.5.1 and Proposition 5.5.2], we can see that S is a Clifford semigroup. Hence, the previous theorem implies that $l^1(S)$ is approximately biflat. □

Data Availability

Data are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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