Hindawi
Journal of Mathematics
Volume 2023, Article ID 9968793, 7 pages
https://doi.org/10.1155/2023/9968793

Research Article

Some Variations of δ-Supplemented Modules with Regard to a Hereditary Torsion Theory

Jian Tian,[1] Esra Ozturk Sozen,[2] and Ali Reza Moniri Hamzekolaee [3]

[1] *Department of Basic Courses Teaching, Software Engineering Institute of Guangzhou, Guangzhou 510990, China*
[2] *Sinop University, Faculty of Sciences and Arts, Department of Mathematics, Sinop, Turkey*
[3] *University of Mazandaran, Department of Mathematics, Faculty of Mathematical Sciences, Babolsar, Iran*

Correspondence should be addressed to Ali Reza Moniri Hamzekolaee; a.monirih@umz.ac.ir

Received 17 February 2023; Revised 5 September 2023; Accepted 12 September 2023; Published 12 October 2023

Academic Editor: Asad Ullah

In present work, we describe and investigate torsion theoretic versions of δ-supplemented modules via a hereditary torsion theory τ. With this aim, first, we define δ_τ-small submodules. On this basis, the concepts of δ_τ-lifting modules, δ_τ-supplemented modules, and amply δ_τ-supplemented modules and their fundamental properties are given, respectively. Furthermore, we present δ_τ-semiperfect modules and give a characterization for them via (amply) δ_τ-supplemented modules. Even we supply binary relations between these new module classes.

1. Introduction

Along this work, an associative ring with a unit is denoted by R, a unitary left R-module is denoted by W, and R-*Mod* is the category of unitary left R-modules. The symbols "$\leq$ and $\leq_\oplus$" will denote a submodule and a direct summand of a module, respectively.

Let us point a community of modules with ζ. The *reject* of ζ in W is described by $\mathrm{Rej}_W(\zeta) = \cap\{\mathrm{Ker}(h) \mid h: W \longrightarrow U \text{ for some } U \in \zeta\}$. The module homomorphism $h: W \longrightarrow U$ satisfies $h(\mathrm{Rej}_W(\zeta)) \leq \mathrm{Rej}_U(\zeta)$. Whenever $h: W \longrightarrow U$ is onto and $\mathrm{Ker}(h) \subseteq \mathrm{Rej}_W(\zeta)$, $h(\mathrm{Rej}_W(\zeta)) = \mathrm{Rej}_U(\zeta)$ is confirmed [1].

A submodule X of W is called *small* in W (denoted by $X \trianglelefteq W$) if $W \neq X + P$ for every proper submodule P of W. A submodule X of W is called *essential* in W (denoted by $X \trianglelefteq W$) if the intersection of X with each submodule of W is nonzero excluding 0. The community of elements of W whose annihilators are essential in ${}_R R$ is described as the *singular submodule* of W (denoted by $Z(W)$). W is said to be *singular* (*nonsingular*) whenever $Z(W) = W$ $(Z(W) = 0)$ [2]. A form of small submodules via singularity was contributed to the literature in [3] by Zhou. For a module W, $X \leq W$ is said to be δ-*small* in W (denoted by $X \ll_\delta W$) in case $X + T = W$ with W/T singular implies that $T = W$. Let Ω be the community of whole singular simple modules. As it is indicated in [3], $\delta(W) = Rej_W(\Omega) = \cap\{X \leq W \mid W/X \in \Omega\} = \sum\{X \leq W \mid X \ll_\delta W\}$. For $X \leq W$, a δ-*supplement* submodule S of X provides $W = X + S$ and $X \cap S \ll_\delta S$. A δ-*supplemented* module W is a module in which each submodule of W is of a δ-supplement. Besides, $X \leq W$ is said to have *ample* δ-*supplements* in W if every submodule S of W with $W = X + S$ involves a δ-supplement of X in W. An amply δ-*supplemented* module W is a module in which each submodule of W is of ample δ-supplements. Even W is called δ-*lifting* if for each $X \leq W$, there exists a decomposition of W such that $W = A \oplus B$ with $A \leq X$ and $X \cap B \ll_\delta B$. W is called *distributive* if for $X, S, Z \leq W$, the statement $(X \cap S) + (X \cap Z) = X \cap (S + Z)$ is verified. If for each $h \in \mathrm{End}(W)$, $h(X) \subseteq X$, we say X is a fully invariant submodule of W. We refer the interested readers to [4–6] for concepts given here.

Now, we give place to fundamental concepts of torsion theory. Let $\tau = (T, F)$ be a torsion theory on R-Mod, where T denotes the community of all modules which are τ-torsion and F denotes the community of all modules which are free of τ-torsion, that is, $T = \{W \in R - \mathrm{Mod} \mid \tau(W) = W\}$ and $F = \{W \in R - \mathrm{Mod} \mid \tau(W) = 0\}$ such that $\tau(W) =$

$\sum\{X \mid X \leq W, X \in T\}$. Ordinarily, T is preserved under homomorphic images, extensions, and direct sums. In response to this, F is preserved by isomorphisms, submodules, extensions, and direct products. If T is preserved by submodules (injective hulls), then T is called a *hereditary (stable)* torsion theory. In the present study, we will accept that τ is a hereditary torsion theory unless otherwise specified. A submodule X of W is defined as *τ-dense (τ-pure)* if W/X is τ-torsion (τ-torsion free), denoted by $X \leq_{\tau-d} W$ ($X \leq_{\tau-p} W$). For further properties associated with the torsion theory, we refer to [7].

In recent years, it is a lifting trend for algebraists to get torsion theoretic forms of known concepts or theories from ring and module theory. In [8], the authors handled lifting modules according to a (hereditary) torsion theory. In 1985, Pardo defined *τ-essential submodules* [9]. By using this fact, in 2017, the authors investigated singular and nonsingular modules according to a hereditary torsion theory to determine the structure of τ-extending modules [10], first defined in [11] according to Bland's τ-essential submodules. They defined the set $Z_\tau(W) = \left\{x \in W \mid \mathrm{I}x \subseteq \tau(W), \exists I \trianglelefteq_{\tau R} R\right\}$. The submodule $Z_\tau(W)$ is called a *τ-singular submodule* of W. W is called a *τ-singular* module provided $W = Z_\tau(W)$, and W is called a non-*τ-singular* module provided $0 = Z_\tau(W)$. Furthermore, in [12], τ-complement submodules of a module are defined as a torsion theoretic version of complement submodules. Dually, supplemented modules, some generalizations, and characterizations of them are handled from this aspect by various authors [13].

In the present study, the structure of δ-supplemented modules is researched by using the concept of τ-singularity of a submodule according to Pardo's τ-essential submodules. Motivated by this idea, we handle the special form of lifting modules given in [8] with respect to τ-singularity. To obtain this, first, we define δ_τ-small submodules and give fundamental properties similar to δ-small submodules. In the light of this fact, we introduce δ_τ-lifting, δ_τ-supplemented, and amply δ_τ-supplemented modules. We also interested in binary relations between these modules. Moreover, δ_τ-semiperfect modules are presented, and characterizations of a δ_τ-semiperfect module are given in view of being (amply) δ_τ-supplemented under special conditions.

2. δ_τ-Small Submodules

Definition 1. Let W be a module and $X \leq W$. If $X + S \neq W$ whenever W/S is τ-singular for any $S \leq W$, then X is said to be δ_τ-small in W. The notation $X \ll_{\delta_\tau} W$ is prefered to point that X is a δ_τ-small submodule of W.

Explicitly, each small submodule of a module is δ_τ-small. Also, note that as τ-singular module classes are different from singular ones, there is not a certain relation between δ_τ-small submodules and δ-small submodules. But if $_RR$ and W are free of τ-torsion, then these concepts coincide.

Following lemma is given for a submodule of a module to be δ_τ-small.

Lemma 2. *For a module W, the listed statements taking place below are equivalent:*

(1) $X \ll_{\delta_\tau} W$

(2) If $X + S = W$, then $W = Z \oplus S$ for a non-τ-singular submodule Z with $Z \subseteq X$

Proof

$(1) \Rightarrow (2)$ Let $X + S = W$. In this case, there subsists a submodule Z of X maximal according to the feature $S \cap Z \subseteq \tau(W)$. Thus, we obtain that $S + Z \trianglelefteq_\tau W$ by [12], Proposition 2.9. Following we have $W/(S + Z)$ is τ-singular by [10], Theorem 3.7. Since $X \ll_{\delta_\tau} W$ and $X + (S \oplus Z) = W$, we have $W = S \oplus Z$. Let $D \leq Z$. Then, $X + D + A = W$. Applying the same way as above by replacing X with $X + D$, we get $X + D = X \oplus D \leq_\oplus W$. Thus, $D \leq_\oplus Z$ that verifies W is semisample. So, we can write $Z = Z_\tau(Z) \oplus Z_n$, where Z_n is non-τ-singular. Then, $W/(S \oplus Z_n) = (S \oplus Z)/(S \oplus Z_n) \cong Z_\tau(Z)$ is τ-singular. Since $X \ll_{\delta_\tau} W$ and $W = X + (S + Z_n)$, we have $S \oplus Z_n = W$. This shows that $Z_\tau(Z) = 0$; that is, Z is non-τ-singular.

$(2) \Rightarrow (1)$: let $X + S = W$ for a submodule S of W with W/S τ-singular. By hypothesis, there subsists $Z \leq X$ with $Z \cong W/S$ is non-τ-singular. This shows that $S = W$.

Now, we list the main features of δ_τ-small submodules in the lemma mentioned as follows. □

Lemma 3. *The following statements given hold for a module W.*

(1) For submodules X, S, and Z of W with $S \subseteq X$, we have

(a) $X \ll_{\delta_\tau} W \Leftrightarrow S \ll_{\delta_\tau} W$ and $X/S \ll_{\delta_\tau} W/S$
(b) $X + Z \ll_{\delta_\tau} W \Leftrightarrow X \ll_{\delta_\tau} W$ and $Z \ll_{\delta_\tau} W$

(2) If $X \ll_{\delta_\tau} W$ and $h\colon W \longrightarrow N$ is a homomorphism, then $h(X) \ll_{\delta_\tau} N$. Most particularly, if $X \ll_{\delta_\tau} W \subseteq N$, then $X \ll_{\delta_\tau} N$.

(3) Let $X_1 \subseteq W_1 \subseteq W$, $X_2 \subseteq W_2 \subseteq W$, and $W = W_1 \oplus W_2$. Then, $X_1 \oplus X_2 \ll_{\delta_\tau} W_1 \oplus W_2 \Leftrightarrow X_1 \ll_{\delta_\tau} W_1$ and $X_2 \ll_{\delta_\tau} W_2$.

Proof. The proofs can be repeated by a similar approach given for small submodules in ([1], 19.3). □

Definition 4. Let ϑ be the community of whole τ-singular simple modules. For a module W, let $\mathrm{Rej}_W(\vartheta) = \delta_\tau(W) = \cap\{X \leq W \mid W/X \in \vartheta\}$ be the reject of ϑ in W. If W does not have any submodule with this type, then we denote $\delta_\tau(W) = W$.

It is an easy fact that $\delta_\tau(W/\delta_\tau(W)) = 0$.

We give a relation between δ_τ-radical of a module and its δ_τ-small submodules in the following lemma.

Lemma 5. *Let W be a module. Then we have, for any module, W holds $\delta_\tau(W) = \sum\left\{X \leq W \mid X \ll_{\delta_\tau} W\right\}$.*

Proof. Let $X \ll_{\delta_\tau} W$. We will show that X is contained in every maximal submodule T of W with W/T τ-singular. Assume that $X \nsubseteq T$ for a maximal submodule of W with W/T τ-singular.

Then, since T is maximal, $X + T = W$. Then, $T = W$, which is a contradiction to the fact that T is maximal in W. Hence, $\sum_{X \ll_{\delta_\tau} W} X \subseteq \delta_\tau(W)$. For any $x \in \delta_\tau(W)$, clearly x is the element of all maximal submodules P of W with W/P being τ-singular. Now, we claim that $Rx \ll_{\delta_\tau} W$. Assume that Rx is not δ_τ-small in W and $\eta = \{T \leq W \mid x \notin T, W/T\ \tau-$ singular and $Rx + T = W\}$. It is clear that $\eta \neq \varnothing$, since Rx is not δ_τ-small in W. By the Zorn Lemma, there exists a maximal element K in η. Accordingly, $x \notin K$ and so we have the contradiction $x \notin \delta_\tau(W)$. Hence, $\delta_\tau(W) = \sum_{X \ll_{\delta_\tau} W} X$.

Now, we give some facts about δ_τ-radical of a module. □

Lemma 6

(1) If $h: W \longrightarrow N$ is a homomorphism, then $h(\delta_\tau(W)) \subseteq \delta_\tau(N)$. So $\delta_\tau(W) \leq W$ is fully invariant.
(2) If $W = \oplus_{i\in I} W_i$, then $\delta_\tau(W) = \oplus_{i\in I} \delta_\tau(W_i)$.
(3) $\delta_\tau(W)$ is the unique largest $\delta\tau$-small submodule of W if every submodule of W is contained in a maximal submodule of W.

Proof. The proof can be repeated alike given in [1]. □

Corollary 7. *If τ is a stable torsion theory or $_RR$ is free of τ-torsion, each δ-small submodule is $\delta\tau$-small in W by ([10], Lemma 3.1).*

As it is understood from the definitions, δ-small submodules need not be $\delta\tau$-small and the converse is also. They are only specialized versions of each other.

3. δ_τ-Lifting Modules

In this department of the article, we introduce δ_τ-lifting modules and present fundamental properties of them. First, we give matching conditions for a module W to be δ_τ-lifting, and afterwards, we handle the other structure theorems for homomorphic images, direct summands, direct sums, etc.

Definition 8. A module W is called δ_τ-lifting if for $N \leq W$ there exists a decomposition $W = X \oplus S$ such that $X \leq N$ and $N \cap S \ll_{\delta_\tau} W$.

If $_RR$ is τ-torsion free or τ is a stable torsion theory, then the case of being δ-lifting satisfies the case of being δ_τ-lifting for a module W since $Z(W) \subseteq Z_\tau(W)$. Even these new concepts coincide for τ-torsion free modules over τ -torsion free rings since $Z(W) = Z_\tau(W)$.

In the following theorem, we list the equivalent conditions for a module to be δ_τ-lifting.

Theorem 9

(1) The following statements given are equivalent for a module W:
  (a) W is δ_τ-lifting
  (b) For each $N \leq W$, there exists submodules $X, S \leq N$ providing $N = X \oplus S$, $X \leq_\oplus W$, and $S \ll_{\delta_\tau} W$
  (c) For each $N \leq W$, there exists $X \leq_\oplus W$, providing $X \leq N$ and $N/X \ll_{\delta_\tau} W/X$

(2) Every direct summand of a $\delta\tau$-lifting module inherits the property.

Proof

(1) ($1a \Rightarrow 1b$) It is obvious. ($1b \Rightarrow 1c$) Let $N \leq W$. By hypothesis, there exists a decomposition of N providing $N = X \oplus S$ with $X \leq_\oplus W$ and $Y \ll_{\delta_\tau} W$. For the natural epimorphism $\pi: W \longrightarrow M/X$, we have $\pi(S) = S + X/X = N/X \ll_{\delta_\tau} W/X$, since $S \ll_{\delta_\tau} W$. ($1c \Rightarrow 1a$) Let W be any submodule of W. By ($1c$), there exists a decomposition of W, providing $W = X \oplus S$ with $X \leq N$ and $N/X \ll_{\delta_\tau} W/X$. Therefore, $W = N + S$ and $N = X \oplus (S \cap N)$. Since $W/X \cong S$ and $N/X \cong N \cap S$, then we get $N \cap S \ll_{\delta_\tau} S$. Hence, W is a δ_τ-lifting module.
(2) Let W be δ_τ-lifting and $N \leq_\oplus W$. In that case, there exists $T \leq W$ with $W = N \oplus T$. For any $X \leq N \leq W$, since W is δ_τ-lifting, there exists a decomposition of W providing $W = Z \oplus S$ with $Z \leq X$ and $X \cap S \ll_{\delta_\tau} S$. Therefore, it is obtained that $N = Z \oplus (N \cap S)$ providing $N \cap (X \cap S) = X \cap (N \cap S) = X \cap S \ll_{\delta_\tau} W$ and so $X \cap S \ll_{\delta_\tau} N$ as $N \leq_\oplus W$. Hence, $X \cap S \ll_{\delta_\tau} N \cap S$ because $N \cap S \leq_\oplus N$.

The following example includes a δ_τ-lifting module. □

Example 1. Let R be a matrix ring in which elements are upper triangular matrices with the form 2×2 and components coming from the field F, $_RW = \begin{bmatrix} 0 & F \\ F & F \end{bmatrix}$ and $X = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$, which is an idempotent ideal of R. Here, τ_x is a hereditary torsion theory with the torsion part $T = \{N \in R\text{-Mod} | \text{IN} = 0\}$. Let us list the all proper submodules of W as follows: $N_1 = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} = \tau_X(W)$, $N_2 = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = \tau_X(N_2)$, $N_3 = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} = \tau_X(N_3)$, and $N_4 = \begin{bmatrix} 0 & 0 \\ F & F \end{bmatrix}$. Since $N_2 \leq_\oplus W$, N_3 is δ_τ-small in W and $N_1 = N_2 \oplus N_3$, and then, W is a δ_τ-lifting module by Theorem 38.

Now, we investigate when the factor module of a δ_τ-lifting module is δ_τ-lifting.

Proposition 10. *Let W be a δ_τ-lifting module. For any $X \leq W$, the module W/X is δ_τ-lifting if one of the following statements is provided:*

(1) For any $N \leq_\oplus W$, $(N + X)/X \leq_\oplus W/X$.

(2) *W is a distributive module.*

(3) *$h(X) \subseteq X$ for any idempotent $h = h^2 \in End(W)$. Most particularly, $X \leq W$ is fully invariant.*

Proof

(1) Let $K/X \leq W/X$. Since $K \leq W$ and W is δ_τ-lifting, there exists $D \leq_\oplus W$ with $D \leq K$ and $K/D \ll_{\delta_\tau} W/D$. It is clear to verify that $(D+X)/X \leq_\oplus W/X$ and $(D+X)/X \leq K/X \leq W/X$. Since $K/D \ll_{\delta_\tau} W/D$, then $K/D + X \ll_{\delta_\tau} W/D + X$ by Lemma 3. Hence, W/X is δ_τ-lifting.

(2) This condition will be proved by using (1). Let $W = S \oplus Z$. We have $W/X = (S+X)/X + (Z+X)/X$, and by hypothesis, $(S+X)/X \cap (Z+X)/X = (S \cap Z) + X/X = 0_{W/X}$. Hence, $(S+X)/X \leq_\oplus W/X$ and so W/X is δ_τ-lifting.

(3) Let $W = A \oplus B$. By (1), we will show that $A + X/X \leq_\oplus W/X$. Let $\pi: A \oplus B \longrightarrow A$ be the projection map where $\mathrm{Ker}(\pi) = (1-\pi)W = B$. Then, $\pi 2 = \pi \in End(W)$ and $\pi(W) = A$. By assumption, $\pi(X) \leq X$ and $(1-\pi)(X) \leq X$. So we have $\pi(X) = X \cap A$ and $(1-\pi)(X) = X \cap B$. Therefore, $X = \pi(X) \oplus (1-\pi)(X) = (X \cap A) \oplus (X \cap B)$. From here, it is clear to see that $(A+X)/X = (A \oplus (X \cap B))/X$ and $B + X/X = B \oplus (X \cap A)/X$. This implies $W/X = (A \oplus (X \cap B))/X + (B \oplus (X \cap A))/X$. In addition, since $[A \oplus (X \cap B)] \cap [B \oplus (X \cap A)] = [A \oplus (X \cap B)] \cap B \oplus (X \cap A) = (X \cap B) \oplus (A \cap B) \oplus (X \cap A) = (X \cap B) \oplus (X \cap A) = X$, we have $(A+X)/X \leq_\oplus W/X$. Hence, M is δ_τ-lifting by (1).

In Lemma 15, we proved that each direct summand of a δ_τ-lifting module is δ_τ-lifting. But the contrast idea is not true generally. By Theorem 12, we present a way verifying this claim by adding suitable conditions. But first, we give the following useful lemma see ([6], 41.14) for completeness. □

Lemma 11. *Let $W = X \oplus S$. Then, the following conditions listed are equivalent:*

(1) *X is S-projective*

(2) *For each $N \leq W$ with $W = T + S$, there exists $T' \leq T$ providing $W = T' \oplus S$*

Theorem 12. *Let $W = X \oplus S$ be a module such that X is both W-projective and S-projective. If X and S are δ_τ-lifting modules, then so is W.*

Proof. Let $N \leq W$. In that case, as X is δ_τ-lifting $X \cap (N+S) \leq W$, there exist direct summands D, D' of X with $D \leq X \cap (N+S)$ and $X \cap (N+S) \cap D' = (N+S) \cap D' \ll_{\delta_\tau} X$. So we have $W = X \oplus S = D \oplus D' \oplus S = N + (D' \oplus S)$. Since X is self and S-projective it is clear to say that X is W-projective. By taking into account the exact sequence $D \longrightarrow D \oplus (D' \oplus S) \longrightarrow D' \oplus S$, it can be seen that D is $D' \oplus S$-projective [[6], 18.1/18.2]. Therefore, by Lemma 18, there exists $N' \leq N$ providing $W = N' \oplus (D' \oplus S)$. Following this, we can say $N \cap (W + D') = W \cap (N + D')$ for any $W \leq S$. In addition, since S is δ_τ-lifting, there exists $Y_1 \leq S \cap (N + D') = N \cap (S + D')$ such that $S = Y_1 \oplus Y_2$ and $N \cap (Y_2 + D') = Y_2 \cap (N + D') \ll_{\delta_\tau} Y_2$ for any $Y_2 \leq S$. Therefore, the fact that $W = N' \oplus (D' \oplus S) = N' \oplus (D' \oplus Y_1 \oplus Y_2) = (N' \oplus Y_1) \oplus (Y_2 \oplus D')$ can be seen easily. Since $N' \leq N$ and $X \leq N \cap (D' \oplus S) \leq N$, we have $N' \oplus Y_1 \leq N$ and so $W = N + (D' \oplus S)$. In addition, $N \cap (Y_2 \oplus D') = Y_2 \cap (N \oplus D') \ll_{\delta_\tau} Y_2 \leq Y_2 \oplus D'$.

Recall that the family of relatively projective modules is defined as a family of modules $\{P_i\}_{i \in I}$ where P_i is P_j-projective for each distinct $i, j \in I$. □

Corollary 13. *Let X be a semisimple module and S be a δ_τ-lifting module which are relatively projective with X, then $W = X \oplus S$ is δ_τ-lifting.*

In the next proposition, we verify that the direct sum of two δ_τ-lifting modules is δ_τ-lifting for a duo module (whose submodules are all fully invariant).

Proposition 14. *Let $W = X \oplus S$ be a duo module. If X and S are δ_τ-lifting modules, then so is W.*

Proof. Let $N \leq W$. Since W is a duo module, it can be written that $N = (N \cap X) \oplus (N \cap S)$. By assumption, for the submodules $N \cap X \leq X$ and $N \cap S \leq S$, there exist submodules X_1, $X_2 \leq X$ and S_1, $S_2 \leq S$, respectively, such that $X = X_1 \oplus X_2$, $X_1 \leq N \cap X$, and $N \cap X_2 \ll_{\delta_\tau} X_2$ and $S = S_1 \oplus S_2$, $S_1 \leq N \cap S$, and $N \cap S_2 \ll_{\delta_\tau} S_2$. Therefore, $W = X \oplus S = (X_1 \oplus X_2) \oplus (S_1 \oplus S_2) = (X_1 \oplus S_1) \oplus (X_2 \oplus S_2)$. So we have $X_1 \oplus S_1 \leq (N \cap X) \oplus (N \cap S) = N \cap (X \oplus S) = N \cap W = N$ and $N \cap (X_2 \oplus S_2) = (N \cap X_2) \oplus (N \cap S_2) \ll_{\delta_\tau} X_2 \oplus S_2$.

In the following example, a type of a module can be seen that is δ -lifting but not δ_τ-lifting. □

Example 2. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F be a field and $W = {}_R R$. Let $I = e_{12}R + e_{22}R$, where e_{ij} is the matrix unit in R. Note that for the idempotent ideal I, we have a hereditary torsion theory τ_I with the torsion part $T_I = \{X \in R\text{-Mod} | XI = 0\}$. Let $X = e_{12}R$. Note that X is simple, and it is not a direct summand of W as $X \trianglelefteq e_{11}R$ which is a direct summand. Also, X is not τ_I-torsion as $XI = e_{12}R$. Thus, X does not involve any direct summand S of W such that X/S is τ_I-torsion. Hence, W is not δ_τ-lifting. However, W is a lifting and so a δ-lifting module by [14].

4. δ_τ-Supplemented Modules

In this part of the study, we define δ_τ-supplemented modules and present basic properties of this type of modules.

Lemma 15. *Let $X, S \leq W$. Then, the statements given below are equivalent:*

(1) *$W = X + S$ and $X \cap S \ll_{\delta_\tau} S$*

(2) $W = X + S$, for any proper $T \leq S$ with S/T being τ-singular, $W \neq X + T$

Proof

(1) $\Rightarrow$ (2) If $W = X + T$, where $T \leq S$ and S/T is τ-singular, then $S = (X + T) \cap S = T + (X \cap S)$. Hence, we have $T = S$ since $X \cap S \ll_{\delta_\tau} S$.

(2) $\Rightarrow$ (1) If $S = T + (X \cap S)$, where $T \leq S$ and S/T is τ-singular, then $W = X + S = X + (X \cap S) + T = X + T$. By (2), $T = S$. So $X \cap S \ll_{\delta_\tau} S$. □

Definition 16. $S \leq W$ is said to be a δ_τ-supplement submodule of X in W if X and S provide one of the equivalent conditions given in Lemma 19. By the way, W is called δ_τ-supplemented if each submodule of W has a δ_τ-supplement in W.

We cannot claim every δ-supplemented module is δ_τ-supplemented or the converse statement directly because of being specialized versions of each other.

It can be seen that in the following proposition, being δ_τ-supplemented is preserved by homomorphic images.

Proposition 17. *Every homomorphic image of a δ_τ-supplemented module is δ_τ-supplemented.*

Proof. Let W be a δ_τ-supplemented module, $f\colon W \longrightarrow G$ be an epimorphism and S be a submodule of G. By assumption, there exists $X \leq W$ providing $f^{-1}(S) + X = W$ and $f^{-1}(S) \cap X \ll_{\delta_\tau} X$. In that case, $f(f^{-1}(S) + X) = f(f^{-1}(S)) + f(X) = [S \cap f(W)] + f(X) = S + f(X) = G$ and $f(f^{-1}(S) \cap X) = S \cap f(X) \ll_{\delta_\tau} f(X)$ by Lemma 5. Thus, $f(X)$ is a δ_τ-supplement of S in G. Hence, G is δ_τ-supplemented. □

Lemma 18. *Let W be a module and $X, S, Z \leq W$. If X is a δ_τ-supplement of S in W and S is a δ_τ-supplement of Z in W, then S is a δ_τ-supplement of X in W.*

Proof. Because X is a δ_τ-supplement of S in W, we get $S + X = W$, $S \cap X \ll_{\delta_\tau} X$, and S is a δ_τ-supplement of Z in W; we have $Z + S = W$, $Z \cap S \ll_{\delta_\tau} S$. It is enough to show that $X \cap S \ll_{\delta_\tau} S$. Let $T \leq W$ with $X \cap S + T = S$ and S/T be τ-singular. Then, $W = Z + S = Z + [(X \cap S) + T] = (X \cap S) + Z + T$. Since $S \cap X \ll_{\delta_\tau} W$, $W = W \oplus (Z + T)$ for a non-τ-singular submodule W with $W \subseteq X \cap S \subseteq S$ by Lemma 3. Hence, $S = [(W \oplus (Z + T)] \cap S = W \oplus [(Z + T) \cap S] = (W \oplus T) + (Z \cap S)$ by the modular law. Since $S/W \oplus T$ is τ-singular and $Z \cap S \ll_{\delta_\tau} S$, we have $S = W \oplus T$. Thus, $W = 0$ as S/T is both τ-singular and non-τ-singular. Finally, $S = T$ is obtained. □

Lemma 19. *For a δ_τ-supplemented module W, $W/\delta_\tau(W)$ is a semisimple module.*

Proof. Let $\delta_\tau(W) \leq X \leq W$. There exists $S \leq W$ providing $W = X + S$ and $X \cap S \ll_{\delta_\tau} S$. So $X \cap S \ll_{\delta_\tau} W$. Thus, $W/\delta_\tau(W) = X/\delta_\tau(W) + (S + \delta_\tau(W))/\delta_\tau(W)$ and $X \cap (S + \delta_\tau(W))/\delta_\tau(W) = (X \cap S) + \delta_\tau(W)/\delta_\tau(W) = \{0_{W/\delta_\tau(W)}\}$. So we have $W/\delta_\tau(W) = X/\delta_\tau(W) \oplus (S + \delta_\tau(W))/\delta_\tau(W)$.

Clearly, each δ_τ-lifting module is δ_τ-supplemented. The converse might be provided under additional conditions as in the following proposition. □

Proposition 20. *A projective δ_τ-supplemented module W is δ_τ-lifting whenever each δ_τ-supplement submodule of W is a direct summand.*

Proof. Let $X \leq W$. By hypothesis, there exists $S \leq W$ with $X + S = W$ and $X \cap S \ll_{\delta_\tau} S$. Since $S \leq_\oplus W$, $W = S \oplus D$ for some $D \leq W$. Following that, we have $D \subseteq X$ as W is projective and $= S + X$. Hence, we obtain a decomposition of W such that $W = D \oplus S$ with $D \subseteq X$ and $X \cap S \ll_{\delta_\tau} S$; that is, W is δ_τ-lifting.

Let W be an R-module and ${}_RR$ and W be τ-torsion free modules. If W is δ-supplemented, then W is also δ_τ -supplemented and vice versa.

Let τ be a stable torsion theory (as Goldie torsion theory) or ${}_RR$ is τ-torsion free (as ${}_{\mathbb{Z}}\mathbb{Z}$). Then, every δ-supplemented module is also δ_τ-supplemented.

Let R be τ-torsion and nonsingular ring. Then, a left R-module W is δ-supplemented module if and only if W is δ_τ-supplemented.

Before giving the finite sum of δ_τ-supplemented modules which is also δ_τ-supplemented, we need the following lemma. □

Lemma 21. *Let X, $U \leq W$, and X be a δ_τ-supplemented module. If $X + U$ has a δ_τ-supplement in W, then so does U.*

Proof. Since $X + U$ has a δ_τ-supplement in W, there exists $S \leq W$ providing $(X + U) + S = W$ and $(X + U) \cap S \ll_{\delta_\tau} S$. Also, there exists $Z \leq X$ providing $[(S + U) \cap X] + Z = X$ and $[(S + U) \cap X] \cap Z = (S + U) \cap Z \ll_{\delta_\tau} Z$. Thus, we have $W = (X + U) + S = [(S + U) \cap X] + Z + U + S = (U + S) + Z$ and $(U + S) \cap Z \ll_{\delta_\tau} Z$; that is, Z is a δ_τ-supplement of $U + S$ in W. Now, we claim that $S + Z$ is a δ_τ-supplement of U in W. It is evident that $(S + Z) + U = W$ and $(S + Z) \cap U \leq S \cap (Z + U) + Z \cap (S + U) \ll_{\delta_\tau} S + Z$ since $S \cap (Z + U) \leq S \cap (X + U) \ll_{\delta_\tau} S$ and $(S + U) \cap Z \ll_{\delta_\tau} Z$. □

Proposition 22. *Let X and S be δ_τ-supplemented modules. If $W = X + S$, then W is a δ_τ-supplemented module.*

Proof. Let $Z \leq W$. Since $X + S + Z = W$ has a trivial δ_τ-supplement 0 in W and X is δ_τ-supplemented, $S + Z$ has a δ_τ-supplement in W by Lemma 21. Thus, Z has a δ_τ -supplement in W as S is δ_τ-supplemented by Lemma 21. So, W is δ_τ-supplemented.

Recall that for a module W a module G is called finitely W-generated if there exists an epimorphism from the sum of finitely many copies of W to G.

As an immediate consequence of the finite sum and homomorphic image property, we give the following proposition. □

Proposition 23. *If W is a δ_τ-supplemented module, then every finitely W-generated module is δ_τ-supplemented.*

Proof. Let W be a δ_τ-supplemented module and G be a finitely W-generated module. Then, there exists an epimorphism h from $W^{(\Lambda)}$ (Λ is a finite index set) to G. Since W is δ_τ-supplemented, then $h(W^{(\Lambda)}) = G$ is a δ_τ-supplemented module by Propositions 17 and 22. □

5. Amply δ_τ-Supplemented Module

In this part of the study, we define amply δ_τ-supplemented modules and give basic properties of them. Also, we present relations between these modules and the modules introduced in previous sections.

Definition 24. A module W is called *amply δ_τ-supplemented* if for any submodules X, S of W with $W = X + S$, there exists a δ_τ-supplement T of X in W contained in S.

Clearly, every δ_τ-lifting module is amply δ_τ-supplemented, and every amply δ_τ-supplemented module is δ_τ-supplemented.

Proposition 25. *Every homomorphic image of an amply δ_τ-supplemented module is amply δ_τ-supplemented.*

Proof. Let W be an amply δ_τ-supplemented module and f be a homomorphism from W to G. We claim that $h(W)$ is amply δ_τ-supplemented. Let $h(W) = X + S$. Then, $W = h^{-1}(X) + h^{-1}(S)$. Thus, there exists a submodule T of W contained in $h^{-1}(S)$ with $h^{-1}(X) + T = W$, $h^{-1}(X) \cap T \ll_{\delta_\tau} T$. Following that, we have $X + h(T) = h(W)$ and $X \cap f(T) = f(f^{-1}(X) \cap T) \ll_{\delta_\tau} f(T) \leq S$. □

Proposition 26. *Let W be a module. If every submodule of W is δ_τ-supplemented, then W is an amply δ_τ-supplemented module.*

Proof. Let $W = X + S$ for $X, S \leq W$. By hypothesis, there exists $T \leq S$ providing $(X \cap S) + T = S$ and $(X \cap S) \cap T = X \cap T \ll_{\delta_\tau} T$. Thus, we have $W = X + S = X + (X \cap S) + T = X + T$. Hence, W is an amply δ_τ-supplemented module. □

Corollary 27. *The following listed statements given are equivalent for a ring R:*

(1) Every R-module is amply δ_τ-supplemented

(2) Every R-module is δ_τ-supplemented

Recall that we say a module is π-projective if there exists a homomorphism $h \in End(W)$ such that $h(W) \subseteq X$ and $(I_W - h)(W) \subseteq S$ for every submodule X, $S \leq W$ which satisfies $W = X + S$.

In general, every projective module is π-projective [1].

Theorem 28. *Let W be a π-projective δ_τ-supplemented module, then W is an amply δ_τ-supplemented module.*

Proof. For any submodule X of W, let $W = X + S$ for $S \leq W$. As W is π -projective, there exists an endomorphism h of W providing $h(W) \leq X$ and $(1-h)(W) \leq S$. Let T be a δ_τ-supplement of X in W. Then, $W = h(W) + (1-h)(W) = h(W) + (1-h)(X+T) \leq X + (1-h)(T) \leq W$, so we have $W = X + (1-h)(T)$ with $(1-h)(T) \leq B$. Also, $X \cap (1-h)(T) = (1-h)(X \cap T) \ll_{\delta_\tau} (1-h)(T)$. Hence, $(1-h)(T)$ is a δ_τ-supplement of X in W contained in S. □

Corollary 29. *If W is a projective and δ_τ-supplemented module, then W is an amply δ_τ-supplemented.*

Theorem 30. *Let W be an amply δ_τ-supplemented module whose δ_τ-supplements are direct summands of W. Then, W is a δ_τ-lifting module.*

Proof. Since W is amply δ_τ-supplemented, there exists a δ_τ-supplement S for every $X \leq W$ and there exists a δ_τ-supplement S' for $S \leq W$ with $S' \leq X$, $W = S' \oplus D$. Then, we have $W = S' + S$, and $X = S' + (S \cap X) = S' \oplus (X \cap D)$ is obtained. For the projection map $\pi\colon S' \oplus D \longrightarrow D$, it is true that $\pi(S \cap X) = \pi(X) = X \cap D$. Moreover, as $S \cap X \ll_{\delta_\tau} S$, $\pi(S \cap X) = X \cap D \ll_{\delta_\tau} \pi(S) \leq D$ and so $X \cap D \ll_{\delta_\tau} D$. Hence, for every $X \leq W$, there exists a decomposition of W providing $W = S' \oplus D$ with $S' \leq X$ and $X \cap D \ll_{\delta_\tau} D$. □

Corollary 31. *Let W be a projective δ_τ-supplemented module whose δ_τ-supplements are direct summands of W. Then, W is a δ_τ-lifting module.*

Proof. It is clear from Theorem 30 and Corollary 31. □

6. δ_τ-Semiperfect Modules

In this section, first, we define the projective δ_τ-cover of a module by means of δ_τ-small submodules to get the concept of δ_τ-semiperfect modules. At the end, we give a characterization theorem between δ_τ-semiperfect modules and (amply) δ_τ-supplemented modules.

Definition 32. Let E be a (projective) module and $f\colon E \longrightarrow W$ be an epimorphism with $\mathrm{Ker}(f) \ll_{\delta_\tau} E$. In this case, (f, E) is called a (projective) δ_τ-cover of W.

Definition 33. A module W is called a δ_τ-semiperfect module if any homomorphic image of W has a projective δ_τ-cover.

Proposition 34. *If $h\colon W \longrightarrow G$ is an epimorphism with* $\mathrm{Ker}(h) \leq \delta_\tau(W)$, *then $h(\delta_\tau(W)) = \delta_\tau(G)$.*

Proof. It is clear from [[15], Corollary, 8.17]. □

Lemma 35. *Let $f\colon W \longrightarrow G$ and $g\colon G \longrightarrow K$ be δ_τ-covers, then $g \circ f\colon W \longrightarrow K$ is a δ_τ-cover.*

Proof. Since f and g are δ_τ-covers, then $\mathrm{Ker}(f) \ll_{\delta_\tau} W$ and $\mathrm{Ker}(g) \ll_{\delta_\tau} G$. Now, we claim that $\mathrm{Ker}(g \circ f) \ll_{\delta_\tau} W$. Let $\mathrm{Ker}(g \circ f) + X = W$ with W/X τ-singular. Following that, we have $\mathrm{Ker}(g) + f(X) = G$. Hence, $f(X) = G$ is obtained as $\mathrm{Ker}(g) \ll_{\delta_\tau} G$, and $G/f(X)$ is τ-singular. This implies that $X + \mathrm{Ker}(f) = W$. Therefore, we have $X = W$ since W/X is τ-singular and $\mathrm{Ker}(f) \ll_{\delta_\tau} W$. □

Lemma 36. *Let $f_i: E_i \longrightarrow M_i$ be δ_τ-covers for every $i = 1, \ldots, n$. Then, $\oplus_{i=1}^n f_i : \oplus_{i=1}^n E_i \to M_i$ is a δ_τ-cover.*

Proof. It can be proved by the standard way. □

Theorem 37. *Let W be a module and $X \leq W$. Then, the following listed statements are equivalent:*

(1) W/X has a projective δ_τ-cover.

(2) If $W = X + S$ for $S \leq W$, then X has a δ_τ-supplement $T \leq S$ such that T has a projective δ_τ-cover.

(3) X has a δ_τ-supplement T which has a projective δ_τ-cover.

Proof

(1) ⇒ (2) Let $f: E \longrightarrow W/X$ be a projective δ_τ-cover. Since $W = X + S$, $g: S \longrightarrow S/(X \cap S) \cong (X + S)/X$ is an epimorphism. Since E is projective, there exists a homomorphism h from E to S satisfying $g \circ h = f$. Following that, we have $W/X = (h(E) + X)/X$ and so $W = X + h(E)$, $h(E) \leq S$. Also, $X \cap h(E) = h(\mathrm{Ker}(f)) \ll_{\delta_\tau} h(E)$, since $\mathrm{Ker}(f) \ll_{\delta_\tau} E$. Hence, $h(E)$ is a δ_τ-supplement of X in W. Thus, $h: P \longrightarrow h(E)$ is a projective δ_τ-cover as $\mathrm{Ker}(h) \subseteq \mathrm{Ker}(f) \ll_{\delta_\tau} P$.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Let $f: E \longrightarrow T$ be a projective δ_τ-cover. By hypothesis, $X + T = W$ and $X \cap T \ll_{\delta_\tau} T$. It follows that the natural epimorphism $g: T \longrightarrow T/(X \cap T) \cong (X + T)/X = W/X$ is a δ_τ-cover. So $g \circ f: E \longrightarrow W/X$ is a projective δ_τ-cover. □

Theorem 38. *The following listed statements are equivalent for a module W:*

(1) W is δ_τ-semiperfect

(2) W is amply δ_τ-supplemented whose δ_τ-supplements have projective δ_τ-covers

(3) W is δ_τ-supplemented whose δ_τ-supplements have projective δ_τ-covers

Proof. It is evident by Theorem 37. □

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

E. O. Sozen conceptualized the study and proposed the methodology. E. O. Sozen and A.R. Moniri Hamzekolaee validated the study. E. O. Sozen carried out the investigation and gathered the resources. E. O. Sozen wrote the original draft. E. O. Sozen, A. R. Moniri Hamzekolaee, and J.Tian wrote the review and edited the manuscript. J. Tian acquired funding.

Acknowledgments

This work was supported by the Software Engineering Institute of Guangzhou, 510990 of P. R. China, under the Grant/Award number (ST202101).

References

[1] R. Wisbauer, "Foundations of Module and Ring Theory," *Revised and Translated from the 1988 German Edition, Algebra, Logic and Applications*, Gordon & Breach, Philedelphia, PA, USA, 1991.

[2] K. R. Goodearl, "Ring Theory: Nonsingular Rings and Modules," Marcel Dekker, New York, NY, USA, 1976.

[3] Y. Zhou, "Generalizations of perfect, semiperfect and semiregular rings," *Algebra Colloquium*, vol. 7, no. 3, pp. 305–318, 2000.

[4] M. T. Koşan, "δ-Lifting and δ-supplemented modules," *Algebra Colloquium*, vol. 14, no. 01, pp. 53–60, 2007.

[5] R. Tribak, Y. Talebi, A. R. Moniri Hamzekolaee, and S. Asgari, "⊕-Supplemented modules relative to an ideal," *Hacettepe Journal of Mathematics and Statistics*, vol. 45, no. 1, pp. 107–120, 2016.

[6] Y. Wang, "δ-Small submodules and δ-supplemented modules," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 58132, pp. 1–8, 2007.

[7] P. E. Bland, "Topics in Torsion Theory," Wiley-VCH Verlag, Berlin, Germany, 1998.

[8] M. T. Koşan, T. C. Quynh, and Y. Talebi, "On a generalization of lifting modules relative to a torsion theory," *Taiwanese Journal of Mathematics*, vol. 17, no. 1, pp. 239–257, 2013.

[9] J. L. Gomez Pardo, "Spectral Gabriel topologies and relative singular functors," *Communications in Algebra*, vol. 13, no. 1, pp. 21–57, 1985.

[10] S. Çeken and M. Alkan, "Singular and nonsingular modules relative to a torsion theory," *Communications in Algebra*, vol. 45, no. 8, pp. 3377–3389, 2017.

[11] S. Charalambides and J. Clark, "CS-modules relative to a torsion theory," *Mediterranean Journal of Mathematics*, vol. 4, no. 3, pp. 291–308, 2007.

[12] S. Çeken and M. Alkan, "On τ-extending modules," *Mediterranean Journal of Mathematics*, vol. 9, no. 1, pp. 129–142, 2012.

[13] M. Alkan, "On τ-lifting modules and τ-semiperfect modules," *Turkish Journal of Mathematics*, vol. 33, no. 2, pp. 117–130, 2009.

[14] S. H. Mohamed and B. J. Müller, "Continuous and Discrete Modules," *London Mathematical Society Lecture Note Series*, vol. 147, 1990.

[15] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," Springer-Verlag, New York, NY, USA, 1974.