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Research Article

Convexity and Monotonicity Preservation of Ternary 4-Point Approximating Subdivision Scheme

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This work discusses a ternary 4-point approximation subdivision technique with two properties, namely, convexity and monotonicity preservation. The fundamental contribution of this research article is to extract the conditions that assure the suggested subdivision scheme's convexity and monotonicity. The methodology for extracting these conditions is explained in two theorems. These theorems prove that if the initial data is strictly convex and monotone and the derived conditions are satisfied, then the limiting curve generated by the proposed subdivision scheme will also be convex and monotone. To show the graphical simulations of results, 2D graphs are plotted. Curvature plots are also drawn to fully comprehend the derived conditions. The entire discourse is backed up by convincing examples.

1. Introduction

The subdivision schemes (SSs) are very indispensable in many fields of science to generate curves and surfaces. Mainly, the SSs are accustomed to computer-aided geometric design such as computer-aided design (CAD), computer-aided manufacturing (CAM), and other interconnected fields. For modeling curves and surfaces, SSs are highly regarded in fields such as computer science, image processing, medical sciences, and surgical simulation.

The credibility of SSs can be estimated by their shape-preserving properties. Positivity, monotonicity, and convexity preservations are the three main properties of shape preservation. In this paper, two properties such as convexity and monotonicity preservation of the SSs in equation (1) are analyzed. In curve and surface modeling, convexity and monotonicity play a vital role. The SSs do not preserve convexity and monotonicity for each type of polygon. If there is no limitation on the initial polygon, then the SSs will not retain shape-preserving properties. For convexity and monotonicity preservation of the limit curve generated by the SS, the initial set of control points is considered convex and monotonic. To sustain convexity and monotonicity, we encounter some conditions on parameters involved in the SS. The particular conditions are derived from the parameters involved in the SS to preserve convexity and monotonicity.

The convexity preservation and monotonicity preservation of SSs have been the subject of many research papers over the last few decades. In 1987, Dyn et al. [1] constructed a 4-point binary interpolating SS. The convexity of this SS was discussed by Cai [2] in 1995. For any convex set of discrete points with no three-point collinearity, Cai concluded that the convex limiting curve can be obtained by 4-point interpolating SS. In 1999, Dyn et al. [3] presented some conditions on the parameter depending upon the initial data, which ensures the convexity preservation of the 4-point SS analyzed in [1]. In a result, this SS generates the C^1 limit function with a second approximation order. In 2022, Yang and Yoon formulated a shape-preserving nonlinear SS, which generalized the B-spline of degree 3 [4]. Monotonicity- and convexity-preserving conditions of this SS were extracted, providing an improved approximation order of four while maintaining the same smoothness as the B-spline of degree 3. In 2002, a class of shape-preserving four-point subdivision schemes was developed by Kuijt and Damme [5]. These schemes are stationary and allow interpolation of nonuniform univariate data. Additional conditions were introduced to ensure convergence to a C^1 limit function. As a result, explicit rational convexity-preserving subdivision schemes were derived, and continuously differentiable limit functions can be generated from initial convex data. In 2009, Cai [6] examined convexity preservation conditions of 4-point C^2 ternary interpolating SS developed by Hassan et al. [7]. Hassan et al. concluded that, in order to maintain convexity in SSs, there should be some restrictions on the initial data points. A 5-point binary SS was developed by Tan et al. [8] for the convexity preservation of this SS. The Hölder exponent and generating polynomial method are used to investigate the uniform convergence. The value of k varies based on the choice of the parameter μ. A new technique was developed by Amat et al. [9] for convexity preservation of interpolating SS. Convexity-preserving properties for interpolatory subdivision schemes are introduced through a new approach. The approach is based on the relationship between subdivision schemes and prediction operators within Harten's multiresolution framework. Certain convexity properties of the reconstruction operator associated with prediction play a key role in this approach. Hao et al. [10] examined the convexity preservation of C^6 approximating SS developed by Siddiqi and Ahmed [11]. This SS with support $[-5,6]$ was proved to have simple and elegant properties. Siddiqi and Noreen [12] examined the convexity preservation of six-point ternary SS with parameter ω. The condition on the tension parameter is determined within the range of $(7/92) < \omega < (11/1215)$. The condition applies when the initial data is strictly convex. Its purpose is to ensure the range of C^2 continuous limit curves. A binary 4-point interpolating SS was developed by Beccari et al. [13] which generates C^1 continuous curve.

Recently, many researchers worked on the constructions, shape-preserving properties, and applications of SSs. Zhao et al. [14] extracted the conditions for the concavity of functions involving the generalized elliptic integral of the first kind. Zhao et al. [15] determined the particular conditions for convexity- and monotonicity-preserving properties of functions involving the generalized elliptic integral of the first kind. Zhao et al. [16] obtained the conditions for preserving convexity and concavity for Bessel functions of the first kind. By introducing local pushback operation, a brand-new interproximate subdivision framework is established which bridges the gap between interpolating schemes and approximating ones [17]. Convexity preservation of the ternary 6-point interpolating subdivision scheme has been presented in [18].

The aim of this paper is to determine the convexity and monotonicity preservation of C^3 ternary four-point stationary SS developed by Siddiqi and Rehan [19]. In Section 2, the ternary approximating SS is introduced, and the monotonicity-preserving property of the scheme in equation (1) is analyzed in Section 2.1. In Section 2.2, the convexity preservation of scheme [19] has been analyzed. In both sections, the significance of convexity- and monotonicity-preserving properties is illustrated through examples. In Section 2.3, the curvature plots of the proposed SS are presented. The whole manuscript is concluded in Section 3.

2. Ternary Four-Point Approximating Subdivision Scheme

Siddiqi and Rehan [19] developed $2N$-point Lagrange SS. These $2N$-Point SSs are extracted by Lagrange polynomial basis functions. The derivative continuity analysis and Laurent polynomial method have been employed. Basically, these are improvements of the ternary $2N$-point SS developed by Dubuc–Deslauriers. A parameter is involved in f_{3i}^{k+1}, which helps in enhancing the smoothness of the limiting curve.

The proposed four-point ternary SS is acquired for $N = 2$ from $2N$-point SS [19]. To define the ternary four-point SS, initial control points of the form $f_j^0 = f_j$: $j \in \mathbb{Z}$ are considered, and for each iteration $\{\kappa = 0,1,2,\ldots\}$, control points are defined as follows:

$$\begin{cases} f_{3j}^{\kappa+1} = -\theta f_{j-2}^{\kappa} + 4\theta f_{j-1}^{\kappa} + (1-6\theta) f_j^{\kappa} + 4\theta f_{j+1}^{\kappa} - \theta f_{j+2}^{\kappa}, \\ f_{2j}^{\kappa+1} = \dfrac{-5}{81} f_{j-1}^{\kappa} + \dfrac{20}{27} f_j^{\kappa} + \dfrac{10}{27} f_{j+1}^{\kappa} - \dfrac{4}{81} f_{j+2}^{\kappa}, \\ f_{2j+1}^{\kappa+1} = \dfrac{-4}{81} f_{j-1}^{\kappa} + \dfrac{10}{27} f_j^{\kappa} + \dfrac{20}{27} f_{j+1}^{\kappa} - \dfrac{5}{81} f_{j+2}^{\kappa}. \end{cases} \tag{1}$$

Siddiqi and Rehan [19] proved that equation (1) generates the converging curve having continuity C^3 but not C^m, $m > 3$. Further, it is also analyzed in [19] that the support and approximation order of equation (1) are $[-3, 3]$ and 4, respectively. This article answers the question of whether the generating curves from equation (1) preserve monotonicity and convexity or not. In the following section, the monotonicity-preserving condition for SS in equation (1) is discussed in detail.

2.1. Monotonicity Preservation. Since the control points are strictly monotonically increasing if their first divided difference $D_i^{\kappa} = (f_{i+1}^{\kappa} - f_i^{\kappa}) > 0$ where $i \in \mathbb{Z}$. Therefore, the first divided difference of equation (1) is evaluated and assumed that the initial control points are strictly monotone increasing, i.e., $D_i^0 > 0$, where $i \in \mathbb{Z}$.

The first-order divided difference of SS in equation (1) can be written as

$$D_{3i}^{\kappa+1} = -\theta D_{i-2}^{\kappa} + \left(\frac{5}{81} + 3\theta\right) D_{i-1}^{\kappa} + \left(\frac{26}{81} - 3\theta\right) D_i^{\kappa} + \left(-\frac{4}{81} + \theta\right) D_{i+1}^{\kappa}, \tag{2}$$

$$D_{3i+1}^{\kappa+1} = -\frac{1}{81} D_{i-1}^{\kappa} + \frac{29}{81} D_i^{\kappa} - \frac{1}{81} D_{i+1}^{\kappa}, \tag{3}$$

$$D_{3i+2}^{\kappa+1} = \left(-\frac{4}{81} + \theta\right) D_{i-1}^{\kappa} + \left(\frac{26}{81} - 3\theta\right) D_i^{\kappa} + \left(\frac{5}{81} + 3\theta\right) D_{i+1}^{\kappa} - \theta D_{i+2}^{\kappa}. \tag{4}$$

To derive the monotonicity-preserving condition, the following theorem is proved.

Remark 1. Denote

$$q_i^{\kappa} = \frac{D_{i+1}^{\kappa}}{D_i^{\kappa}},$$
$$Q^{\kappa} = \max_i, \quad \forall \kappa \in \{0\} \cup \mathbb{Z}^+. \tag{5}$$

The definition of q_i^{κ} and equation (7) give

$$\frac{1}{\lambda} < Q^0 < \lambda. \tag{6}$$

Since $(4/243) < \theta < (7/405)$, it is obtained that $\theta, \lambda > 1$. This is necessary for inequality equation (6). Simultaneously, it is noted that $\lambda < (27/4)$, i.e., $Q^0 < (27/4)$. Therefore, such conditions should be satisfied by initial control points; the difference of the first-order divided difference between every two neighboring initial control points cannot be too large. This can be easily seen from the examples at the end of the theorem. If the difference of the first-order divided difference is too large, whatever the parameter is, the scheme may not preserve monotonicity.

Theorem 2. *If the initial control points are all monotonically increasing,* i.e., $D_i^0 > 0$, $i \in \mathbb{Z}$, *furthermore, the parameter* θ *for* $(4/243) < \theta < (7/405)$ *satisfies*

$$Q^0 < \frac{5 + 243\theta}{81\theta} = \lambda. \tag{7}$$

Then, $D_i^{\kappa} > 0$, $Q^{\kappa} < \lambda$, $\forall \kappa \in \mathbb{Z}$, i.e., the four-point ternary SS in equation (1) preserves monotonicity.

Proof. In order to prove that $D_i^{\kappa} > 0$ and $(1/\lambda) < Q^{\kappa} < \lambda$, the mathematical induction is used.

It is obvious that when $\kappa = 0$, then $D_i^0 = f_{i+1}^0 - f_i^0 > 0$ and $(1/\lambda) < Q^0 < \lambda$ hold.

By inductive part, suppose $D_i^{\kappa} > 0$ and $(1/\lambda) < Q^{\kappa} < \lambda$ for some $\kappa \geq 0$. It will be shown that it also holds for $\kappa + 1$, and equation (2) is considered as follows:

$$\begin{aligned}
D_{3i}^{\kappa+1} &= f_{3i+1}^{\kappa+1} - f_{3i}^{\kappa+1} \\
&= -\theta D_{i-2}^{\kappa} + \left(\frac{5}{81} + 3\theta\right) D_{i-1}^{\kappa} + \left(\frac{26}{81} - 3\theta\right) D_i^{\kappa} + \left(-\frac{4}{81} + \theta\right) D_{i+1}^{\kappa} \\
&= \frac{D_{i-1}^{\kappa}}{81}\left[-\frac{81\theta}{q_{i-2}^{\kappa}} + 5 + 243\theta + (26 - 243\theta) q_{i-1}^{\kappa} + (-4 + 81\theta) q_i^{\kappa} q_{i-1}^{\kappa}\right] \\
&> \frac{D_{i-1}^{\kappa}}{81}\left[-81\theta\lambda + 5 + 243\theta + (26 - 243\theta) q_{i-1}^{\kappa} + (-4 + 81\theta)\lambda q_{i-1}^{\kappa}\right] \\
&> \frac{D_{i-1}^{\kappa}}{81}\left[-81\theta\lambda + 5 + 243\theta + [(-4 + 81\theta)\lambda + 26 - 243\theta] q_{i-1}^{\kappa}\right] \\
&> \frac{D_{i-1}^{\kappa}}{81}\left[-81\theta\lambda + 5 + 243\theta + \frac{(-4 + 81\theta)\lambda + 26 - 243\theta}{\lambda}\right] \\
&= \frac{D_{i-1}^{\kappa}}{81\lambda}\left[-81\theta\lambda^2 + (1 + 324\theta)\lambda + 26 - 243\theta\right] \\
&= \frac{D_{i-1}^{\kappa}}{81}\left(\frac{-20 + 1539\theta}{5 + 243\theta}\right) \\
&> 0.
\end{aligned} \tag{8}$$

From equation (3), we get

$$\begin{aligned}
D_{3i+1}^{\kappa+1} &= f_{3i+2}^{\kappa+1} - f_{3i+1}^{\kappa+1} \\
&= -\frac{1}{81}D_{i-1}^{\kappa} + \frac{29}{81}D_i^{\kappa} - \frac{1}{81}D_{i+1}^{\kappa} \\
&= \frac{D_i^{\kappa}}{81}\left[-\frac{1}{q_{i-1}^{\kappa}} + 29 - q_i^{\kappa}\right] \\
&> \frac{D_i^{\kappa}}{81}[29 - 2\lambda] \\
&= \frac{D_i^{\kappa}}{81}\left(23 - \frac{10}{81\theta}\right) \\
&> 0.
\end{aligned} \tag{9}$$

From equation (4), we get

$$\begin{aligned}
D_{3i+2}^{\kappa+1} &= f_{3i+3}^{\kappa+1} - f_{3i+2}^{\kappa+1} \\
&= \left(-\frac{4}{81} + \theta\right)D_{i-1}^{\kappa} + \left(\frac{26}{81} - 3\theta\right)D_i^{\kappa} + \left(\frac{5}{81} + 3\theta\right)D_{i+1}^{\kappa} - \theta D_{i+2}^{\kappa} \\
&= \frac{D_i^{\kappa}}{81}\left[\frac{-4 + 81\theta}{q_{i-1}^{\kappa+1}} + 26 - 243\theta + (5 + 243\theta)q_i^{\kappa} - 81\theta q_i^{\kappa} q_{i+1}^{\kappa}\right] \\
&> \frac{D_i^{\kappa}}{81}\left[(-4 + 81\theta)\lambda + 26 - 243\theta + (5 + 243\theta)q_i^{\kappa} - 81\theta\lambda q_i^{\kappa}\right] \\
&> \frac{D_i^{\kappa}}{81}\left[(-4 + 81\theta)\lambda + 26 - 243\theta + (5 + 243\theta - 81\theta\lambda)q_i^{\kappa}\right] \\
&> \frac{D_i^{\kappa}}{81}\left[(-4 + 81\theta)\lambda + 26 - 243\theta + \frac{5 + 243\theta - 81\theta\lambda}{\lambda}\right] \\
&= \frac{D_i^{\kappa}}{81\lambda}\left[(-4 + 81\theta)\lambda^2 + (26 - 243\theta)\lambda + 5 + 243\theta - 81\theta\lambda\right] \\
&= \frac{D_i^{\kappa}}{81\lambda}\left[(-4 + 81\theta)\lambda^2 + (26 - 324\theta)\lambda + 5 + 243\theta\right] \\
&= \frac{D_i^{\kappa}}{81}\left(19 - \frac{20}{81\theta}\right) \\
&> 0.
\end{aligned} \tag{10}$$

Thus, $D_i^{\kappa+1} > 0, \forall \kappa \geq 0, \kappa \in \mathbb{Z}$.
Hence, by induction, $D_i^{\kappa} > 0, \forall \kappa \in \{0\} \cup \mathbb{Z}^+$.

To prove $(1/\lambda) < Q^{\kappa} < \lambda$, it is sufficient to show that $q_i^{\kappa} < \lambda$ and $(1/q_i^{\kappa}) < \lambda$, $\forall \kappa \geq 0$ and $\forall \kappa \in \mathbb{Z}, \forall i \in \mathbb{Z}$.

Since

$$q_{3i}^{\kappa+1} = \frac{D_{3i+1}^{\kappa+1}}{D_{3i}^{\kappa+1}} = \frac{-\left(D_{i-1}^{\kappa}/81\right) + \left(29D_i^{\kappa}/81\right) - \left(D_{i+1}^{\kappa}/81\right)}{-\theta D_{i-2}^{\kappa} + (243\theta + 5/81)D_{i-1}^{\kappa} + (26 - 243\theta/81)D_i^{\kappa} + (81\theta - 4/81)D_{i+1}^{\kappa}}, \tag{11}$$

therefore,

$$q_{3i}^{\kappa+1} - \lambda = \frac{N}{D}, \tag{12}$$

where

$$\begin{aligned} N &= \frac{81\theta\lambda}{q_{i-2}^{\kappa}} + [29 - (26 - 243\theta)\lambda]q_{i-1}^{\kappa} - 1 - (243\theta + 5)\lambda - [1 + (81\theta - 4)\lambda]q_i^{\kappa} q_{i-1}^{\kappa}, \\ D &= -\frac{81\theta}{q_{i-2}^{\kappa}} + 243\theta + 5 + (26 - 243\theta)q_{i-1}^{\kappa} + (81\theta - 4)q_i^{\kappa} q_{i-1}^{\kappa}. \end{aligned} \tag{13}$$

It can be seen that $D = (81D_{3i}^{\kappa+1}/D_{i-1}^{\kappa}) > 0$, and thus, to prove $N < 0$, it is considered that

$$\begin{aligned} N &= -1 + \frac{81\theta\lambda}{q_{i-2}^{\kappa}} + [29 - (26 - 243\theta)\lambda]q_{i-1}^{\kappa} - (243\theta + 5)\lambda - [1 + (81\theta - 4)\lambda]q_i^{\kappa} q_{i-1}^{\kappa} \\ &< 81\theta\lambda^2 - (243\theta + 5)\lambda - 1 + [29 - (26 - 243\theta)\lambda]q_{i-1}^{\kappa} - \left[\lambda + (81\theta - 4)\lambda^2\right]q_{i-1}^{\kappa} \\ &< 81\theta\lambda^2 - (243\theta + 5)\lambda - 1 + \left[(4 - 81\theta)\lambda^2 + (243\theta - 27)\lambda + 29\right]q_{i-1}^{\kappa}. \end{aligned} \tag{14}$$

Since

$$(4 - 81\theta)\lambda^2 + (243\theta - 27)\lambda + 29 = -31 - \frac{20(-5 + 162\theta)}{6561\theta^2} < 0, \tag{15}$$

then

$$\begin{aligned} N &< 81\theta\lambda^2 - (243\theta + 5)\lambda + \frac{(4 - 81\theta)\lambda^2 + (243\theta - 27)\lambda + 29}{\lambda} - 1 \\ &= \frac{1}{\lambda}\left[81\theta\lambda^3 - (1 + 324\theta)\lambda^2 + (243\theta - 28)\lambda + 29\right] \\ &= \frac{20}{81\theta} - \frac{3(35 + 918\theta)}{5 + 243\theta} \\ &< 0. \end{aligned} \tag{16}$$

Thus, $q_{3i}^{\kappa+1} - \lambda = (N/D) < 0$, i.e.,

$$q_{3i}^{\kappa+1} < \lambda. \tag{17}$$

Since

$$q_{3i+1}^{\kappa+1} = \frac{(-(4/81) + \theta)D_{i-1}^{\kappa} + ((26/81) - 3\theta)D_i^{\kappa} + ((5/81) + 3\theta)D_{i+1}^{\kappa} - \theta D_{i+2}^{\kappa}}{-(D_{i-1}^{\kappa}/81) + (29D_i^{\kappa}/81) - (D_{i+1}^{\kappa}/81)}, \tag{18}$$

therefore,

$$q_{3i+1}^{\kappa+1} - \lambda = \frac{N}{D}, \tag{19}$$

where

$$\begin{aligned} N &= \frac{81\theta - 4 + \lambda}{q_{i-1}^{\kappa}} + 26 - 243\theta - 29\lambda + (5 + 243\theta + \lambda)q_i^{\kappa} - 81\theta q_i^{\kappa} q_{i+1}^{\kappa}, \\ D &= -\frac{1}{q_{i-1}^{\kappa}} + 29 - q_i^{\kappa}. \end{aligned} \tag{20}$$

It can be seen that $D = (81D_{3i+1}^{\kappa+1}/D_i^{\kappa}) > 0$, and thus, to prove $N < 0$, it is considered that

$$\begin{aligned} N &= \frac{81\theta - 4 + \lambda}{q_{i-1}^{\kappa}} + 26 - 243\theta - 29\lambda + (5 + 243\theta + \lambda)q_i^{\kappa} - 81\theta q_i^{\kappa} q_{i+1}^{\kappa} \\ &< (81\theta - 4 + \lambda)\lambda + 26 - 243\theta - 29\lambda + (5 + 243\theta + \lambda)q_i^{\kappa} - \frac{81\theta q_i^{\kappa}}{\lambda} \\ &< \lambda^2 + (81\theta - 33)\lambda + 26 - 243\theta + \left[\frac{\lambda^2 + (5 + 243\theta)\lambda - 81\theta}{\lambda}\right] q_i^{\kappa}. \end{aligned} \tag{21}$$

Since

$$\lambda^2 + (5 + 243\theta)\lambda - 81\theta = 39 + \frac{25}{6561\theta^2} + \frac{55}{81\theta} + 648\theta > 0, \tag{22}$$

therefore,

$$\begin{aligned} N &< \lambda^2 + (81\theta - 33)\lambda + 26 - 243\theta + \lambda^2 + (5 + 243\theta)\lambda - 81\theta \\ &= 2\lambda^2 + (324\theta - 28)\lambda + 26 - 324\theta \\ &= -10 + \frac{5}{81\theta} + 162\theta \\ &< 0. \end{aligned} \tag{23}$$

Thus, $q_{3i+1}^{\kappa+1} - \lambda = (N/D) < 0$, i.e.,

$$q_{3i+1}^{\kappa+1} < \lambda. \tag{24}$$

Since

$$q_{3i+2}^{\kappa+1} = \frac{-\theta D_{i-1}^{\kappa} + (3\theta + (5/81))D_i^{\kappa} + ((26/81) - 3\theta)D_{i+1}^{\kappa} + (\theta - (4/81))D_{i+2}^{\kappa}}{(-(4/81) + \theta)D_{i-1}^{\kappa} + ((26/81) - 3\theta)D_i^{\kappa} + ((5/81) + 3\theta)D_{i+1}^{\kappa} - \theta D_{i+2}^{\kappa}}, \tag{25}$$

therefore,

$$q_{3i+2}^{\kappa+1} - \lambda = \frac{N}{D}, \tag{26}$$

where

$$\begin{aligned} N &= \frac{-81\theta - (81\theta - 4)\lambda}{q_{i-1}^{\kappa} q_i^{\kappa}} + \frac{5 + 243\theta - (26 - 243\theta)\lambda}{q_i^{\kappa}} + 26 - 243\theta - (5 + 243\theta)\lambda \\ &\quad + (81\theta - 4 + 81\theta\lambda) q_{i+1}^{\kappa} \\ D &= \frac{81\theta - 4}{q_{i-1}^{\kappa} q_i^{\kappa}} + \frac{(26 - 243\theta)}{q_i^{\kappa}} + 5 + 243\theta - 81\theta q_{i+1}^{\kappa}. \end{aligned} \tag{27}$$

It can be seen that $D = (81D_{3i+2}^{\kappa+1}/D_{i+1}^{\kappa}) > 0$, and thus, to prove $N < 0$, it is considered that

$$\begin{aligned} N &= \frac{-81\theta - (81\theta - 4)\lambda}{q_{i-1}^{\kappa} q_i^{\kappa}} + \frac{5 + 243\theta - (26 - 243\theta)\lambda}{q_i^{\kappa}} \\ &\quad + 26 - 243\theta - (5 + 243\theta)\lambda + (81\theta - 4 + 81\theta\lambda) q_{i+1}^{\kappa} \\ &< \frac{(4 - 81\theta)\lambda^2 + (162\theta - 26)\lambda + 243\theta + 5}{q_i^{\kappa}} \\ &\quad + 81\theta\lambda^2 - (9 + 162\theta)\lambda + 26 - 243\theta. \end{aligned} \tag{28}$$

Since

$$\begin{aligned} &(4 - 81\theta)\lambda^2 + (162\theta - 26)\lambda + 243\theta + 5 \\ &\quad = -57 - \frac{5(-20 + 567\theta)}{6561\theta^2} < 0, \end{aligned} \tag{29}$$

therefore,

$$\begin{aligned} N &< \frac{(4 - 81\theta)\lambda^2 + (162\theta - 26)\lambda + 243\theta + 5}{\lambda} + 81\theta\lambda^2 - (9 + 162\theta)\lambda + 26 - 243\theta \\ &= \frac{1}{\lambda}\left[81\theta\lambda^3 - (5 + 243\theta)\lambda^2 - 81\theta\lambda + 243\theta + 5\right] \\ &= \frac{1}{\lambda}(\lambda - 1)(\lambda + 1)\left(\lambda - \frac{5 + 243\theta}{81\theta}\right) \\ &= 0. \end{aligned} \tag{30}$$

Thus, $q_{3i+2}^{\kappa+1} - \lambda = (N/D) < 0$, i.e.,

$$q_{3i+2}^{\kappa+1} < \lambda. \tag{31}$$

Combining equations (17), (24), and (31), it can be observed that

$$q_i^{\kappa+1} < \lambda, \forall \kappa \in \{0\} \cup \mathbb{Z}^{+}. \tag{32}$$

Hence, by induction,

$$q_i^{\kappa} < \lambda, \forall \kappa \in \{0\} \cup \mathbb{Z}^{+}. \tag{33}$$

To prove $(1/q_i^{\kappa}) < \lambda$, it is considered that

$$\frac{1}{q_{3i}^{\kappa+1}} = \frac{D_{3i}^{\kappa+1}}{D_{3i+1}^{\kappa+1}} = \frac{-\theta D_{i-2}^{\kappa} + (3\theta + (5/81)) D_{i-1}^{\kappa} + ((26/81) - 3\theta) D_i^{\kappa} + (\theta - (4/81)) D_{i+1}^{\kappa}}{-\left(D_{i-1}^{\kappa}/81\right) + \left(29 D_i^{\kappa}/81\right) - \left(D_{i+1}^{\kappa}/81\right)}. \tag{34}$$

This implies that

$$\frac{1}{q_{3i}^{\kappa+1}} - \lambda = \frac{N}{D}, \quad (35)$$

where

$$\begin{aligned} N &= \frac{-81\theta}{q_{i-1}^{\kappa} q_{i-2}^{\kappa}} + \frac{(243\theta + 5 + \lambda)}{q_{i-1}^{\kappa}} + 26 - 243\theta - 29\lambda + (81\theta - 4 + \lambda)q_i^{\kappa}, \\ D &= -\frac{1}{q_{i-1}^{\kappa}} + 29 - q_i^{\kappa}. \end{aligned} \quad (36)$$

It can be seen that $D = (81D_{3i+1}^{\kappa+1}/D_i^{\kappa}) > 0$, and thus, to prove $N < 0$, it is considered that

$$\begin{aligned} N &= \frac{-81\theta}{q_{i-1}^{\kappa} q_{i-2}^{\kappa}} + \frac{(243\theta + 5 + \lambda)}{q_{i-1}^{\kappa}} + 26 - 243\theta - 29\lambda + (81\theta - 4 + \lambda)q_i^{\kappa} \\ &< \left[\frac{\lambda^2 + (243\theta + 5)\lambda - 81\theta}{\lambda}\right] \frac{1}{q_{i-1}^{\kappa}} + 26 - 243\theta - 29\lambda + (81\theta - 4 + \lambda)q_i^{\kappa} \\ &< \lambda^2 + (243\theta + 5)\lambda - 81\theta + 26 - 243\theta - 29\lambda + (81\theta - 4 + \lambda)q_i^{\kappa}. \end{aligned} \quad (37)$$

Since

$$81\theta - 4 + \lambda = -1 + \frac{5}{81\theta} + 81\theta > 0, \quad (38)$$

therefore,

$$\begin{aligned} N &< \lambda^2 + (243\theta - 24)\lambda - 324\theta + 26 + (81\theta - 4 + \lambda)\lambda \\ &= 2\lambda^2 + (324\theta - 28)\lambda - 324\theta + 26 \\ &= -10 + \frac{5}{81\theta} + 162\theta \\ &< 0. \end{aligned} \quad (39)$$

Thus, $(1/q_{3i}^{\kappa+1}) - \lambda = (N/D) < 0$, i.e.,

$$\frac{1}{q_{3i}^{\kappa+1}} < \lambda. \quad (40)$$

Since

$$\frac{1}{q_{3i+1}^{\kappa+1}} = \frac{-\left(D_{i-1}^{\kappa}/81\right) + \left(29D_i^{\kappa}/81\right) - \left(D_{i+1}^{\kappa}/81\right)}{(\theta - (4/81))D_{i-1}^{\kappa} + ((26/81) - 3\theta)D_i^{\kappa} + ((5/81) + 3\theta)D_{i+1}^{\kappa} - \theta D_{i+2}^{\kappa}}, \quad (41)$$

therefore,

$$\frac{1}{q_{3i+1}^{\kappa+1}} - \lambda = \frac{N}{D}, \quad (42)$$

where

$$\begin{aligned} N &= \frac{-1 - (81\theta - 4)\lambda}{q_{i-1}^{\kappa}} + 29 - (26 - 243\theta)\lambda + [-1 - (5 + 243\theta)\lambda]q_i^{\kappa} + 81\theta\lambda q_i^{\kappa} q_{i+1}^{\kappa}, \\ D &= \frac{81\theta - 4}{q_{i-1}^{\kappa}} + 26 - 243\theta + (5 + 243\theta)q_i^{\kappa} - 81\theta q_i^{\kappa} q_{i+1}^{\kappa}. \end{aligned} \quad (43)$$

It can be seen that $D = (81D_{3i+2}^{\kappa+1}/D_i^{\kappa}) > 0$, and thus, to prove $N < 0$, it is considered that

$$
\begin{aligned}
N &= \frac{-1-(81\theta-4)\lambda}{q_{i-1}^{\kappa}} + 29 - (26-243\theta)\lambda + [-1-(5+243\theta)\lambda]q_i^{\kappa} + 81\theta\lambda q_i^{\kappa} q_{i+1}^{\kappa} \\
&< \frac{-1-(81\theta-4)\lambda}{q_{i-1}^{\kappa}} + 29 - (26-243\theta)\lambda + \left[81\theta\lambda^2 - (5+243\theta)\lambda - 1\right]q_i^{\kappa} \\
&< \frac{-1-(81\theta-4)\lambda}{q_{i-1}^{\kappa}} + \frac{81\theta\lambda^2 - (5+243\theta)\lambda - 1}{\lambda} - (26-243\theta)\lambda + 29 \\
&< \frac{-1-(81\theta-4)\lambda}{q_{i-1}^{\kappa}} + \frac{(324\theta-26)\lambda^2 - (243\theta-24)\lambda - 1}{\lambda}.
\end{aligned} \tag{44}
$$

Since

$$
-1-\lambda(81\theta-4) = 6 + \frac{20}{81\theta} - 243\theta > 0. \tag{45}
$$

It follows that

$$
\begin{aligned}
N &< [-1-(81\theta-4)\lambda]\lambda + \frac{(324\theta-26)\lambda^2 - (243\theta-24)\lambda - 1}{\lambda} \\
&= \frac{1}{\lambda}\left[(4-81\theta)\lambda^3 + (324\theta-27)\lambda^2 + (24-243\theta)\lambda - 1\right] \\
&= -\frac{94}{3} - \frac{20(-5+162\theta)}{6561\theta^2} + \frac{5}{15+729\theta} \\
&< 0.
\end{aligned} \tag{46}
$$

Thus, $(1/q_{3i+1}^{\kappa+1}) - \lambda = (N/D) < 0$, i.e.,

$$
\frac{1}{q_{3i+1}^{\kappa+1}} < \lambda. \tag{47}
$$

Since

$$
\frac{1}{q_{3i+2}^{\kappa+1}} = \frac{(-(4/81)+\theta)D_{i-1}^{\kappa} + ((26/81)-3\theta)D_i^{\kappa} + ((5/81)+3\theta)D_{i+1}^{\kappa} - \theta D_{i+2}^{\kappa}}{-\theta D_{i-1}^{\kappa} + ((5/81)+3\theta)D_i^{\kappa} + ((26/81)-3\theta)D_{i+1}^{\kappa} + (-(4/81)+\theta)D_{i+2}^{\kappa}}, \tag{48}
$$

therefore,

$$\frac{1}{q_{3i+2}^{\kappa+1}}-\lambda=\frac{N}{D}, \tag{49}$$

where

$$\begin{aligned} N &= \frac{81\theta-4+81\theta\lambda}{q_{i-1}^{\kappa}}+26-243\theta-(5+243\theta)\lambda+(5+243\theta-(26-243\theta)\lambda)q_i^{\kappa} \\ &\quad +[-81\theta-(81\theta-4)\lambda]q_i^{\kappa}q_{i+2}^{\kappa} \\ D &= -\frac{81\theta}{q_{i-1}^{\kappa}}+5+243\theta+(26-243\theta)q_i^{\kappa}+(-4+81\theta)q_i^{\kappa}q_{i+1}^{\kappa}. \end{aligned} \tag{50}$$

It can be seen that $D=(81D_{3i+3}^{\kappa+1}/D_i^{\kappa})>0$, and thus, to prove $N<0$, consider

$$N=\frac{81\theta-4+81\theta\lambda}{q_{i-1}^{\kappa}}+26-243\theta-(5+243\theta)\lambda+[5+243\theta-(26-243\theta)\lambda]q_i^{\kappa}+[-81\theta-(81\theta-4)\lambda]q_i^{\kappa}q_{i+1}^{\kappa}. \tag{51}$$

Since

$$\begin{aligned} 81\theta-4+81\theta\lambda &= 1+324\theta>0, \\ -81\theta-(81\theta-4)\lambda &= 7+\frac{20}{81\theta}-324\theta>0, \end{aligned} \tag{52}$$

therefore,

$$\begin{aligned} N &< (81\theta-4+81\theta\lambda)\lambda+26-243\theta-(5+243\theta)\lambda+[5+243\theta-(26-243\theta)\lambda]q_i^{\kappa} \\ &\quad +[-81\theta-(81\theta-4)\lambda]\lambda q_i^{\kappa} \\ &< 81\theta\lambda^2-(9+162\theta)\lambda+26-243\theta+\left[(4-81\theta)\lambda^2+(162\theta-26)\lambda+5+243\theta\right]q_i^{\kappa}. \end{aligned} \tag{53}$$

Since

$$\begin{aligned} &(4-81\theta)\lambda^2+(162\theta-26)\lambda+5+243\theta \\ &\quad =-57-\frac{5(-20+567\theta)}{6561\theta^2}<0, \end{aligned} \tag{54}$$

therefore,

$$\begin{aligned} N &< 81\theta\lambda^2-(9+162\theta)\lambda+\frac{(4-81\theta)\lambda^2+(162\theta-26)\lambda+5+243\theta}{\lambda}+26-243\theta \\ &= \frac{1}{\lambda}\left[81\theta\lambda^3-(5+243\theta)\lambda^2-81\theta\lambda+5+243\theta\right] \\ &= \frac{1}{\lambda}(\lambda-1)(\lambda+1)\left(\lambda-\frac{5+243\theta}{81\theta}\right) \\ &= 0. \end{aligned} \tag{55}$$

Thus, $(1/q_{3i+2}^{\kappa+1}) - \lambda = (N/D) < 0$, i.e.,

$$\frac{1}{q_{3i+2}^{\kappa+1}} < \lambda. \tag{56}$$

Combining equations (40), (47), and (56), it can be seen that $(1/q_i^{\kappa+1}) < \lambda$, $\forall \kappa \in \{0\} \cup \mathbb{Z}^+$.

Hence, by induction,

$$\frac{1}{q_i^{\kappa}} < \lambda, \forall \kappa \in \{0\} \cup \mathbb{Z}^+. \tag{57}$$

Using equations (33) and (57) gives

$$\frac{1}{\lambda} < Q^{\kappa} < \lambda, \tag{58}$$

which completes the proof. □

In the following examples, the monotonicity preservation for 4-point ternary SS in equation (1) has been analyzed.

Example 1. Consider the set of initial control points and the limit curve which is generated without satisfying the derived sufficient condition of monotonicity. Figure 1(a) displays the initial control polygon using the monotonic increasing set {(1,1.5), (2,2), (3,4), (4,20.5), (5,21), (6,23), (7,36), (8,38), (9,40)} in which the monotonicity additional condition is not satisfied. Figure 1(b) displays the limit curve generated by the scheme in equation (1) for $\theta = 0.0209$ with a blue solid line. Figure 1(c) displays the limit curve generated by the scheme in equation (1) for $\theta = 0.039$ with a red solid line. It can be observed that the limit curves generated by the scheme in equation (1) are not monotonic, as shown in Figure 1(d). The limiting curve after three iterations shows that the curve is not monotonically increasing for these initial control points because the limit curve does not satisfy the derived conditions.

Example 2. To highlight the significance of the sufficient condition of monotonicity, considering a set of initial control points, the limit curve is generated which preserves the monotonicity of the initial data. Figure 2(a) displays the initial control polygon with the set {(1,1.5), (2,2), (3,4), (4,6), (5,7.5), (6,8.2), (7,10.5), (8,12), (9,14)} in which the control points are increasing monotonically, but the monotonicity additional condition is satisfied. Figure 2(b) displays the limit curve generated by the scheme in equation (1) for $\theta = 0.0209$ with blue solid line. Figure 2(c) displays the limit curve generated by the scheme in equation (1) for $\theta = 0.039$ with red solid line. It can be observed that the limit curves generated by the scheme in equation (1) are monotonic, as shown in Figure 2(d). The final limiting curve after three iterations shows that the curve is monotonically increasing for these initial control points.

The convexity preservation of SS in equation (1) has been analyzed in the following section.

2.2. Convexity Preservation. In order to maintain convexity, the initial control points must be convex. The convexity of control points is determined by the second-order divided difference (SODD). The SODD is represented by d_j^{κ} and defined by $d_j^{\kappa} = 3^{2\kappa}(f_{j-1}^{\kappa} - 2f_j^{\kappa} + f_{j+1}^{\kappa})$. Suppose initial control points for ternary four-point SS (1) are convex, i.e., $d_j^0 > 0$. The SODD of four-point SS in equation (1) is given as follows:

$$d_{3j}^{\kappa+1} = \frac{1}{9}\left[(-4+162\theta)d_{j+1}^{\kappa} + (17-324\theta)d_j^{\kappa} + (-4+162\theta)d_{j-1}^{\kappa}\right], \tag{59}$$

$$d_{3j+1}^{\kappa+1} = \frac{1}{3}\left[(1-27\theta)d_{j+1}^{\kappa} + (2+54\theta)d_j^{\kappa} - 27\theta d_{j-1}^{\kappa}\right], \tag{60}$$

$$d_{3j+2}^{\kappa+1} = \frac{1}{3}\left[-27\theta d_{j+2}^{\kappa} + (2+54\theta)d_{j+1}^{\kappa} + (1-27\theta)d_j^{\kappa}\right]. \tag{61}$$

The convexity-preserving condition that gives surety of generating convex limit curve has been derived in the following theorem.

Theorem 3. *Consider $\gamma_j^{\kappa} = d_{j+1}^{\kappa}/d_j^{\kappa}$ and $R^{\kappa} = \max_j\{\gamma_j^{\kappa}, (1/\gamma_j^{\kappa})\}$, $\forall \kappa$, $j \in \mathbb{Z}$. If the initial control points are strictly convex, i.e., $d_j^0 > 0$ such that parameter θ satisfies $\theta \in]4/243, 7/405[$ and $R^0 < \sigma$, where $1 < \sigma < (324\theta - 17/+324\theta - 8)$, then the ternary approximating SS in equation (1) preserves convexity.*

Proof. To get the result $d_j^{\kappa} > 0$ and $R^{\kappa} < \sigma$, $\forall \kappa \in \{0\} \cup \mathbb{Z}^+$ and $\forall j \in Z$, mathematical induction is used on κ.

(1) For $\kappa = 0$, $d_j^0 > 0 \forall j \in \mathbb{Z}$, and $R^0 < \sigma$, it clearly holds.
(2) It is assumed that $d_j^{\kappa} > 0$ and $R_j^{\kappa} < \sigma$, and then, by definition of mathematical induction, it follows that $(1/\sigma) \le R_j^{\kappa} < \sigma \,\forall\, \kappa \in \mathbb{Z}$ and $\forall j \in \mathbb{Z}$. The result is to be proved for $\kappa + 1$.

From equation (59), it is implies that

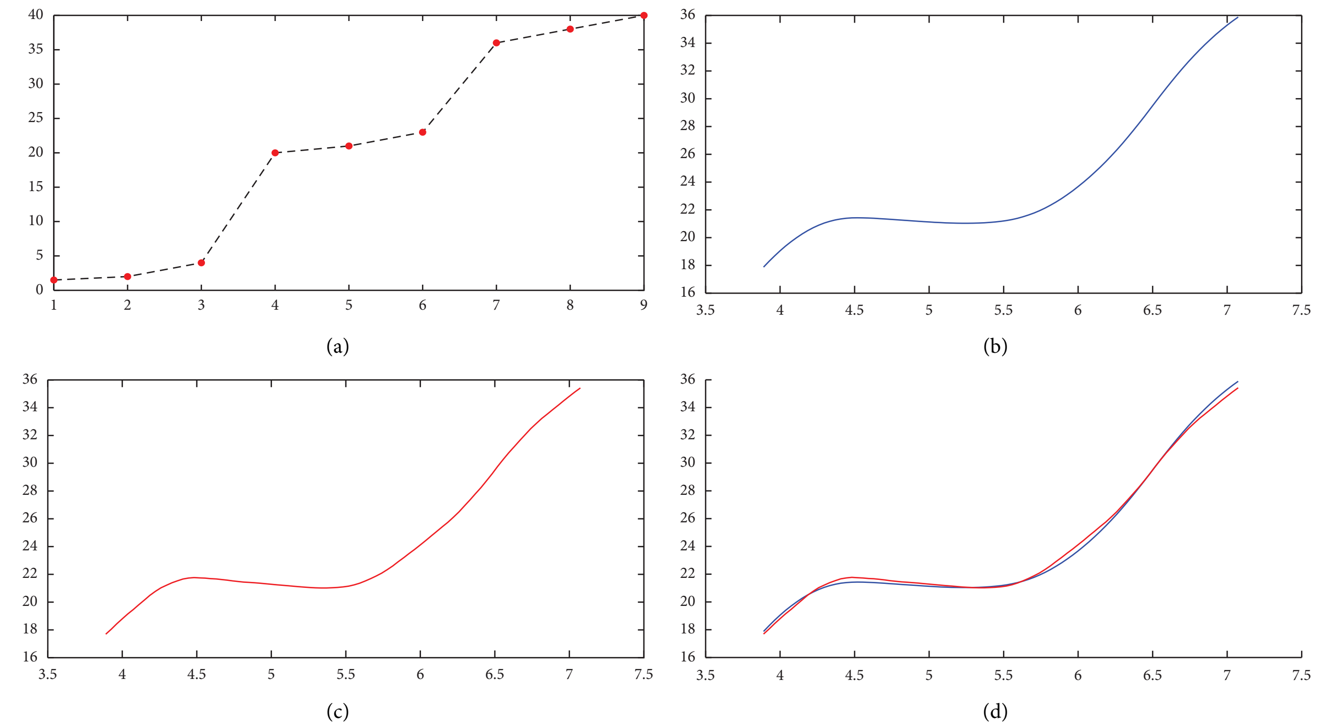

Figure 1: Illustration for Example 1 limit curves after three steps of refinement. (a) Initial control polygon. (b) Limit curve for $\theta = 0.0209$. (c) Limit curve for $\theta = 0.039$. (d) Limit curves.

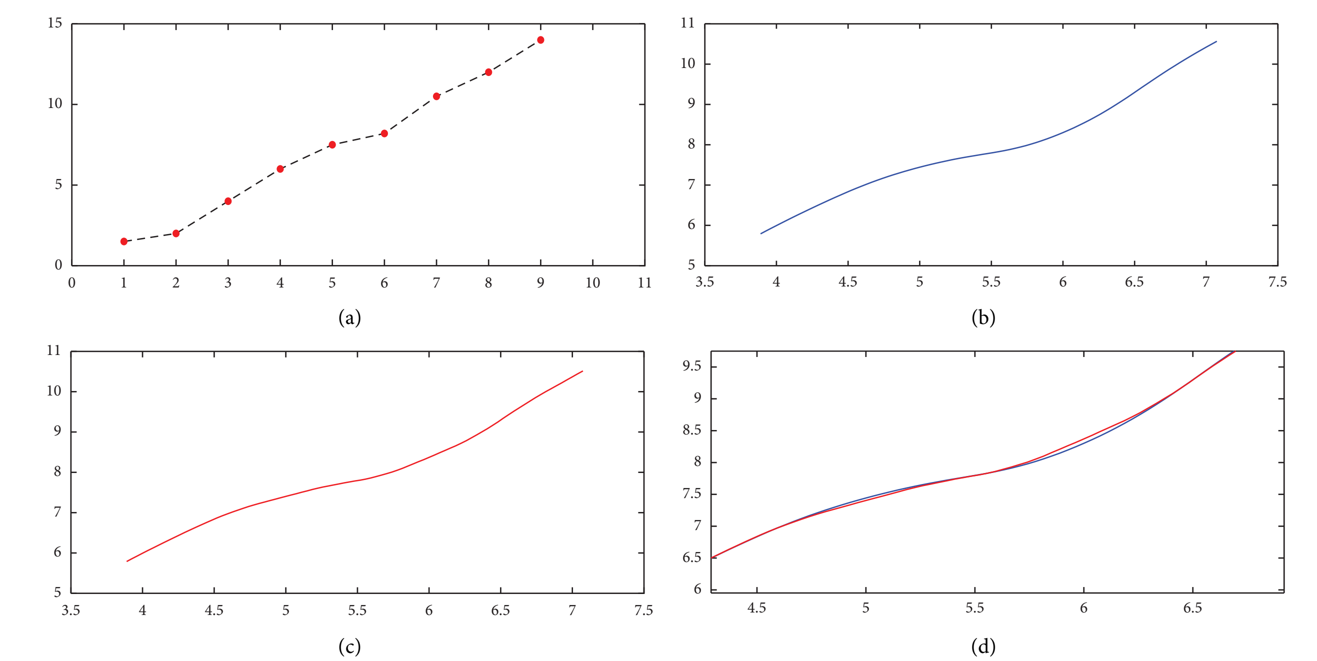

Figure 2: Illustration for Example 2 limit curves after three steps of refinement. (a) Initial control polygon. (b) Limit curve for $\theta = 0.0209$. (c) Limit curve for $\theta = 0.039$. (d) Limit curves.

$$\begin{aligned}
d_{3j}^{\kappa+1} &= \frac{1}{9}\left[(-4+162\theta)d_{j-1}^{\kappa} + (17-324\theta)d_j^{\kappa} + (-4+162\theta)d_{j+1}^{\kappa}\right] \\
&= \frac{d_j^{\kappa}}{9}\left[(-4+16\theta)\frac{1}{\gamma_{j-1}^{\kappa}} + (17-324\theta) + (-4+162\theta)\gamma_j^{\kappa}\right] \\
&> \frac{d_{j-1}^{\kappa}}{9}\left[(-4+16\theta)\sigma + (17-324\theta) + (-4+162\theta)\sigma\right] \\
&= \frac{d_j^{\kappa}}{9}\left(\sigma - \frac{17-324\theta}{-8+324\theta}\right) \\
&> 0.
\end{aligned} \tag{62}$$

It shows that $d_{3j}^{\kappa+1} > 0$ for $1 < \sigma \leq (17 - 324\theta/8 - 324\theta)$ (equation. (60))

$$\begin{aligned}
d_{3j+1}^{\kappa+1} &= \frac{1}{3}\left[(-27\theta)d_{j-1}^{\kappa} + (2+54\theta)d_j^{\kappa} + (1-27\theta)d_{j+1}^{\kappa}\right] \\
&= \frac{d_{j-1}^{\kappa}}{3}\left[(-27\theta) + (2+54\theta)\gamma_{j-1}^{\kappa} + (1-27\theta)\gamma_{j-1}^{\kappa}\gamma_j^{\kappa}\right] \\
&> \frac{d_{j-1}^{\kappa}}{3}\left[-27\theta + \left((2+54\theta) + (1-27\theta)\frac{1}{\sigma}\right)\gamma_{j-1}^{\kappa}\right] \\
&= \frac{d_{j-1}^{\kappa}}{3\sigma^2}\left(\sigma - \frac{1+27\theta+\sqrt{1+81\theta}}{27\theta}\right)\left(\sigma - \frac{1+27\theta+\sqrt{1+81\theta}}{27\theta}\right) \\
&> 0
\end{aligned} \tag{63}$$

which depicts that $d_{3j+1}^{\kappa+1} > 0$ for $1 < \sigma \leq (17 - 324\theta/8 - 324\theta)$. Similarly, it can be shown that $d_{3j+2}^{\kappa+1} > 0$ for $1 < \sigma \leq (17 - 324\theta/8 - 324\theta)$ which results in $d_j^{\kappa+1} > 0$, $\forall j \in \mathbb{Z}$. Therefore, by induction, $d_j^{\kappa} > 0 \forall j, \kappa \in \mathbb{Z}$.

To show $R^{\kappa+1} < \sigma$, it is sufficient to prove $(1/\sigma) < \gamma_j^{\kappa+1} < \sigma$, $\forall j \in \mathbb{Z}, \forall \kappa \in \{0\} \cup \mathbb{Z}^+$. Since

$$\begin{aligned}
\gamma_{3j}^{\kappa+1} &= \frac{d_{3j+1}^{\kappa+1}}{d_{3j}^{\kappa+1}}, \\
&= \frac{3\left[(-27\theta)d_{j-1}^{\kappa} + (2+54\theta)d_j^{\kappa} + (1-27\theta)d_{j+1}^{\kappa}\right]}{\left[(-4+162\theta)d_{j-1}^{\kappa} + (17-324\theta)d_j^{\kappa} + (-4+162\theta)d_{j+1}^{\kappa}\right]},
\end{aligned} \tag{64}$$

therefore,

$$\begin{aligned}\gamma_{3j}^{\kappa+1}-\sigma &= \left[(-4+162\theta)+(17-324\theta)\gamma_{j-1}^{\kappa}+(-4+162\theta)\gamma_{j-1}^{\kappa}\gamma_{j}^{\kappa}\right]^{-1}\\ &\quad\cdot\left[-81\theta-(-4+162\theta)\sigma+\gamma_{j-1}^{\kappa}(6+162\theta-17\sigma+324\sigma\theta)+(3-81\theta+4\sigma-162\theta\sigma)\gamma_{j-1}^{\kappa}\gamma_{j}^{\kappa}\right]\\ &= \frac{A}{B},\end{aligned}\tag{65}$$

where $B=(9d_{3j}^{\kappa+1}/d_{j-1}^{\kappa})>0$, To prove $A>0$, it is assumed that

$$\begin{aligned}A &= \left[-81\theta-(-4+162\theta)\sigma+\gamma_{j-1}^{\kappa}(6+162\theta-17\sigma+324\sigma\theta)+(3-81\theta+4\sigma-162\theta\sigma)\gamma_{j-1}^{\kappa}\gamma_{j}^{\kappa}\right]\\ &< \left[-81\theta-(-4+162\theta)\sigma+\gamma_{j-1}^{\kappa}(6+162\theta-17\sigma+324\sigma\theta)+(3-81\theta+4\sigma-162\theta\sigma)\sigma\gamma_{j-1}^{\kappa}\right]\\ &= \left[-81\theta-(-4+162\theta)\sigma+\left[6-14\sigma+243\sigma\theta+(4-162\theta)\sigma^{2}\right]\gamma_{j-1}^{\kappa}\right]\\ &= \left[-81\theta-(-4+162\theta)\sigma+\left[6-14\sigma+243\sigma\theta+(4-162\theta)\sigma^{2}\right]\sigma\right]\\ &= \left[(\sigma-1)\left(-81\theta+(10-81\theta)\sigma+2(-2+81\theta)\sigma^{2}\right)\right]\\ &< 0,\end{aligned}\tag{66}$$

which proves that $A<0$ for $1<\sigma<(17-324\theta/8-324\theta)$.

Thus, $\gamma_{3j}^{\kappa+1}-\sigma=(A/B)<0$, i.e., $\gamma_{3j}^{\kappa+1}<\sigma$. Since

$$\begin{aligned}\gamma_{3j+1}^{\kappa+1} &= \frac{d_{3j+2}^{\kappa+1}}{d_{3j+1}^{\kappa+1}}\\ &= \frac{\left[(1-27\theta)d_{j}^{\kappa}+(2+54\theta)d_{j+1}^{\kappa}+(-27\theta)d_{j+2}^{\kappa}\right]}{\left[(-27\theta)d_{j}^{\kappa}+(2+54\theta)d_{j}^{\kappa}+(1-27\theta)d_{j+1}^{\kappa}\right]},\end{aligned}\tag{67}$$

therefore,

$$\begin{aligned}\gamma_{3j+1}^{\kappa+1}-\sigma &= \frac{\left[(1-27\theta)-(2+54\theta)\sigma+(2+54\theta-\sigma+27\theta\sigma)\gamma_{j}^{\kappa}+27\left(\theta\sigma/\gamma_{j-1}^{\kappa}\right)-27\theta\gamma_{j}^{\kappa}\gamma_{j+1}^{\kappa}\right]}{\left[(-27\theta)\left(1/\gamma_{j-1}^{\kappa}\right)+(2+54\theta)+(1-27\theta)\gamma_{j}^{\kappa}\right]}\\ &= \frac{A}{B}.\end{aligned}\tag{68}$$

Since the denominator $B=(3d_{3j+1}^{\kappa+1}/d_{j}^{\kappa})>0$, therefore, to show $A<0$, it is considered that

$$\begin{aligned} A &= \left[(1-27\theta)-(2+54\theta)\sigma+(2+54\theta-\sigma+27\theta\sigma)\gamma_j^{\kappa}+27\frac{\theta\sigma}{\gamma_{j-1}^{\kappa}}-27\theta\gamma_j^{\kappa}\gamma_{j+1}^{\kappa}\right] \\ &< \left[(1-27\theta)-(2+54\theta)\sigma+(2+54\theta-\sigma+27\theta\sigma)\gamma_j^{\kappa}+27\frac{\theta\sigma}{\sigma}-27\frac{\theta}{\sigma}\gamma_j^{\kappa}\right] \\ &= \left[\left(1-(2+54\theta)\sigma+\left(2+54\theta-\sigma+27\theta\sigma-27\frac{\theta}{\sigma}\right)\gamma_j^{\kappa}\right.\right] \\ &= \left[\left(1-(2+54\theta)\sigma+\left(2+54\theta-\sigma+27\theta\sigma-27\frac{\theta}{\sigma}\right)\sigma\right.\right] \\ &= \left[1-27\theta-\sigma^2+27\theta\sigma\right] \\ &< 0, \end{aligned} \tag{69}$$

which shows that $A<0$ for $1<\sigma<(324\theta-17/+324\theta-8)$. Thus, $\gamma_{3j+1}^{\kappa+1}-\sigma=(A/B)<0$, i.e., $\gamma_{3j+1}^{\kappa+1}<\sigma$.

Similarly, it can be proved that $\gamma_{3j+2}^{\kappa+1}<\sigma$ for $1<\sigma<(324\theta-17/+324\theta-8)$.

Thus,

$$\gamma_j^{\kappa}<\sigma\forall\kappa\geq 0, \kappa\in Z. \tag{70}$$

To prove $(1/\gamma_j^{k})<\sigma$, it is considered that

$$\begin{aligned} \frac{1}{\gamma_{3j}^{\kappa+1}} &= \frac{d_{3j}^{\kappa+1}}{d_{3j+1}^{\kappa+1}} \\ &= \frac{\left[(-4+162\theta)d_j^{\kappa}+(17-324\theta)d_{j+1}^{\kappa}+(-4+162\theta)d_{j+2}^{\kappa}\right]}{3\left[(-27\theta)d_{j-1}^{\kappa}+(2+54\theta)d_j^{\kappa}+(1-27\theta)d_{j+1}^{\kappa}\right]}, \end{aligned} \tag{71}$$

which gives

$$\begin{aligned} \frac{1}{\gamma_{3j}^{\kappa+1}}-\sigma &= \left[(-27\theta)+(2+54\theta)\gamma_j^{\kappa}+(1-27\theta)\gamma_j^{\kappa}\gamma_{j+1}^{\kappa}\right]^{-1} \\ &\quad\times\left[(-4+162\theta)+27\theta\sigma+(17-324\theta-2\sigma-54\theta\sigma)\gamma_{j-1}^{\kappa}(-4+162\theta-\sigma-27\theta\sigma)\gamma_{j-1}^{\kappa}\gamma_j^{\kappa}\right] \\ &= \frac{A}{B}. \end{aligned} \tag{72}$$

Since the denominator of the above equation $(3d_{3j+1}^{\kappa+1}/d_j^{\kappa})>0$, therefore, $B>0$. To show that $A>0$, it is considered that

$$\begin{aligned}A &= \left[(-4+162\theta)+27\theta\sigma+(17-324\theta-2\sigma-54\theta\sigma)\gamma_{j-1}^{\kappa}+(-4+162\theta-\sigma-27\theta\sigma)\gamma_{j-1}^{\kappa}\gamma_{j}^{\kappa}\right]\\ &< \left[(-4+162\theta)+27\theta\sigma+(17-324\theta-2\sigma-54\theta\sigma)\gamma_{j-1}^{\kappa}+(-4+162\theta-\sigma-27\theta\sigma)\frac{1}{\sigma}\gamma_{j-1}^{\kappa}\right]\\ &= \left[(-4+162\theta)+27\theta\sigma+\left(16-351\theta-2\sigma-\frac{4}{\sigma}+\frac{162\theta}{\sigma}-54\theta\sigma\right)\gamma_{j-1}^{\kappa}\right]\\ &= \left[(-4+162\theta)+27\theta\sigma+\left(16-351\theta-2\sigma-\frac{4}{\sigma}+\frac{162\theta}{\sigma}-54\theta\sigma\right)\sigma\right]\\ &= \left[-2\left(-2\left(4-162\theta+\sigma^{2}(1+27\theta)+2\sigma(-4+81\theta)\right)\right)\right]\\ &< 0,\end{aligned} \tag{73}$$

which shows that $A<0$ for $1<\sigma<(324\theta-17/+324\theta-8)$. Thus, $(1/\gamma_{3j}^{\kappa+1})-\sigma=(A/B)<0$, i.e., $(1/\gamma_{3j}^{\kappa+1})<\sigma$. It is considered that

$$\frac{1}{\gamma_{3j+1}^{\kappa+1}}=\frac{d_{3j+1}^{\kappa+1}}{d_{3j+2}^{\kappa+1}}=\frac{\left[-27\theta d_{j-1}^{\kappa}+(2+54\theta)d_{j}^{\kappa}+(1-27\theta)d_{j+1}^{\kappa}\right]}{\left[(1-27\theta)d_{j}^{\kappa}+(2+54\theta)d_{j+1}^{\kappa}-27\theta d_{j+2}^{\kappa}\right]}, \tag{74}$$

which gives

$$\begin{aligned}\frac{1}{\gamma_{3j+1}^{\kappa+1}}-\sigma &= \frac{\left[-27\theta\left(1/\gamma_{j-1}^{\kappa}\right)+(2+54\theta)+(1-27\theta)\gamma_{j}^{\kappa}-(1-27\theta)\sigma-(2+54\theta)\gamma_{j}^{\kappa}\sigma+27\theta\sigma\gamma_{j}^{\kappa}\gamma_{j+1}^{\kappa}\right]}{\left[(1-27\theta)+(2+54\theta)\gamma_{j}^{\kappa}-27\theta\gamma_{j+1}^{\kappa}\gamma_{j}^{\kappa}\right]}\\ &= \frac{A}{B}.\end{aligned} \tag{75}$$

Since the denominator of the above equation $B=(d_{3j+2}^{\kappa+1}/d_{j}^{\kappa})>0$, therefore, to show $A<0$, it is considered that

$$\begin{aligned}A &= \left[-27\theta\frac{1}{\gamma_{j-1}^{\kappa}}+(2+54\theta)+(1-27\theta)\gamma_{j}^{\kappa}-(1-27\theta)\sigma-(2+54\theta)\gamma_{j}^{\kappa}\sigma+27\theta\sigma\gamma_{j}^{\kappa}\gamma_{j+1}^{\kappa}\right]\\ &< \left[-27\theta\sigma+(2+54\theta)+(1-27\theta)\gamma_{j}^{\kappa}-(1-27\theta)\sigma-(2+54\theta)\gamma_{j}^{\kappa}\sigma+27\theta\sigma^{2}\gamma_{j}^{\kappa}\right]\\ &= \left[2+54\theta-\sigma+\left(1-27\theta-2\sigma-54\theta\sigma+27\theta\sigma^{2}\right)\gamma_{j}^{\kappa}\right]\\ &= \left[2+54\theta-\sigma+\left(1-27\theta-2\sigma-54\theta\sigma+27\theta\sigma^{2}\right)\sigma\right]\\ &= \left[2+54\sigma-2\sigma^{2}-27\theta\sigma-54\theta\sigma^{2}+27\theta\sigma^{3}\right]\\ &< 0,\end{aligned} \tag{76}$$

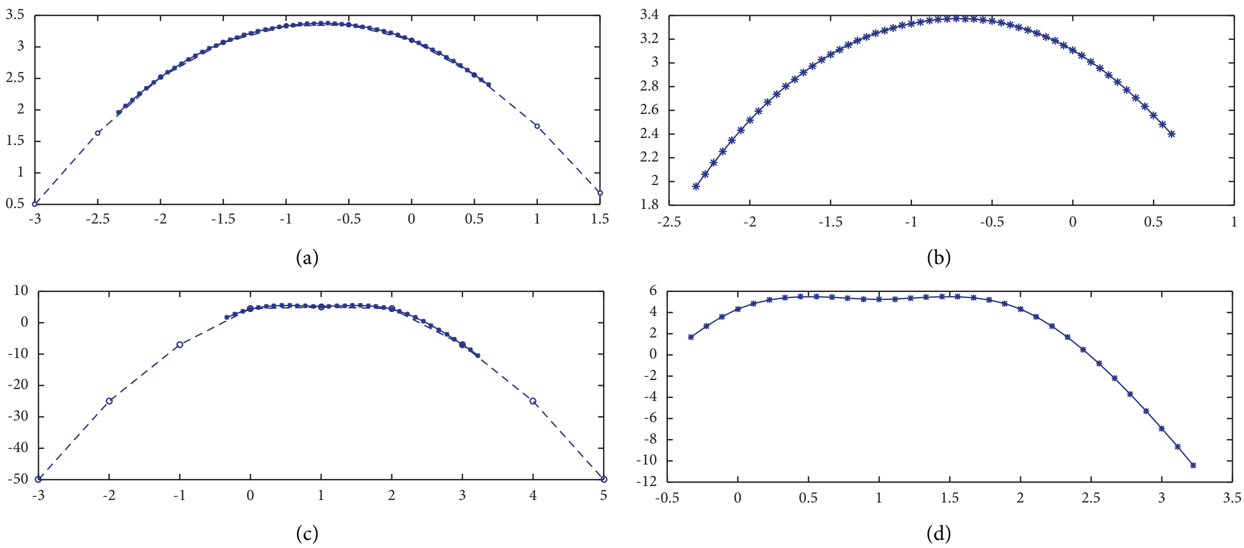

Figure 3: Limit curves after three steps of refinement. (a) Initial control polygon with limit curve. (b) Limit curve for $\theta = (41/1215)$. (c) Initial control polygon with limit curve. (d) Limit curve for $\theta = (41/1215)$.

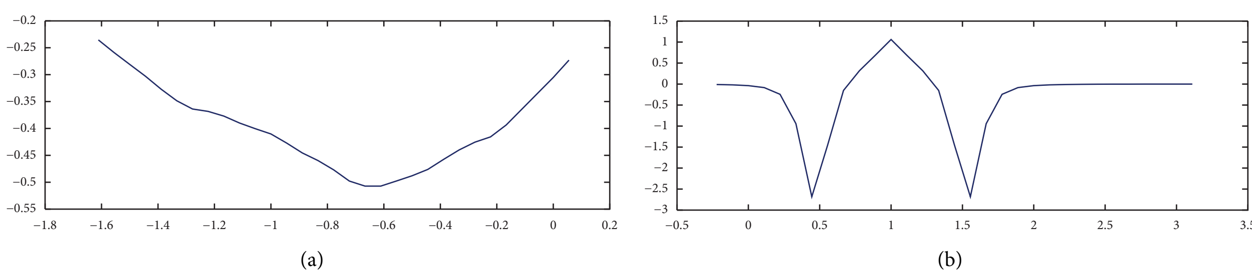

Figure 4: Curvature plots of the limit curve in Figure 3(b) (a) and Figure 3(d) (b).

which shows that $A<0$ for $1<\sigma<(324\theta-17/+324\theta-8)$. Thus, $(1/\gamma^{\kappa+1}_{3j+1})-\sigma=(A/B)<0$, i.e., $(1/\gamma^{\kappa+1}_{3j+1})<\sigma$. Similarly, it can be shown that $(1/\gamma^{\kappa+1}_{3j+2})<\sigma$. Thus,

$$\frac{1}{\gamma^{\kappa}_{j}}<\sigma,\quad \forall\kappa\in\{0\}\cup\mathbb{Z}^{+}. \tag{77}$$

Combining equations (28) and (29), it can be written as $(1/\sigma)<\gamma^{\kappa+1}_{j}<\sigma$.

By induction, $(1/\lambda)<\gamma^{\kappa}_{j}<\lambda, \forall\kappa\in\{0\}\cup\mathbb{Z}^{+}$ i.e., $R^{\kappa}<\lambda$. This completes the proof. □

The productivity of convexity preservation for ternary SS in Eq. (1) is determined in the following examples. The tension parameter $(4/243)<\theta<(7/405)$ is involved in ternary SS in equation (1). After three iterations of refinements, the limit curves are achieved. Each case has a set of strictly convex initial control points.

Example 3. Assume that the initial control points are strictly convex $\{(-3,0.5),(-2.5,1.6319),(-2,2.5174),(-1.5,3.0709),(-1,3.3316),(-0.5,3.3514),(0,3.1073),(0.5,2.5587),(1,1.7444),(1.5,0.6798)\}$. With the control polygon, the limiting curve looks like in Figure 3(a). Control points are denoted by "∘" in the figure, while newly generated control points are represented by "⋆". There are two lines in this figure: one representing the initial control polygon and the other representing the limit curve after three iterations. There is also a separate illustration of the limit curve in Figure 3(b). The derived condition $R^{0}<\sigma$ is satisfied by a set of control points for $\theta=(41/1215)$. As a result, the limiting curve is obtained by equation (1), which is preserving the convexity.

Another set of strictly convex control points $\{(-3,-50.5),(-2,-25.5),(-1,-7.5),(0,5),(1,5.5),(2,5),(3,-6.5),(4,-24.5),(5,-49.5)\}$ are considered. The derived condition $R^{0}<\sigma$ is not satisfied by this set of control points. As a result, the limiting curve obtained by SS in equation (1) does not preserve the convexity. The initial control polygon with the limit curve is represented in Figure 3(c). There are two lines in this figure: one representing the initial control polygon and the other representing the limit curve after three iterations. There is also a separate illustration of the limit curve in Figure 3(d). The limiting curve is not preserving the convexity, and it is shown for $\theta=(41/1215)$.

2.3. Curvature Analysis. In this section, the curvature plots are drawn from limiting curves.

From the above examples, it is concluded that the limiting curves generated by the convex data points may not retain the convexity preservation. Therefore, in Theorem 3, the additional condition $R^0 < \sigma$ is extracted. If convex data points satisfy this condition, then the limit curve will certainly preserve convexity.

Curvature plots are drawn for the limiting curve in Figures 3(b) and 3(d), which are presented in Figures 4(a) and 4(b), respectively. The importance of the derived condition $R^0 < \sigma$ is depicted by these curvature plots in Figures 4(a) and 4(b). Two sets of convex initial control points in Example 3 are given. The first set of convex points satisfies the derived condition; therefore, the curvature plot, as depicted in Figure 4(a), shows that the limiting curve does not alter its direction, which gives the guarantee of not involving inflection points. The second set of convex control points does not satisfy the extracted condition $R^0 < \sigma$. Therefore, the curvature plot in Figure 4(b) shows that the limiting curve corresponding to the second set of convex control points is altering its direction, which depicts that there must be inflection points in the limiting curve. As a result, the limit curve does not retain convexity preservation.

3. Conclusion

In this paper, two shape-persevering properties such as monotonicity and convexity preservation of ternary four-point approximating SS involving tension parameter $(4/243) < \theta < (7/405)$ are determined. The additional conditions for the monotonicity and convexity preservation of four-point ternary SS in equation (1) are extracted in Theorems 2 and 3, respectively. Initially, monotonically increasing data does not ensure the extraction of a monotone limiting curve. From Theorem 2, it has been concluded that if the initial data is monotonically increasing and satisfies the derived condition from equation (7), then the four-point ternary SS in equation (1) must preserve the monotonicity. Since initial convex data may not lead to generating a convex limiting curve, therefore, an additional condition $1 < \sigma < (324\theta - 17/324\theta - 8)$ has been derived. Theorem 3 concludes that, if the initial data is convex and satisfies the derived additional condition $1 < \sigma < (324\theta - 17/324\theta - 8)$, then four-point ternary SS in equation (1) must preserve the convexity. To demonstrate the importance of deriving conditions, curvature plots are also drawn, showing that if initial control points do not satisfy the conditions of convexity, the scheme in equation (1) may give rise to unwanted twists and inflection points in the limit curve.

The research findings demonstrate the convexity and monotonicity preservation of the proposed SS in equation (1) under specific conditions, which has significant academic implications and practical applications. However, there might be some limitations and areas for further exploration to enhance the scheme's versatility and robustness in different contexts. For further research, an additional condition can be further generalized or relaxed while still ensuring the convexity and monotonicity of the subdivision scheme. This could potentially make the scheme applicable to a broader range of initial data.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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