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Research Article

New Local Fractional Mohand–Adomian Decomposition Method for Solving Nonlinear Fractional Burger's Type Equation

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In this article, we address the challenge of solving the nonlinear fractional Burger's KdV equation, time-fractional Burger's equation, and the fractional modified Burger's equation. This is achieved by employing the Caputo and conformable derivatives. To tackle these equations, we introduce a new numerical method which is the combination of the local fractional Mohand transform and the Adomian decomposition method. This choice is driven by its straightforward methodology and reduced computational complexity. Moreover, to demonstrate the versatility of this technique, we provide several illustrative examples along with their corresponding exact or approximate solutions. These solutions are accompanied by graphical representations, further enhancing the clarity of the presented approach.

1. Introduction

Fractional differential equations are widely used in many fields of science and engineering to describe physical phenomena that exhibit nonlocal and nonlinear behavior. Burger's type equation is one such equation that describes the dynamics of a wide range of phenomena including gas dynamics, fluid mechanics, and population dynamics [1]. However, the solution of Burger's type equation becomes challenging due to its nonlinearity and fractional derivatives [2]. To address this challenge, various numerical methods have been proposed, and one such method is the local fractional Mohand–Adomian decomposition method (LFMADM). In general, fractional partial differential equations do not have exact solutions, and only approximate and numerical methods can be employed to obtain solutions.

Burger's equation:

$$\frac{\partial \mathbb{U}(y,t)}{\partial t} + \mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} = \frac{\partial^2 \mathbb{U}(y,t)}{\partial^2 y^2}, \qquad (1)$$

is a fundamental nonlinear partial differential equation in fluid mechanics. Burger's equation is a crucial model utilized in various fields of applied mathematics, including gas dynamics, heat conduction, traffic flow, and acoustic waves. Its importance was initially highlighted by Bateman in 1915 [3] who identified its steady solutions as worthy of exploration. Later, Burger proposed it as one of the equations describing mathematical models of turbulence in 1948 [4]. Benton and Platzman [5] conducted a comprehensive survey of one-dimensional Burger's equation in 1972, examining its exact solutions. The pursuit of numerical or analytical solutions to such equations is of significant importance in applied mathematics, leading to numerous studies by scientists to determine these solutions [6, 7].

The area of fractional calculus finds extensive applications across a wide range of engineering and scientific disciplines. These include viscoelasticity, fluid mechanics, biological population modeling, electrochemistry, and optics. Fractional calculus is particularly valuable when it comes to modeling physical and engineering systems that are most accurately described by fractional differential equations. These models are employed to provide precise representations of systems with specific damping requirements. Recently, various numerical and analytical methods have been introduced in these domains, often applied to tackle novel and challenging problems. Mathematical modeling frequently leads to the formulation of fractional differential equations and a range of problems that encompass special functions of mathematics, along with their generalizations in more variables. Furthermore, many physical phenomena in fields such as quantum mechanics, fluid dynamics, electricity, and ecological systems are governed by fractional-order partial differential equations (PDEs) within their applicable scope. Consequently, it is of growing significance to have a comprehensive understanding of both traditional and newly devised techniques for solving fractional PDEs and the practical applications of these methods [8–10].

Numerous investigations have been conducted by researchers to acquire both numerical and analytical solutions for Burger's equation. Ozis and Ozdes [11] employed a direct variational approach, whereas Aksan and Ozdes [12] introduced a variational method based on time discretization. Numerous wavelet-based approaches have been meticulously examined to evaluate their precision and efficiency in tackling both linear and nonlinear fractional differential equations. The authors have delved into the challenges faced by researchers in this particular field and emphasized the importance of interdisciplinary collaboration to advance the exploration of various wavelets for solving differential equations spanning diverse orders. Several wavelet methodologies, such as the cubic B-spline wavelet, Haar wavelet, Legendre wavelet, Legendre multiwavelet, and Chebyshev wavelet methods, have been subjected to scrutiny for their applicability in solving fractional differential equations. Among these techniques, the Legendre multiwavelet method, when combined with the Galerkin method, has demonstrated effectiveness in providing approximate solutions to initial value problems associated with fractional nonlinear partial differential equations. These wavelet-based approaches adeptly convert fractional differential equations into systems of algebraic equations, facilitating their straightforward resolution through conventional methods. Kutluay et al. [13] derived a numerical solution for Burger's equation using finite difference methods, while Varoḡlu and Liam Finn [14] utilized a weighted residue method. Caldwell et al. [15] employed finite elements, and Evans and Abdullah [16] utilized the group explicit method. Mittal and Singhal [17] employed the Galerkin method to compute numerical solutions for Burger's equation. In 2005, Gorguis [18] presented a comparison between the Cole–Hopf transformation and the decomposition method for solving Burger's equation. Thus, it can be inferred that researchers have dedicated significant attention to obtaining solutions for the fractional Burger's equation.

Nonhomogenous fractional Burger's equation:

$$\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha} y^{2}} - 2\mathbb{U}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} + \frac{\partial\mathbb{U}^{2}(y,t)}{\partial y} = f(y,t). \tag{2}$$

Time-fractional Burger's Kdv equation:

$$\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial t^{\alpha}} - \frac{\partial^{3}\mathbb{U}(y,t)}{\partial y^{3}} + 6u(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} = 0. \tag{3}$$

Fractional modified Burger's equation:

$$\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial t^{\alpha}} + \mathbb{U}^{2}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} + \frac{\partial^{2}\mathbb{U}(y,t)}{\partial y^{2}} = 0. \tag{4}$$

Considering $0 < \alpha \leq 1$, the symbol $\mathbb{U}(y,t)$ signifies the velocity in the spatial dimension y and at time t. Due to its significant relevance, researchers have focused extensively on acquiring both precise and numerical solutions for equations resembling Burger's equation.

The objective of this study is to create a dependable and effective numerical technique, utilizing the LFMADM, for solving nonlinear fractional Burger's equation. The proposed method will undergo validation and analysis, assessing its convergence behavior, computational efficiency, and sensitivity to parameters and boundary conditions. The method will be applied to practical problems in fluid mechanics and population dynamics. The results of this research are anticipated to enhance the development of more precise and effective numerical methods for solving nonlinear fractional differential equations.

Furthermore, the proposed method will be applied to solve practical problems related to fluid mechanics or population dynamics, which will serve as a demonstration of its effectiveness in real-world scenarios. Through the outcomes of this research, the development of more accurate and efficient numerical methods for solving nonlinear fractional differential equations is expected to be advanced. The successful development of this numerical technique will provide valuable insights into the behavior and solutions of nonlinear fractional Burger's equation, which can be utilized in various fields that require a precise understanding of complex mathematical models, such as engineering and physics.

This paper is organized as follows. In Section 2, we highlight some basic definitions which are used. In Section 3, we develop the numerical method and present the graphical representation of the proposed methods. In Section 4, we draw the conclusion and give the future direction.

2. Definitions

Definition 1. The CFD of a function $\mathscr{F}$: $\mathfrak{R}^+ \longrightarrow \mathfrak{R}$ is defined by

$$ {}^{c}D_{a+}^{\gamma}\mathscr{F}(\ell) = \frac{1}{\Gamma(n-\gamma)}\int_a^{\ell} (\ell-\eta)^{n-\gamma-1}\mathscr{F}^{(n)}(\eta)\mathrm{d}\eta, \quad (5) $$

provided the right side is pointwise defined on $(0,\infty)$, where $n = [\gamma] + 1$ in case γ is not an integer and $n = \gamma$ in case γ is an integer.

2.1. Mohand Transform. Integral transforms provide a systematic approach for resolving challenges within engineering sciences, such as heat conduction, radioactive decay, beam vibration, and population growth problems. Numerous researchers have employed a wide range of integral transforms, including Fourier, Laplace, Mahgoub, Kamal, Mohand, Sumudu, and Elzaki transforms, to provide solutions for various classes of equations such as delay differential equations, integral equations, partial differential equations, and partial integrodifferential equations. One particularly powerful technique in this regard is the Mohand transform, which finds its roots in the classical Fourier integral. This transformative method, pioneered by Mohand Mahgoub, simplifies the resolution of both partial and ordinary differential equations within the time domain, leveraging its mathematical elegance and inherent properties. While Laplace, Fourier, Elzaki, Aboodh, Sumudu, and Kamal transforms are established mathematical tools for addressing differential equations, the Mohand transform and its fundamental characteristics also offer valuable resources for tackling such mathematical challenges.

The operator denoted as $M(.)$, commonly referred to as the Mohand transform, is rigorously defined through integral equations as follows:

$$ \begin{aligned} M[f(t)] &= R(s) \\ &= s^2\int_0^{\infty} f(t)\exp^{-st}\,\mathrm{d}t, \end{aligned} \quad (6) $$

and here, for $t \geq 0$ and $k1 \leq s \leq k2$, the variable $k1 \leq s \leq k2$ plays a role in factoring the variable $k1 \leq s \leq k2$ within the argument of the function f. This transformation exhibits notable connections with the Laplace, Elzaki, and Aboodh transforms.

2.2. Local Fractional Mohand–Adomian Decomposition Method. The combination of the Mohand transform and Adomian decomposition method, known as the Mohand–Adomian decomposition method, offers notable advantages in solving differential equations. This approach not only provides algebraic values but also delivers closed-form solutions in the form of power series. It proves to be a straightforward technique for obtaining solutions to both linear and nonlinear fractional differential equations, characterized by its efficiency in terms of computational workload. An advantageous feature of this method lies in its capability to address nonlinear fractional differential equations without the necessity of employing He's or Adomian's polynomials for handling nonlinear terms. This proposed methodology stands out due to its absence of restrictive assumptions and linearization requirements. As a result, it proves instrumental in analytically addressing a wide array of practical problems represented by fractional-order ordinary and partial differential equations.

Definition 2. The Mohand transform, as proposed by Mahgoub in 2017 [19], is defined as the transformation of the function $f_\alpha(p^*)$ in the following manner:

$$ \begin{aligned} M[f_\alpha(p^*)] &= N_\alpha(s^*) \\ &= s^{2\alpha}\int_0^{\infty} E_\alpha(-s^\alpha p^{\alpha*})f_\alpha(p^*)\mathrm{d}p^{\alpha*}. \end{aligned} \quad (7) $$

Additionally, the inverse Mohand transform is expressed as

$$ M^{-1}[N_\alpha(s^*)] = f_\alpha(p^*). \quad (8) $$

Definition 3. Local fractional convolution: when dealing with the Mohand transform of functions, $f_{1,\alpha}(p^*), f_{2,\alpha}(p^*)$ are $N_1(s^*), N_2(s^*)$, respectively; then,

$$ M\left[f_{1,\alpha}(p^*) * f_{2,\alpha}(p^*)\right] = \frac{1}{s^{2\alpha}}[N_1(s^*) * N_2(s^*)], \quad (9) $$

where $f_{1,\alpha}(p^*) * f_{2,\alpha}(p^*)$ is defined as

$$ f_{1,\alpha}(p^*) * f_{2,\alpha}(p^*) = \int_0^p f_{1,\alpha}(p^* - y^*)f_{2,\alpha}(p^*)\mathrm{d}x^*. \quad (10) $$

Definition 4. The Mittag-Leffler function, denoted as $E_\alpha(y)$ for $\alpha > 0$, is defined as follows:

$$ E_\alpha(y) = \sum_{i=0}^{\infty}\frac{y^i}{\Gamma(\alpha i + 1)}, \quad \alpha > 0, \quad (11) $$

where α is a member of a complex number set.

2.3. Mohand Transform of Some Functions. Considering an arbitrary function $f(t)$, it is assumed that integral (6) holds true. The presence of the Mohand transform is contingent upon specific conditions being met. Sufficient criteria for the Mohand transform's existence dictate that the function $f(t)$, valid for $t \geq 0$, must exhibit properties of being piecewise continuous and having exponential order. Otherwise, the existence of the Mohand transform remains uncertain.

(i) Let $f(t) = 1$; then, by the definition, we have

$$\begin{aligned} M[1] &= R(s) \\ &= s^2 \int_0^\infty \exp^{-st}\,\mathrm{d}t \\ &= s^2 \left[\frac{-1}{s}\exp^{-st}\right]_0^\infty \\ &= s. \end{aligned} \tag{12}$$

(ii) Let $f(t) = t$; then:

$$\begin{aligned} M[t] &= R(s) \\ &= s^2 \int_0^\infty t\exp^{-st}\,\mathrm{d}t \\ &= 1. \end{aligned} \tag{13}$$

In the typical scenario, when n is a positive integer, then

$$\begin{aligned} M[t^n] &= R(s) \\ &= \frac{n!}{s-1}. \end{aligned} \tag{14}$$

(iii) Let $f(t) = \exp^{at}$; then,

$$\begin{aligned} M\left[\exp^{at}\right] &= s^2 \int_0^\infty \exp^{at}\exp^{-st}\,\mathrm{d}t \\ &= \frac{s^2}{s-a}. \end{aligned} \tag{15}$$

3. Numerical Methods

In this section, we proposed a numerical Mohand–Adomian decomposition method for homogenous fractional Burger's equation.

3.1. Mohand–Adomian Decomposition Method. To illustrate the method described above, we have chosen a homogenous fractional Burger's equation:

$$\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha \mathbb{U}(y,t)}{\partial^\alpha y^2} - 2\mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} + \frac{\partial \mathbb{U}^2(y,t)}{\partial y} = 0, \tag{16}$$

where $(0 < \alpha < 1)$ and the initial condition is

$$\mathbb{U}(y,0) = \mathbb{V}(y). \tag{17}$$

Upon applying the Mohand transform to (16), we obtain:

$$M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial t^\alpha}\right] - M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial^\alpha y^2}\right] - 2M\left[\mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y}\right] + M\left[\frac{\partial \mathbb{U}^2(y,t)}{\partial y}\right] = 0. \tag{18}$$

Now using the definition of the Mohand transform, we get

$$N(s) = s\mathbb{U}(0) + \frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} - \frac{\partial \mathbb{U}^2(y,t)}{\partial y}\right]\right\}, \tag{19}$$

and taking inverse Mohand transform,

$$\mathbb{U}(y,t) = \mathbb{U}(y,0) + M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} - \frac{\partial \mathbb{U}^2(y,t)}{\partial y}\right]\right\}\right]. \tag{20}$$

Therefore, the initial condition will be utilized to determine the first term of $u(y,t)$.

$$\mathbb{U}_0(y,t) = \mathbb{U}(y,0). \tag{21}$$

Ultimately, we derive the recursive general relation in the following form:

$$\mathbb{U}_{m+1}(y,t) = M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_m(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_m(y,t)\frac{\partial \mathbb{U}_m(y,t)}{\partial y} - \frac{\partial \mathbb{U}_m^2(y,t)}{\partial y}\right]\right\}\right], \tag{22}$$

which gives

$$\mathbb{U}_1(y,t) = M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_0(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_0(y,t)\frac{\partial \mathbb{U}_0(y,t)}{\partial y} - \frac{\partial \mathbb{U}_0^2(y,t)}{\partial y}\right]\right\}\right]. \tag{23}$$

u_0 is given, so we get u_1 easily.

Similarly, we have

$$\begin{aligned}
\mathbb{U}_2(y,t) &= M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_1(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_1(y,t)\frac{\partial \mathbb{U}_1(y,t)}{\partial y} - \frac{\partial \mathbb{U}_1^2(y,t)}{\partial y}\right]\right\}\right],\\
\mathbb{U}_3(y,t) &= M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_2(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_2(y,t)\frac{\partial \mathbb{U}_2(y,t)}{\partial y} - \frac{\partial \mathbb{U}_2^2(y,t)}{\partial y}\right]\right\}\right],\\
\mathbb{U}_4(y,t) &= M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_3(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_3(y,t)\frac{\partial \mathbb{U}_3(y,t)}{\partial y} - \frac{\partial \mathbb{U}_3^2(y,t)}{\partial y}\right]\right\}\right],\\
&\vdots,\\
\mathbb{U}_n(y,t) &= M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_{n-1}(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_{n-1}(y,t)\frac{\partial \mathbb{U}_{n-1}(y,t)}{\partial y} - \frac{\partial \mathbb{U}_{n-1}^2(y,t)}{\partial y}\right]\right\}\right].
\end{aligned} \tag{24}$$

Hence, the solution to (21) can be expressed as

$$\mathbb{U}(y,t) = \sum_{n=0}^{\infty} \mathbb{U}_n(y,t). \tag{25}$$

3.2. Examples. Next, we will employ the previously mentioned method in specific scenarios.

Example 1. Consider the following homogenous fractional Burger's equation:

$$\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha \mathbb{U}(y,t)}{\partial^\alpha y^2} - 2\mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} + \frac{\partial \mathbb{U}^2(y,t)}{\partial y} = 0, \tag{26}$$

where $(0 < \alpha < 1)$ and initial condition

$$\mathbb{U}(y,0) = \sin(y), \tag{27}$$

where $0 < \alpha \leq 1, t > 0, y \in R,$ and $(\partial^\alpha \mathbb{U}(y,t)/\partial t^\alpha)$ is the conformable derivative of the function $\mathbb{U}(y,t)$ with respect to t.

Apply the Mohand transform to (26):

$$M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial t^\alpha}\right] - M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial^\alpha y^2}\right] - 2M\left[\mathbb{U}(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y}\right] + M\left[\frac{\partial \mathbb{U}^2(y,t)}{\partial y}\right] = 0. \tag{28}$$

Now using the definition of the Mohand transform, we get

$$R(y,t) = s\mathbb{U}(y,0) + \frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha}y^{2}} + 2u(y,t)\frac{\partial u(y,t)}{\partial y} - \frac{\partial\mathbb{U}^{2}(y,t)}{\partial y}\right]\right\}. \tag{29}$$

Applying the inverse Mohand transform yields

$$\mathbb{U}(y,t) = \mathbb{U}(y,0) + M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha}y^{2}} + 2\mathbb{U}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} - \frac{\partial\mathbb{U}^{2}(y,t)}{\partial y}\right]\right\}\right]. \tag{30}$$

In conclusion, we arrive at the recursive general relation in the following form:

$$\mathbb{U}_{m+1}(y,t) = M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}_{m}(y,t)}{\partial^{\alpha}y^{2}} + 2\mathbb{U}_{m}(y,t)\frac{\partial\mathbb{U}_{m}(y,t)}{\partial y} - \frac{\partial\mathbb{U}_{m}^{2}(y,t)}{\partial y}\right]\right\}\right]. \tag{31}$$

Therefore, from (31), we get

$$\begin{aligned}\mathbb{U}_{1}(y,t) &= M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}_{0}(y,t)}{\partial^{\alpha}y^{2}} + 2u_{0}(y,t)\frac{\partial\mathbb{U}_{0}(y,t)}{\partial y} - \frac{\partial\mathbb{U}_{0}^{2}(y,t)}{\partial y}\right]\right\}\right],\\ \mathbb{U}_{1}(y,t) &= -\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(y).\end{aligned} \tag{32}$$

In a similar manner, we obtain that

$$\begin{aligned}\mathbb{U}_{2}(y,t) &= M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}_{1}(y,t)}{\partial^{\alpha}y^{2}} + 2\mathbb{U}_{1}(y,t)\frac{\partial\mathbb{U}_{1}(y,t)}{\partial y} - \frac{\partial\mathbb{U}_{1}^{2}(y,t)}{\partial y}\right]\right\}\right],\\ \mathbb{U}_{2}(y,t) &= \frac{t^{2\alpha}}{\Gamma(2\alpha+2)}\sin(y).\end{aligned} \tag{33}$$

Also,

$$\begin{aligned}\mathbb{U}_{3}(y,t) &= M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}_{2}(y,t)}{\partial^{\alpha}y^{2}} + 2\mathbb{U}_{2}(y,t)\frac{\partial\mathbb{U}_{2}(y,t)}{\partial y} - \frac{\partial\mathbb{U}_{2}^{2}(y,t)}{\partial y}\right]\right\}\right],\\ \mathbb{U}_{3}(y,t) &= -\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(y),\end{aligned} \tag{34}$$

and so on, and we get the solution of (26):

$$\begin{aligned}
\mathbb{U}(y,t) &= \mathbb{U}_0(y,t) + \mathbb{U}_1(y,t) + \mathbb{U}_2(y,t) + \mathbb{U}_3(y,t),\\
\mathbb{U}(y,t) &= \sin(y) - \frac{t^{\alpha}}{\alpha}\sin(y) + \frac{t^{2\alpha}}{2!\alpha^2}\sin(y) - \frac{t^{3\alpha}}{3!\alpha^2}\sin(y),\\
\mathbb{U}(y,t) &= \sin(y)\left\{1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right\},\\
\mathbb{U}(y,t) &= \mathrm{Sin}(y)E_{\alpha}(-t).
\end{aligned} \tag{35}$$

Figure 1 represents the line graph plotting of homogenous fractional Burger equation (26), while Figure 2 illustrates the 3D surface plots of (26) using various fractional orders.

3.3. Nonhomogenous Fractional Burger's Equation. In this section, we developed a numerical method based on the Mohand–Adomian decomposition method for nonhomogenous fractional Burger's equation. Let us now consider the following nonhomogenous fractional Burger's equation:

$$\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha} y^2} - 2\mathbb{U}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} + \frac{\partial\mathbb{U}^2(y,t)}{\partial y} = y^2 - 2\frac{t^{\alpha}}{\alpha}, \tag{36}$$

with initial condition:

$$\mathbb{U}(y,0) = 0, \tag{37}$$

where $0 < \alpha \leq 1, t > 0, y \in R$, and $(\partial^{\alpha}\mathbb{U}(y,t)/\partial t^{\alpha})$ is the conformable derivative of the function $u(y,t)$ with respect to t.

Applying the Mohand transform to (36), we get

$$M\left[\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial t^{\alpha}}\right] - M\left[\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha} y^2}\right] - 2M\left[\mathbb{U}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y}\right] + M\left[\frac{\partial\mathbb{U}^2(y,t)}{\partial y}\right] = M\left[y^2 - 2\frac{t^{\alpha}}{\alpha}\right]. \tag{38}$$

Now using the definition of the Mohand transform, we get

$$R(y,t) = s\mathbb{U}(y,0) + \frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha} y^2} + 2\mathbb{U}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} - \frac{\partial\mathbb{U}^2(y,t)}{\partial y} - y^2 + 2\frac{t^{\alpha}}{\alpha}\right]\right\}. \tag{39}$$

Taking inverse Mohand transform, we get

$$\mathbb{U}(y,t) = \mathbb{U}(y,0) + M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}(y,t)}{\partial^{\alpha} y^2} + 2\mathbb{U}(y,t)\frac{\partial\mathbb{U}(y,t)}{\partial y} - \frac{\partial\mathbb{U}^2(y,t)}{\partial y} - y^2 + 2\frac{t^{\alpha}}{\alpha}\right]\right\}\right]. \tag{40}$$

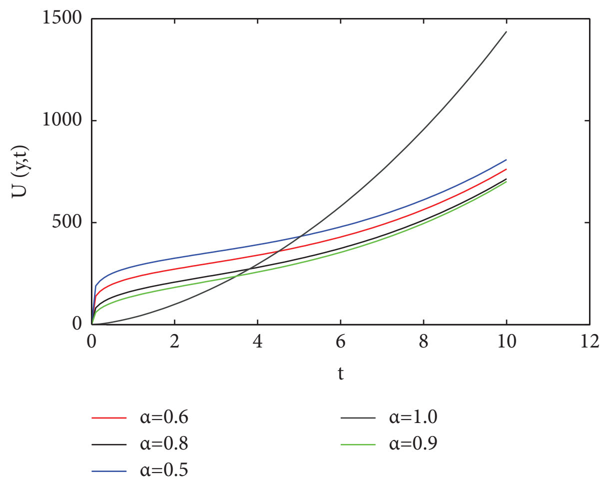

Figure 1: Temperature plots for equation (13) at various α in Caputo sense.

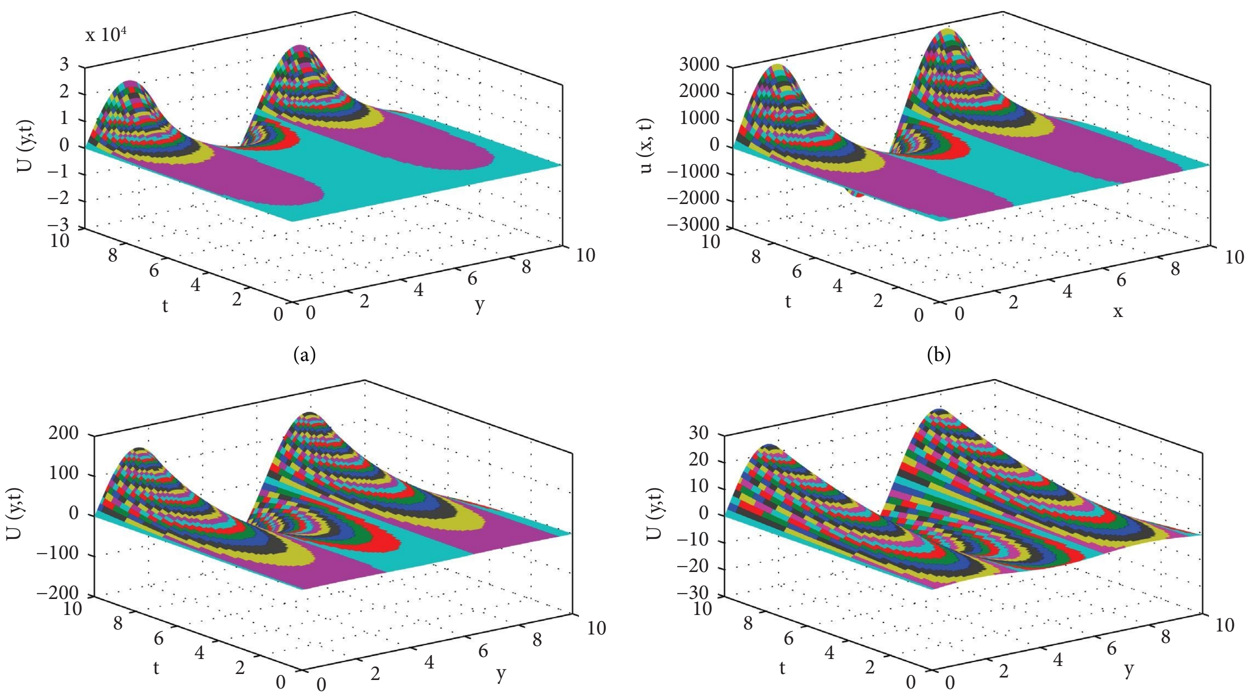

Figure 2: The approximate solution of $\mathbb{U}(y,t)$ of equation (13) for different values of α, $0 \le y \ge 10$ and $0 \le t \ge 10$. (a) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 1$. (b) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.9$. (c) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.8$. (d) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.7$.

Finally, we obtain the recursive general relation form as

$$\mathbb{U}_{m+1}(y,t) = M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{\partial^{\alpha}\mathbb{U}_m(y,t)}{\partial^{\alpha}y^{2}} + 2\mathbb{U}_m(y,t)\frac{\partial\mathbb{U}_m(y,t)}{\partial y} - \frac{\partial\mathbb{U}_m^{2}(y,t)}{\partial y} - y^{2} + 2\frac{t^{\alpha}}{\alpha}\right]\right\}\right]. \tag{41}$$

Therefore, from (41), we get

$$\mathbb{U}_1(y,t) = M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_0(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_0(y,t)\frac{\partial \mathbb{U}_0(y,t)}{\partial y} - \frac{\partial \mathbb{U}_0^2(y,t)}{\partial y} - y^2 + 2\frac{t^\alpha}{\alpha}\right]\right\}\right]. \quad (42)$$

Since $u_0 = 0$, we get

$$\begin{aligned}
\mathbb{U}_1(y,t) &= M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[y^2 - 2\frac{t^\alpha}{\alpha}\right]\right\}\right],\\
\mathbb{U}_1(y,t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}y^2 + 2\Gamma 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
\mathbb{U}_2(y,t) &= M^{-1}\left[\frac{1}{s^\alpha}\left\{M\left[\frac{\partial^\alpha \mathbb{U}_1(y,t)}{\partial^\alpha y^2} + 2\mathbb{U}_1(y,t)\frac{\partial \mathbb{U}_1(y,t)}{\partial y} - \frac{\partial \mathbb{U}_1^2(y,t)}{\partial y} - y^2 + 2\frac{t^\alpha}{\alpha}\right]\right\}\right],\\
\mathbb{U}_2(y,t) &= \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned} \quad (43)$$

Similarly,

$$\mathbb{U}_3(y,t) = 0. \quad (44)$$

By adding all the terms, the series solution of (36) can be found as

$$\mathbb{U}(y,t) = \frac{t^\alpha}{\Gamma(\alpha+1)}y^2 + 2\Gamma 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (45)$$

Figure 3 depicts a line graph illustrating the plotting of solutions derived from nonhomogenous fractional Burger equation (36). This visual representation offers a clear insight into the behavior and trends depicted by these solutions. Moreover, in Figure 4, there are 3D surface plots showcasing (36) at work, demonstrating its behavior under different fractional orders. This visual exploration allows for a comprehensive understanding of how varying fractional orders impacts the overall solution landscape. Additionally, the exact solution of (36) is presented in Figure 5, providing a comparative perspective to the plotted solutions. This figure serves as a reference point to understand the characteristics and deviations of the exact solution concerning the solutions portrayed in Figure 4.

3.4. Another Type of Fractional Burger's Equation. In this section, we developed a numerical method based on the Mohand–Adomian decomposition method for another type of fractional Burger's equation.

We examine the subsequent fractional Burger's equation:

$$\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial t^\alpha} + \mathbb{U}^2(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}(y,t)}{\partial y^2} = 0, \quad (46)$$

with initial condition

$$\mathbb{U}(y,0) = y. \quad (47)$$

Applying the Mohand transform to (46), we get

$$M\left[\frac{\partial^\alpha \mathbb{U}(y,t)}{\partial t^\alpha}\right] + M\left[\mathbb{U}^2(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y}\right] + M\left[\frac{\partial^2 \mathbb{U}(y,t)}{\partial y^2}\right] = 0. \quad (48)$$

Now using the definition of the Mohand transform, we get

$$R(y,t) = s\mathbb{U}(y,0) - \frac{1}{s^\alpha}\left\{M\left[\mathbb{U}^2(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}(y,t)}{\partial y^2}\right]\right\}. \quad (49)$$

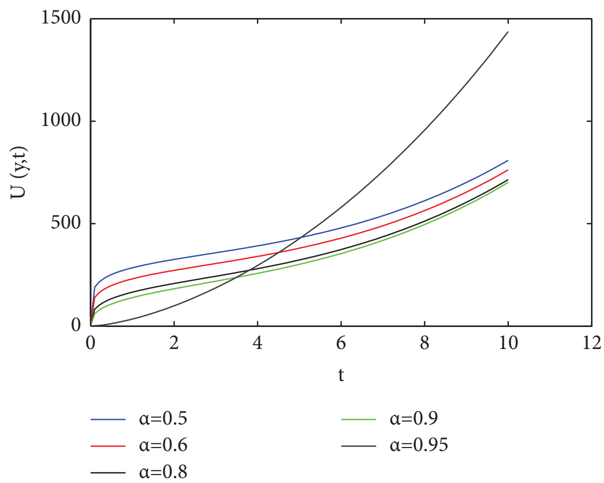

FIGURE 3: Temperature plots for equation (18) at various α in Caputo sense.

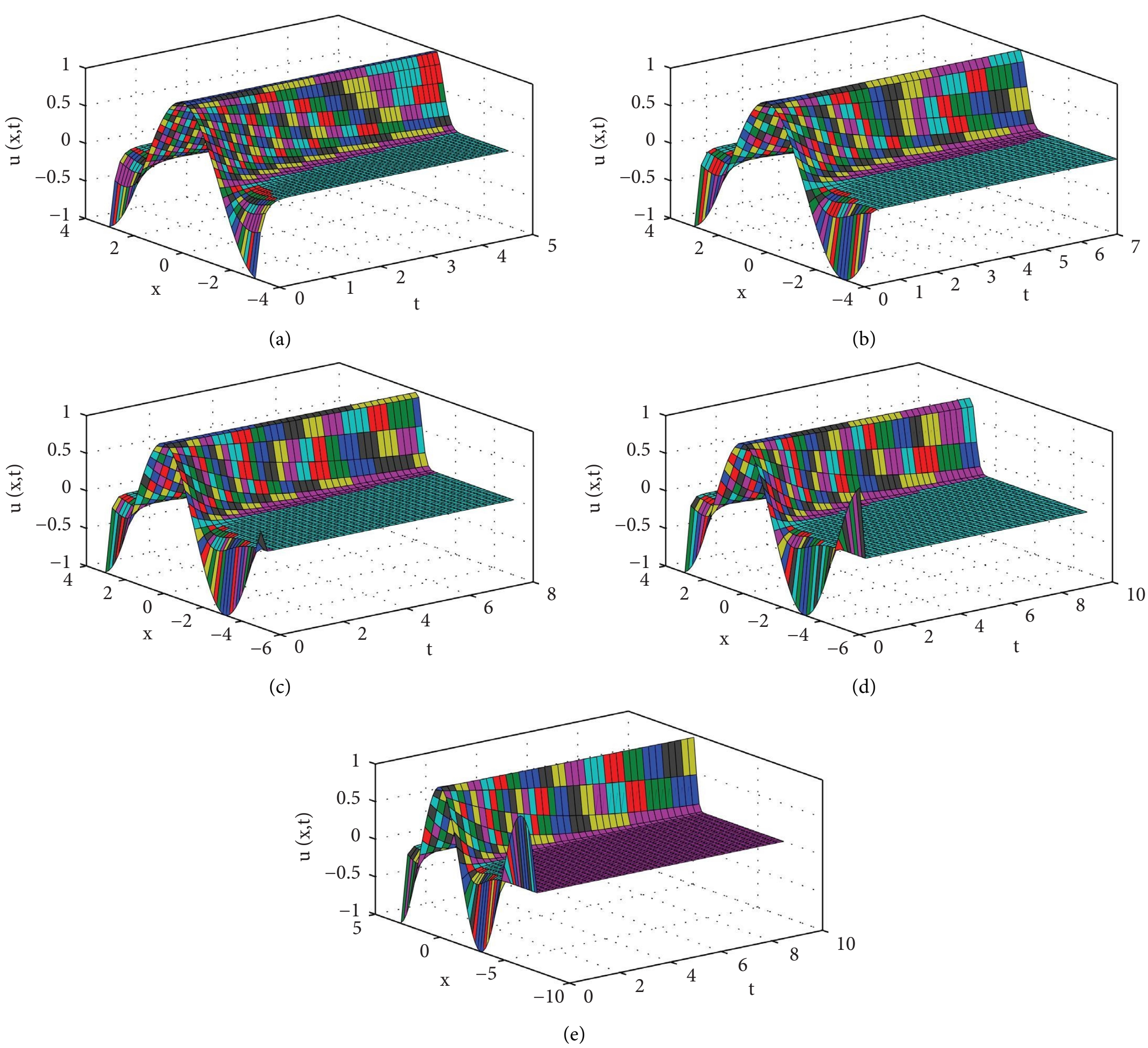

FIGURE 4: The approximate solution of $\mathbb{U}(y,t)$ of equation (18) for different values of α, $-10 \leq y \geq 10$ and $0 \leq t \geq 5$. (a) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 1$. (b) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.9$. (c) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.8$. (d) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.6$. (e) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.5$.

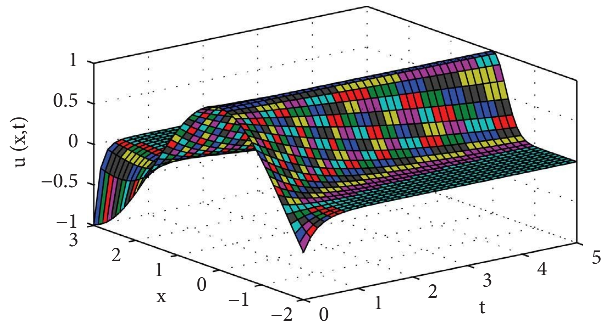

Figure 5: The exact solution for equation (18).

Taking inverse Mohand transform, we get

$$\mathbb{U}(y,t) = \mathbb{U}(y,0) - M^{-1} \cdot \left[\frac{1}{s^{\alpha}}\left\{M\left[\mathbb{U}^2(y,t)\frac{\partial \mathbb{U}(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}(y,t)}{\partial y^2}\right]\right\}\right]. \tag{50}$$

We derive the general recursive relation in the following form:

$$\mathbb{U}_{m+1}(y,t) = -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\mathbb{U}_m^2(y,t)\frac{\partial \mathbb{U}_m(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}_m(y,t)}{\partial y^2}\right]\right\}\right]. \tag{51}$$

Therefore, from (46), we get

$$\begin{aligned}
\mathbb{U}_1(y,t) &= -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\mathbb{U}_0^2(y,t)\frac{\partial \mathbb{U}_0(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}_0(y,t)}{\partial y^2}\right]\right\}\right],\\
\mathbb{U}_1(y,t) &= -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[y^2\right]\right\}\right],\\
\mathbb{U}_1(y,t) &= -\frac{t^{\alpha}}{\Gamma(\alpha+1)}y^2.
\end{aligned} \tag{52}$$

Similarly,

$$\begin{aligned}
\mathbb{U}_2(y,t) &= -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\mathbb{U}_1^2(y,t)\frac{\partial \mathbb{U}_1(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}_1(y,t)}{\partial y^2}\right]\right\}\right],\\
\mathbb{U}_2(y,t) &= -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[-y^4\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\right\}\right],\\
\mathbb{U}_2(y,t) &= \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}y^4,\\
\mathbb{U}_3(y,t) &= -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\mathbb{U}_2^2(y,t)\frac{\partial \mathbb{U}_2(y,t)}{\partial y} + \frac{\partial^2 \mathbb{U}_2(y,t)}{\partial y^2}\right]\right\}\right],\\
\mathbb{U}_3(y,t) &= -M^{-1}\left[\frac{1}{s^{\alpha}}\left\{M\left[\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}\left(y^8+4\right)\right]\right\}\right],\\
\mathbb{U}_3(y,t) &= -6y^8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.
\end{aligned} \tag{53}$$

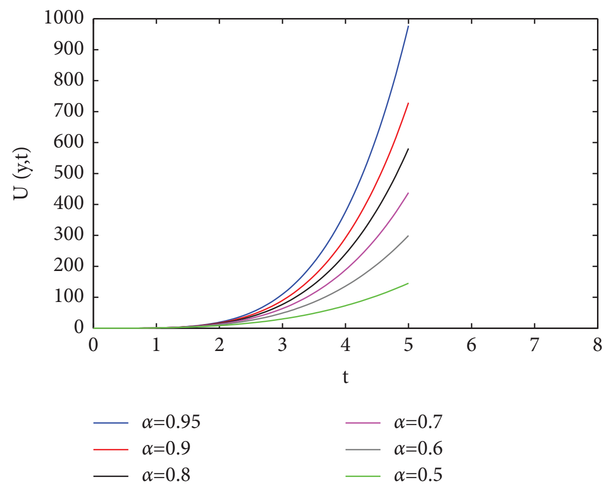

FIGURE 6: Temperature plots for equation (20) at various α in Caputo sense.

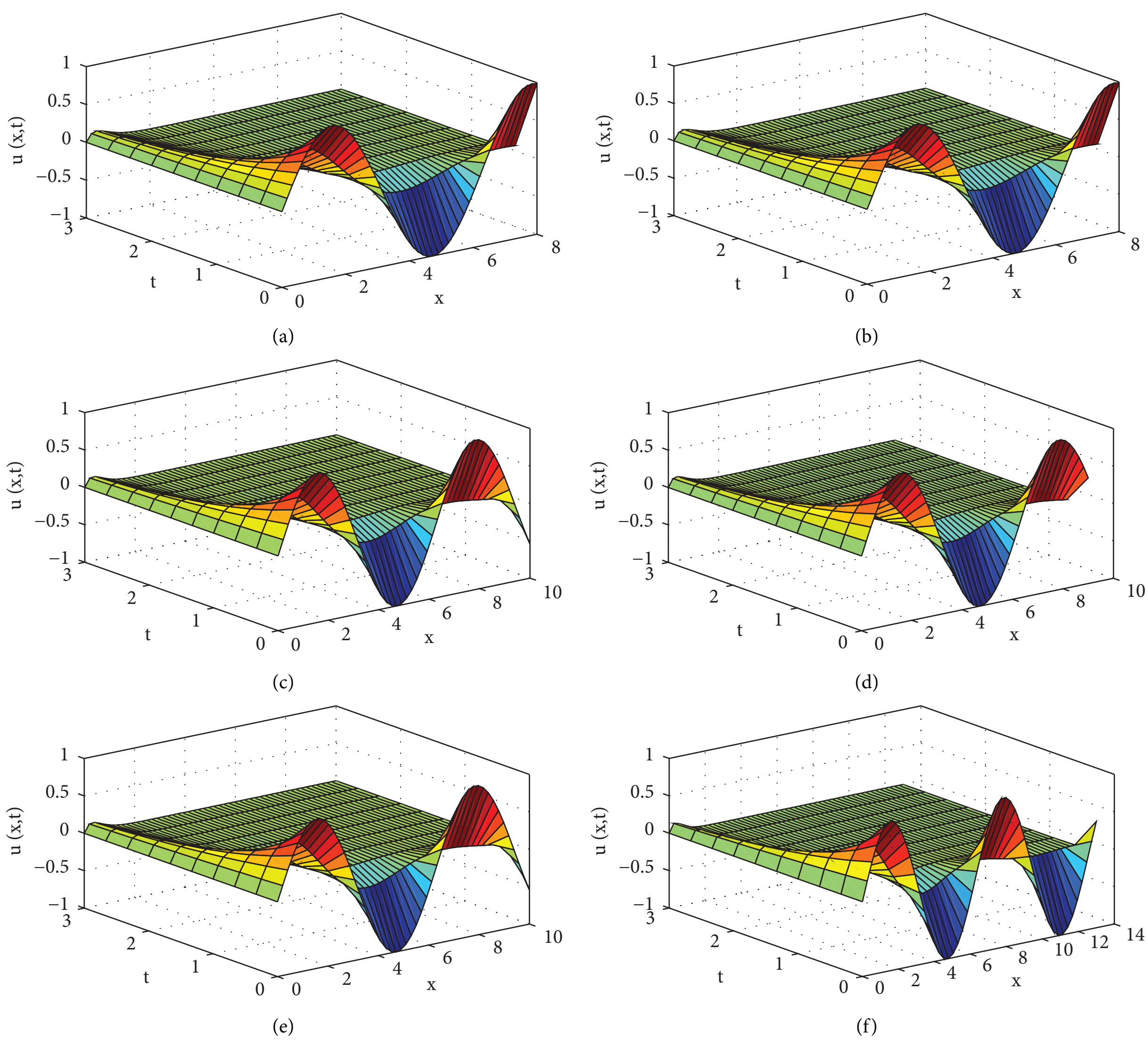

FIGURE 7: The approximate solution of $\mathbb{U}(y,t)$ of equation (20) for different values of α, $0 \le y \ge 500$ and $0 \le t \ge 5$. (a) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 1$. (b) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.9$. (c) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.8$. (d) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.7$. (e) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.6$. (f) Approximate solution of $\mathbb{U}(y,t)$ for $\alpha = 0.5$.

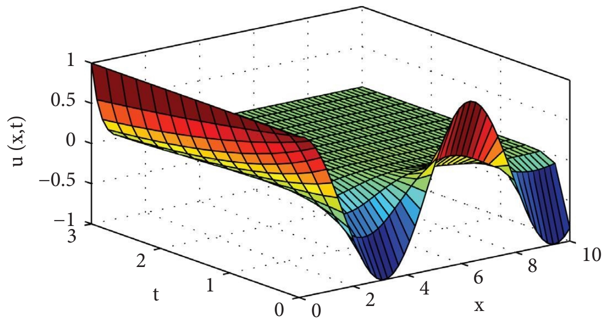

FIGURE 8: The exact solution for equation (20).

Summing the above terms yields

$$\mathbb{U}(y,t) = y - \frac{t^{\alpha}}{\Gamma(\alpha+1)}y^2 + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}y^4 - -6y^8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \tag{54}$$

In Figure 6, a precise line graph illustrates the solutions obtained from fractional type Burger equation (46). This graph offers a clear and detailed view of the complex behavior and trends displayed by these solutions. Further, Figure 7 provides intricate 3D surface plots that demonstrate (46) under various fractional orders. These visuals allow a comprehensive understanding of how altering fractional orders intricately influences the overall solution patterns.

Moreover, Figure 8 exhibits the exact solution derived from (46) for comparison purposes. This figure acts as a standard reference, aiding in a nuanced examination of specific traits and deviations when contrasted with the solutions depicted in Figure 7.

4. Conclusion

In this research, we have applied the Mohand transform and Adomian decomposition method to address fractional Burger's equations, fractional Burger's Kdv equations, and fractional modified Burger's equations. The objective of this investigation is to showcase the effectiveness and simplicity of the Mohand transform as a viable approach for obtaining precise and approximate solutions to these equations. Furthermore, we have illustrated several applications to underscore the versatility of this methodology.

By analyzing the graphs of the solutions, we have observed that the behavior of the solution varies significantly with different values of the fractional-order derivative. This indicates the importance of considering different fractional-order values when studying these equations. Notably, when setting $\alpha = 1$ in the given examples, we have obtained the exact solutions that have been previously investigated in references [20, 21].

It is worth noting a few crucial points regarding the Mohand transform. Firstly, this method provides the solution in terms of easily computable components, making it highly practical for real-world applications. The solutions obtained using the Mohand transform exhibit rapid convergence, which is beneficial for solving physical problems accurately. The numerical results obtained from this approach have shown excellent agreement with their respective exact solutions, further validating its effectiveness.

Secondly, the methods employed in this study were applied directly, without resorting to linearization, perturbation, or restrictive assumptions. This direct approach demonstrates the broad applicability of the Mohand transform in solving various linear and nonlinear fractional problems encountered in applied science.

For the purpose of presenting graphical representations of the solutions, we have utilized MATLAB in this thesis. In future work, we can further explore the capabilities of the Mohand transform by employing it to solve nonlinear differential equations and systems of linear equations and compare its outcomes with other existing methods to evaluate its efficiency and performance.

Overall, the results obtained in this study highlight the effectiveness and practicality of the Mohand transform and Adomian decomposition method in solving fractional differential equations. The potential of this approach for solving a wide range of linear and nonlinear fractional problems demonstrates its significance in applied science and paves the way for future research and applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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