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Research Article

Hankel Determinants for the Logarithmic Coefficients of a Subclass of Close-to-Star Functions

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Suppose that $\mathcal{ST}(1)$ is a class of close-to-star functions. In this paper, we investigated the estimate of Zalcman functional on the logarithmic coefficients and the third Hankel determinant for the class $\mathcal{ST}(1)$ with the determinant entry of logarithmic coefficients. Also, we obtained the sharp bounds of Zalcman functional $J_{2,4}(f)$ and $J_{3,3}(f)$ for the class $\mathcal{ST}(1)$.

1. Introduction

Let $\mathcal{U}$ be the unit disk $\{z\colon\ |z|<1\}$, $\mathcal{A}$ be the class of functions analytic in $\mathcal{U}$, satisfying the conditions

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1. \end{aligned} \tag{1}$$

Then, each functions f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Let $\mathcal{S}$ denote a class of analytic and univalent functions in $\mathcal{U}$. Pommerenke (see [1, 2]) defined the k-th Hankel determinant for a function f as

$$H_{k,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}, \tag{3}$$

where $a_1 = 1$ and $n, k \in \{1, 2, \cdots\}$. Note that the Fekete–Szeg ö functional is actually Hankel determinant with $k = 2$ and $n = 1$, where

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2. \tag{4}$$

Then, the second Hankel determinant with $k = 2$ and $n = 2$ gives

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{5}$$

The third Hankel determinant $H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3 H_{2,2}(f) + a_4 I + a_5 H_{2,1}(f), \tag{6}$$

where $I = a_2 a_3 - a_4$.

In recent years, many mathematicians have investigated Hankel determinants for various classes of functions contained in $\mathcal{A}$. These studies focus on the main subclasses of class $\mathcal{S}$ consisting of univalent functions. For $f \in S$, the two determinants $H_{2,1}(f)$ and $H_{2,2}(f)$ have been extensively studied in the literature for various subfamilies of univalent functions. The sharp bounds for the second determinant were obtained, which are particularly noteworthy. A few papers were devoted to the estimation of sharp upper bound to $H_{3,1}(f)$. Namely, for starlike functions the upper bounds of the third order Hankel determinant $H_{3,1}(f)$ is 4/9 (see [3]), respectively, while for the same bounds for the convex functions, the upper bound is 4/135 (see [4]).

Robertson [5] defined and studied a subclass of close-to-star functions, which is defined as follows:

$$\mathcal{ST}(1) = \left\{ f \in \mathcal{S}\colon \operatorname{Re}\left\{ (1-z)^2 \frac{f(z)}{z} \right\} > 0, \quad z \in \mathcal{U} \right\}. \tag{7}$$

Associated with each $f \in \mathcal{S}$ is a well-defined logarithmic function

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n \quad (z \in \mathcal{U}). \tag{8}$$

The number γ_n are called logarithmic coefficients of f. Differentiating (8) and using (2), we have

$$\begin{cases} \gamma_1 = \frac{1}{2} a_2, \\ \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \\ \gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}a_2^3\right), \\ \gamma_4 = \frac{1}{2}\left(a_5 - a_4 a_2 + a_3 a_2^2 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \\ \gamma_5 = \frac{1}{2}\left(a_6 - a_2 a_5 - a_3 a_4 + a_4 a_2^2 + a_2 a_3^2 - a_2^3 a_3 + \frac{1}{5}a_2^5\right). \end{cases} \tag{9}$$

The logarithmic coefficients are very essential in the problems of univalent functions coefficients. For instance, Milin's conjecture highly depends on logarithmic coefficients (see [6, 7]). Logarithmic coefficients are a hot topic for various authors. For instance, Lecko and Sim [8] studied logarithmic coefficients' problems in families related to close-to-star functions, while the Hermitian Toeplitz determinants of the second- and third-order for classes of close-to-star functions was studied by Jastrzębski et al. [9]. In [10], it was shown that the logarithmic coefficients γ_n of $f \in \mathcal{G}(1)$ satisfy the inequality $\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \leq 1/12$. Also, the early bounds of γ_n for functions in the class of close-to-convex functions, starlike functions related to the vertical strip, and functions starlike with respect to symmetric points were examined in [11–13].

Very recently, Kowalczyk and Lecko [14] introduced the Hankel determinant $H_{k,n}(F_f/2)$, whose elements are logarithmic coefficients of f, that is,

$$H_{k,n}\left(F_f/2\right) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+k-1} & \gamma_{n+k} & \cdots & \gamma_{n+2k-2} \end{vmatrix}. \tag{10}$$

Kowalczyk and Lecko [15] obtained sharp bounds for $H_{2,1}(F_f/2)$ for the classes of starlike and convex functions of order α. The problem of computing the sharp bounds of Hankel determinant on logarithmic coefficients for functions of bounded turning associated with petal-shaped domain has been considered by Shi et al. [16]. In this paper, we investigate the estimate of Zalcman inequality on the logarithmic coefficients and the third Hankel determinant for the class $\mathcal{ST}(1)$ with the determinant entry of logarithmic coefficients. Also, we obtain the sharp estimate of Zalcman inequality for the class $\mathcal{ST}(1)$.

2. Preliminaries

$\mathcal{B}_0$ denotes the class of Schwarz functions, i.e., analytic functions $\omega\colon \mathcal{U} \longrightarrow \mathcal{U}, \omega(0) = 0$. The function $\omega \in \mathcal{B}_0$ has the Taylor series expansion

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{11}$$

In order to establish our main results, we need the following Lemmas.

Lemma 1 (see [17]). *If $\omega \in \mathcal{B}_0$ is given by (11), then the sharp estimate $|c_n| \leq 1$ holds for $n \geq 1$.*

Lemma 2 (see [18]). *Let $\omega \in \mathcal{B}_0$ be the analytic function of the form (11). Then, for any real numbers μ and ν such that*

$$(\mu, \nu) \in \left\{ |\mu| \leq \frac{1}{2}, \quad -1 \leq \nu \leq 1 \right\}, \tag{12}$$

the following sharp estimate holds:

$$\left| c_3 + \mu c_1 c_2 + \nu c_1^3 \right| \leq 1. \tag{13}$$

Lemma 3 (see [19]). *Let $\omega \in \mathcal{B}_0$ be given by (11). Then,*

$$\begin{aligned} |c_2| &\leq 1 - |c_1|^2, \\ |c_3| &\leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \\ |c_4| &\leq 1 - |c_1|^2 - |c_2|^2. \end{aligned} \tag{14}$$

3. Hankel Determinant with Logarithmic Coefficients for the Class $\mathcal{ST}(1)$

We begin this section by finding the absolute values of the first five initial logarithmic coefficients for the function of class $\mathcal{ST}(1)$.

Theorem 4. *Let $f \in \mathcal{ST}(1)$, then*

$$\begin{aligned} |\gamma_1| &\leq 2, \\ |\gamma_2| &\leq \frac{3}{2}, \\ |\gamma_3| &\leq \frac{4}{3}, \\ |\gamma_4| &\leq \frac{5}{4}, \\ |\gamma_5| &\leq \frac{6}{5}. \end{aligned} \tag{15}$$

These inequalities are sharp.

Proof. Let $f \in \mathcal{ST}(1)$. It follows from (2) that

$$(1-z)^2 \frac{f(z)}{z} = \frac{1+\omega(z)}{1-\omega(z)}. \tag{16}$$

From (16), we get

$$\begin{aligned} (1-z)^2 \frac{f(z)}{z} &= 1 + (a_2 - 2)z + (a_3 - 2a_2 + 1)z^2 \\ &\quad + (a_4 - 2a_3 + a_2)z^3 + (a_5 - 2a_4 + a_3)z^4 \\ &\quad + (a_6 - 2a_5 + a_4)z^5 + \ldots, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \frac{1+\omega(z)}{1-\omega(z)} &= 1 + 2c_1 z + 2\left(c_2 + c_1^2\right)z^2 + 2\left(c_3 + 2c_1c_2 + c_1^3\right)z^3 \\ &\quad + 2\left(c_4 + 2c_3c_1 + c_2^2 + 3c_2c_1^2 + c_1^4\right)z^4 \\ &\quad + 2\left(c_5 + 2c_1c_4 + 2c_2c_3 + 3c_1c_2^2 + 3c_1^2c_3 + 4c_1^3c_2 + c_1^5\right)z^5 + \ldots. \end{aligned} \tag{18}$$

Hence, comparing (17) and (18), we achieve

$$\begin{cases} a_2 = 2 + 2c_1, \\ a_3 = 3 + 4c_1 + 2c_2 + 2c_1^2, \\ a_4 = 4 + 6c_1 + 4c_2 + 4c_1^2 + 2c_2 + 4c_1c_2 + 2c_1^3, \\ a_5 = 5 + 8c_1 + 6c_2 + 6c_1^2 + 4c_3 + 8c_2c_1 + 4c_1^3 + 2\left(c_4 + 2c_1c_3 + c_2^2 + 3c_1^2c_2 + c_1^4\right), \\ a_6 = 6 + 10c_1 + 8c_2 + 8c_1^2 + 6c_3 + 12c_1c_2 + 6c_1^3 + 4c_4 + 8c_1c_3 + 4c_2^2 + 12c_1^2c_2 + 4c_1^4 + 2\left(c_5 + 2c_1c_4 + 2c_2c_3 + 3c_1c_2^2 + 3c_1^2c_3 + 4c_1^3c_2 + c_1^5\right). \end{cases} \tag{19}$$

From (9) and (19), we obtain

$$\begin{cases} \gamma_1 = 1 + c_1, \\ \gamma_2 = \frac{1}{2} + c_2, \\ \gamma_3 = \frac{1}{3} + c_3 + \frac{1}{3}c_1^3, \\ \gamma_4 = \frac{1}{4} + c_4 + c_1^2 c_2, \\ \gamma_5 = \frac{1}{5} + c_5 + c_2^2 c_1 + c_1^2 c_3 + \frac{1}{5}c_1^5. \end{cases} \tag{20}$$

Since $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$, the bonds of γ_1 and γ_2 are obvious. Taking $\mu = 0$ and $\nu = 1/3$ in Lemma 2, the bound of γ_3 are as follows. By (20) and Lemma 3, we have

$$\begin{aligned} |\gamma_4| &= \frac{1}{4}\left|1 + 4c_4 + 4c_1^2 c_2\right| \\ &\leq \frac{1}{4}\left[1 + 4\left(1 - |c_1|^2 - |c_2|^2\right) + 4|c_1|^2|c_2|\right] \\ &= \frac{1}{4}\left(5 - 4c^2 - 4d^2 + 4c^2 d\right) = \varphi_1(c,d), \end{aligned} \tag{21}$$

where $c = |c_1|$ and $d = |c_2| \leq 1 - c^2$. Calculus of functions of two variables easily leads to conclusion that $\varphi_1(c,d)$ attains its maximal value 5/4 on $[0,1] \times [0, 1 - c^2]$.

From (20) and Lemma 3 for γ_5, we get

$$|\gamma_5| = \left|\frac{1}{5} + c_5 + c_2^2 c_1 + c_1^2 c_3 + \frac{1}{5}c_1^5\right| \le \frac{1}{5} + 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1+|c_1|} + |c_2|^2|c_1| + |c_1|^2|c_3| + \frac{1}{5}|c_1|^5. \quad (22)$$

The expression on the right side of the above inequality takes its greatest value with respect to $|c_3|$ when $|c_3| = (|c_1|^2(1+|c_1|)/2)$, so

$$|\gamma_5| \le \varphi_2(c,d), \quad (23)$$

where $c = |c_1|$, $d = |c_2|$, and

$$\varphi_2(c,d) = \frac{6}{5} - c^2 - d^2 + \frac{1}{4}c^4 + cd^2 + \frac{9}{20}c^5. \quad (24)$$

A simple algebraic computation shows that the critical points of φ_2 in $\Omega = \{(c,d)\colon 0 \le c \le 1, 0 \le d \le 1 - c^2\}$ satisfy

$$\begin{cases} \dfrac{\partial \varphi_2}{\partial c} = -2c + c^3 + d^2 + \dfrac{9}{4}c^4 = 0, \\ \dfrac{\partial \varphi_2}{\partial d} = -2d + 2cd = 0. \end{cases} \quad (25)$$

By a numerical computation,

$$\begin{cases} c_1 \approx 0, \\ d_1 \approx 0, \end{cases} \quad \begin{cases} c_2 \approx 0.8339, \\ d_2 \approx 0. \end{cases} \quad (26)$$

Thus, there are critical points in Ω. On the boundary of Ω, we get

$$\varphi_2(0,d) = \frac{6}{5} - d^2 \le \frac{6}{5},$$
$$\varphi_2(c,0) = \frac{6}{5} - c^2 + \frac{1}{4}c^4 + \frac{9}{20}c^5 \le \varphi_2(0,0) = \frac{6}{5}, \quad (27)$$
$$\varphi_2\left(c, 1-c^2\right) = \frac{1}{5} + c + c^2 - 2c^3 - \frac{3}{4}c^4 + \frac{29}{20}c^5.$$

Since the functions $\phi_1(c) = c^2 - (3/4)c^4$ and $\phi_2(c) = (1/5) + c - 2c^3 + (29/20)c^5$ reach their greatest values for $c = \sqrt{(2/3)}$ and $c \approx 0.4810$, respectively. $\phi_1(c) \le \phi_1(2/3) = (1/3)$ and $\phi_2(c) \le \phi_2(0.4810) \approx 0.4211$, and it follows that

$$\varphi_2\left(c, 1-c^2\right) \le \frac{1}{3} + 0.4211 \approx 0.75433. \quad (28)$$

The equalities in Theorem 4 hold for the functions f given by (16) with $\omega(z) = z$, $\omega(z) = z^2$, $\omega(z) = z^3$, $\omega(z) = z^4$, and $\omega(z) = z^5$, respectively. □

Theorem 5. *Let $c_0 = 0.2978$ be a zero of the polynomial*

$$4 - 12c - 6c^2 + 4c^3. \quad (29)$$

If $f \in \mathcal{ST}(1)$, then

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \frac{1}{12}\left(25 + 16c_0 - 24c_0^2 - 8c_0^3 + 4c_0^4\right) = 2.28804\ldots. \quad (30)$$

Proof. By (20), we get

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{12}\left|1 + 12c_3 + 4c_1^3 + 4c_1 + 12c_1c_3 + 4c_1^4 - 12c_2 - 12c_2^2\right|. \quad (31)$$

Applying the triangle inequality and Lemma 3 in (31), we obtain

$$\begin{aligned}|\gamma_1\gamma_3 - \gamma_2^2| &\le \frac{1}{12}\left[1 + 12\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + 4|c_1|^3 + 4|c_1| + 12|c_1|\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + 4|c_1|^4 + 12|c_2| + 12|c_2|^2\right] \\ &= \frac{1}{12}\left(13 + 16|c_1| - 12|c_1|^2 - 8|c_1|^3 + 4|c_1|^4 + 12|c_2|\right) \\ &\le \frac{1}{12}\left(13 + 16|c_1| - 12|c_1|^2 - 8|c_1|^3 + 4|c_1|^4 + 12\left(1 - |c_1|^2\right)\right) \\ &= \frac{1}{12}\left(25 + 16|c_1| - 24|c_1|^2 - 8|c_1|^3 + 4|c_1|^4\right) = \varphi_3(c),\end{aligned} \quad (32)$$

where $c = |c_1|$. Obviously, we have

$$\varphi_3'(c) = \frac{1}{3}\left(4 - 12c - 6c^2 + 4c^3\right), \tag{33}$$

the critical point of $\varphi_3'(c)$ is $c = 0.2978\ldots$, and we find

$$\varphi_3''(c) = -4 - 4c + 4c^2, \varphi_3''(0.2978\ldots) = -1.61217\ldots < 0. \tag{34}$$

As a result, at $c = 0.2978\ldots$, $\varphi_3(c)$ reaches its maximum value.

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \varphi_3(0.2978\ldots) = \frac{27.4564\ldots}{12} = 2.28804\ldots. \tag{35}$$

Hence, the proof is completed. □

Theorem 6. *Let* $(C_0, d_0) \approx (0.012165, 0.150977)$ *be the approximate root of the system of linear equations*

$$\begin{cases} 288d^3 - 216\left(c^2 + 2c + 1\right)d^2 + \left(48c^4 + 480c^3 + 360c^2 - 648c - 576\right)d - 36c^4 - 72c^3 + 54c^2 + 180c + 90 = 0, \\ -144d^4 + \left(48c^4 - 240c^3 + 144c^2 + 648c + 252\right)d^2 + \left(-72c^4 - 216c^3 - 216c^2 - 72c\right)d \\ +48c^8 - 96c^7 - 288c^6 + 384c^5 + 744c^4 - 504c^3 - 1176c^2 - 456c = 0. \end{cases} \tag{36}$$

If $f \in \mathcal{ST}(1)$, *then*

$$\begin{aligned} \left|\gamma_2\gamma_4 - \gamma_3^2\right| \le \frac{1}{72}\Bigg[&\frac{72}{\left(1 + c_0\right)^2}d_0^4 - 72d_0^3 + \frac{24c_0^3 + 216c_0^2 - 36c_0 - 288}{1 + c_0}d_0^2 + \left(-36c_0^2 + 90\right)d_0 \\ &+ 8c_0^6 - 48c_0^5 + 72c_0^4 + 64c_0^3 - 228c_0^2 + 157\Bigg] = \frac{163.897337\ldots}{72} = 2.276352\ldots. \end{aligned} \tag{37}$$

Proof. From (20), we achieve

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| = \frac{1}{72}\left|1 - 72c_3^2 - 48c_1^3c_3 - 48c_3 + 72c_2c_4 + 36c_4 + 36c_1^2c_2 + 18c_2 + 72c_1^2c_2^2 - 16c_1^3 - 8c_1^6\right|. \tag{38}$$

Applying the triangle inequality and Lemma 3 to the equation above, we obtain

$$\begin{aligned} \left|\gamma_2\gamma_4 - \gamma_3^2\right| &\le \frac{1}{72}\left(1 + 72|c_3|^2 + 48|c_1|^3|c_3| + 48|c_3| + 72|c_2||c_4| + 36|c_4| + 36|c_1|^2|c_2| + 18|c_2| + 72|c_1|^2|c_2|^2 + 16|c_1|^3 + 8|c_1|^6\right) \\ &\le \frac{1}{72}\Bigg[1 + 72\left(1 - |c_1|^2 - \frac{|c_2|}{1 + |c_1|}\right)^2 + 48|c_1|^3\left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}\right) + 48\left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}\right) \\ &\quad + 72|c_2|\left(1 - |c_1|^2 - |c_2|^2\right) + 36\left(1 - |c_1|^2 - |c_2|^2\right) + 36|c_1|^2|c_2| + 18|c_2| + 72|c_1|^2|c_2|^2 + 16|c_1|^3 + 8|c_1|^6\Bigg] \\ &= \frac{1}{72}\Bigg[\frac{72}{\left(1 + |c_1|\right)^2}|c_2|^4 - 72|c_2|^3 + \frac{24|c_1|^3 + 216|c_1|^2 - 36|c_1| - 288}{1 + |c_1|}|c_2|^2 + \left(-36|c_1|^2 + 90\right)|c_2| \\ &\quad + 8|c_1|^6 - 48|c_1|^5 + 72|c_1|^4 + 64|c_1|^3 - 228|c_1|^2 + 157\Bigg]. \end{aligned} \tag{39}$$

Setting $c = |c_1|$ and $d = |c_2|$, we have

$$72\left|\gamma_2\gamma_4 - \gamma_3^2\right| \le \frac{72}{(1+c)^2}d^4 - 72d^3 + \frac{24c^3 + 216c^2 - 36c - 288}{1+c}d^2 + \left(-36c^2 + 90\right)d + 8c^6 - 48c^5 + 72c^4 + 64c^3 - 228c^2 + 157 = \varphi_4(c,d). \tag{40}$$

We need to find the maximum value of $\varphi_4(c,d)$ on $\Omega = \{(c,d)\colon 0 \le c \le 1, 0 \le d \le 1 - c^2\}$. First, assume that there is a maximum at an interior point (c_0, d_0) of Ω. Differentiating $\varphi_4(c,d)$ with respect to c and d, we get

$$\frac{\partial\varphi_4}{\partial d} = \frac{288}{(1+c)^2}d^3 - 216d^2 + \frac{48c^3 + 432c^2 - 72c - 576}{1+c}d - 36c^2 + 90, \tag{41}$$

and

$$\frac{\partial\varphi_4}{\partial c} = \frac{-144}{(1+c)^3}d^4 + \frac{48c^3 - 288c^2 + 432c + 252}{(1+c)^2}d^2 - 72cd + 8c^6 + 48c^5 - 240c^4 + 288c^3 + 192c^2 - 456c. \tag{42}$$

Setting $\partial\varphi_4/\partial d = 0, \partial\varphi_4/\partial c = 0$ and simplifying, we obtain

$$\begin{cases} 288d^3 - 216\left(c^2 + 2c + 1\right)d^2 + \left(48c^4 + 480c^3 + 360c^2 - 648c - 576\right)d - 36c^4 - 72c^3 + 54c^2 + 180c + 90 = 0 \\ -144d^4 + \left(48c^4 - 240c^3 + 144c^2 + 648c + 252\right)d^2 + \left(-72c^4 - 216c^3 - 216c^2 - 72c\right)d \\ +48c^8 - 96c^7 - 288c^6 + 384c^5 + 744c^4 - 504c^3 - 1176c^2 - 456c = 0. \end{cases} \tag{43}$$

By a numerical computation,

$$\begin{cases} c_0 \approx 0.012165, \\ d_0 \approx 0.150977, \end{cases} \quad \begin{cases} c_1 = -1, \\ d_1 = 0, \end{cases} \quad \begin{cases} c_2 \approx 0.659797, \\ d_2 \approx -1.015659, \end{cases} \quad \begin{cases} c_3 \approx 2.781615, \\ d_3 \approx 0.207693, \end{cases} \quad \begin{cases} c_4 \approx -0.164927, \\ d_4 \approx -1.104444, \end{cases} \quad \begin{cases} c_5 \approx -0.985977, \\ d_5 \approx 0.591726. \end{cases} \tag{44}$$

Thus, in Ω, there is a critical point $(c_0, d_0) \approx (0.012165, 0.150977)$. For the point, we have $\varphi_4(c_0, d_0) = 163.897337\ldots$.

On the boundary of Ω, we get

(1) $c = 0$. Then,

$$\varphi_4(0,d) = 72d^4 - 72d^3 - 288d^2 + 90d + 157 \le \varphi_4(0, 0.1495) = 163.8135\ldots. \tag{45}$$

(2) $d = 0$. Then,

$$\varphi_4(c,0) = 8c^6 - 48c^5 + 72c^4 + 64c^3 - 228c^2 + 157 \le \varphi_4(0,0) = 157. \tag{46}$$

(3) $d = 1 - c^2$. Then,

$$\varphi_4\left(c, 1 - c^2\right) = 176c^6 - 456c^4 - 92c^3 + 330c^2 + 108c - 41 = \tau_1(c) \le \tau_1(0.6882) = 70.0435\ldots. \tag{47}$$

Thus, we get

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \le \frac{\varphi_4(c_0, d_0)}{72} = \frac{163.897337\ldots}{72} = 2.276352\ldots. \tag{48}$$

We complete the proof of Theorem 6. □

Theorem 7. *Let* $(c_0, d_0) \approx (0.330472, 0.327078)$ *be the approximate root of the system of linear equations*

$$\begin{cases} -36d^2 - \left(24c^2 + 48c + 36\right)d + 8c^4 + 8c^3 + 16c + 16 = 0, \\ 12d^3 - \left(12c^2 + 24c + 6\right)d^2 + \left(24c^4 + 48c^3 + 24c^2\right)d - 30c^4 - 96c^3 - 87c^2 - 6c + 15 = 0. \end{cases} \quad (49)$$

If $f \in \mathcal{ST}$ *(1), then*

$$\left|\gamma_1\gamma_4 - \gamma_2\gamma_3\right| \leq \frac{1}{12}\left[-\frac{12}{1+c_0}d_0^3 - \frac{12c_0^2 + 24c_0 + 18}{1+c_0}d_0^2 + \left(8c_0^3 + 16\right)d_0 + 19 + 15c_0 - 18c_0^2 - 10c_0^3\right] = 2.042277. \quad (50)$$

Proof. From (20), we get

$$\left|\gamma_1\gamma_4 - \gamma_2\gamma_3\right| = \frac{1}{12}\left|-12c_2c_3 - 6c_3 + 1 + 12c_4 + 12c_1c_4 + 12c_1^2c_2 + 3c_1 + 8c_1^3c_2 - 2c_1^3 - 4c_2\right|. \quad (51)$$

Applying the triangle inequality and Lemma 3 to the equation above, we obtain

$$\begin{aligned} \left|\gamma_1\gamma_4 - \gamma_2\gamma_3\right| \leq{}& \frac{1}{12}\Bigg[12\left|c_2\right|\left(1 - \left|c_1\right|^2 - \frac{\left|c_2\right|^2}{1+\left|c_1\right|}\right) + 6\left(1 - \left|c_1\right|^2 - \frac{\left|c_2\right|^2}{1+\left|c_1\right|}\right) + 1 + 12\left(1 - \left|c_1\right|^2 - \left|c_2\right|^2\right) \\ & + 12\left|c_1\right|\left(1 - \left|c_1\right|^2 - \left|c_2\right|^2\right) + 12\left|c_1\right|^2\left|c_2\right| + 3\left|c_1\right| + 8\left|c_1\right|^3\left|c_2\right| + 2\left|c_1\right|^3 + 4\left|c_2\right|\Bigg] \\ ={}& \frac{1}{12}\left[-\frac{12}{1+\left|c_1\right|}\left|c_2\right|^3 - \frac{12\left(1+\left|c_1\right|\right)^2 + 6}{1+\left|c_1\right|}\left|c_2\right|^2 + \left(8\left|c_1\right|^3 + 16\right)\left|c_2\right| + 19 + 15\left|c_1\right| - 18\left|c_1\right|^2 - 10\left|c_1\right|^3\right]. \end{aligned} \quad (52)$$

Setting $c = \left|c_1\right|$ and $d = \left|c_2\right|$, we obtain

$$12\left|\gamma_1\gamma_4 - \gamma_2\gamma_3\right| \leq -\frac{12}{1+c}d^3 - \frac{12c^2 + 24c + 18}{1+c}d^2 + \left(8c^3 + 16\right)d + 19 + 15c - 18c^2 - 10c^3 = \varphi_5(c, d). \quad (53)$$

We need to find the maximum value of $\varphi_5(c, d)$ on $\Omega = \left\{(c, d)\colon 0 \leq c \leq 1, 0 \leq d \leq 1 - c^2\right\}$. First, assume that there is a maximum at an interior point (c_0, d_0) of Ω. Differentiating $\varphi_4(c, d)$ with respect to c and d, we get

$$\frac{\partial\varphi_5}{\partial d} = -\frac{36}{1+c}d^2 - \frac{24c^2 + 48c + 36}{1+c}d + 8c^3 + 16, \quad (54)$$

and

$$\begin{aligned} \frac{\partial\varphi_5}{\partial c} ={}& \frac{12}{(1+c)^2}d^3 - \frac{12c^2 + 24c + 6}{(1+c)^2}d^2 \\ & + 24c^2d + 15 - 36c - 30c^2. \end{aligned} \quad (55)$$

Setting $\partial\varphi_5/\partial d = 0$, $\partial\varphi_5/\partial c = 0$, and simplifying, we obtain

$$\begin{cases} -36d^2 - \left(24c^2 + 48c + 36\right)d + 8c^4 + 8c^3 + 16c + 16 = 0, \\ 12d^3 - \left(12c^2 + 24c + 6\right)d^2 + \left(24c^4 + 48c^3 + 24c^2\right)d - 30c^4 - 96c^3 - 87c^2 - 6c + 15 = 0. \end{cases} \tag{56}$$

By a numerical computation,

$$\begin{cases} c_0 \approx 0.330472, \\ d_0 \approx 0.327078, \end{cases}$$
$$\begin{cases} c_1 = -1, \\ d_1 = 0, \end{cases}$$
$$\begin{cases} c_2 \approx 2.908356, \\ d_2 \approx 1.865770, \end{cases} \tag{57}$$
$$\begin{cases} c_3 \approx -0.330825, \\ d_3 \approx -0.941903, \end{cases}$$
$$\begin{cases} c_4 \approx -1.257837, \\ d_4 \approx -0.376144. \end{cases}$$

Thus, in Ω, there is a critical point $(c_0, d_0) \approx (0.330472, 0.327078)$. For the point, we have $\varphi_5(c_0, d_0) = 24.507320\ldots$

On the boundary of Ω, we get

(1) $c = 0$. Then,

$$\varphi_5(0, d) = -12d^3 - 18d^2 + 16d + 19 \le \varphi_5\left(0, \frac{1}{3}\right) = \frac{591}{27} = 21.888889\ldots. \tag{58}$$

(2) $d = 0$. Then,

$$\varphi_5(c, 0) = -10c^3 - 18c^2 + 15c + 19 \le \varphi_5\left(\frac{-6 + \sqrt{86}}{10}, 0\right) = 21.6306\ldots. \tag{59}$$

(3) $d = 1 - c^2$. Then,

$$\varphi_5\left(c, 1 - c^2\right) = -18c^5 - 24c^4 + 2c^3 + 20c^2 + 21c + 45 = \tau_2(c) \le \tau_2(0.6877) = 21.4142\ldots. \tag{60}$$

Thus, we get

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \le \frac{\varphi_5(c_0, d_0)}{12} = \frac{24.507320\ldots}{12} = 2.042277\ldots. \tag{61}$$

Hence, the proof is completed. □

Theorem 8. *Let $f \in \mathcal{ST}(1)$, then*

$$\left|H_{3,1}\left(F_f/2\right)\right| \le 8.33363\ldots. \tag{62}$$

Proof. Since

$$\left|H_{3,1}\left(F_f/2\right)\right| = \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{vmatrix} \le |\gamma_3|\left|\gamma_2\gamma_4 - \gamma_3^2\right| + |\gamma_4|\left|\gamma_1\gamma_4 - \gamma_2\gamma_3\right| + |\gamma_5|\left|\gamma_1\gamma_3 - \gamma_2^2\right|. \tag{63}$$

Using the above results, we achieve the required results. □

4. Zalcman Functional

Lecko and Sim [8] obtained the sharp bound of the Zalcman functional $J_{2,3}(f)$ being a special case of the generalized Zalcman functional $J_{n,m}(f) = a_{n+m-1} - a_n a_m$ for functions from $\mathcal{ST}(1)$. We will compute also the sharp bounds of $J_{2,4}(f)$ and $J_{3,3}(f)$ for the family $\mathcal{ST}(1)$.

Theorem 9. *Let $f \in \mathcal{ST}(1)$, then*

$$J_{2,4}(f) = \left|a_5 - a_2a_4\right| \le 39, J_{3,3}(f) = \left|a_5 - a_3^2\right| \le 56. \tag{64}$$

The inequalities are sharp with the extremal function

$$f(z) = \frac{z(1+z)}{(1-z)^3}, (z \in \mathcal{U}). \tag{65}$$

Proof. From (19), we have

$$\left|a_5 - a_2a_4\right| = \left|-3 - 12c_1 - 2c_2 - 14c_1^2 - 8c_1c_2 - 8c_1^3 + 2c_4 + 2c_2^2 - 2c_2c_1^2 - 2c_1^4\right|. \tag{66}$$

Applying the triangle inequality and Lemma 3 to the equation above, we obtain

$$\begin{aligned}\left|a_5 - a_2a_4\right| &\leq 3 + 12\left|c_1\right| + 2\left|c_2\right| + 14\left|c_1\right|^2 + 8\left|c_1\right|\left|c_2\right| + 8\left|c_1\right|^3 + 2\left(1 - \left|c_1\right|^2 - \left|c_2\right|^2\right) \\ &\quad + 2\left|c_2\right|^2 + 2\left|c_2\right|\left|c_1\right|^2 + 2\left|c_1\right|^4 \\ &= 5 + 12\left|c_1\right| + 2\left|c_2\right| + 12\left|c_1\right|^2 + 8\left|c_1\right|\left|c_2\right| + 8\left|c_1\right|^3 + 2\left|c_2\right|\left|c_1\right|^2 + 2\left|c_1\right|^4 = \varphi_6(c,d),\end{aligned} \tag{67}$$

where $c = \left|c_1\right|$ and $d = \left|c_2\right|$. Differentiating partially with respect to d, we obtain

$$\frac{\partial \varphi_6}{\partial d} = 2 + 8c + 2c^2. \tag{68}$$

Clearly, $\partial \varphi_6/\partial d > 0$ and then $\varphi_6(c,d)$ is increasing in d for fixed c. $\varphi_6(c,d)$ attains its maximum at $d = 1 - c^2$, so

$$\varphi_6(c,d) \leq \varphi_6\left(c, 1 - c^2\right) = 7 + 20c + 12c^2 \leq \varphi_6(1,0) = 39. \tag{69}$$

From (19) by using Lemma 3, it follows that

$$\begin{aligned}\left|a_5 - a_3^2\right| &= \left|-4 - 16c_1 - 6c_2 - 22c_1^2 + 4c_3 - 8c_1c_2 - 12c_1^3 + 2c_4 + 4c_1c_3 - 2c_2^2 - 2c_2c_1^2 - 2c_1^4\right| \\ &\leq 4 + 16\left|c_1\right| + 6\left|c_2\right| + 22\left|c_1\right|^2 + 4\left(1 - \left|c_1\right|^2 - \frac{\left|c_2\right|^2}{1 + \left|c_1\right|}\right) + 8\left|c_1\ \right|c_2\right| + 12\left|c_1\right|^3 \\ &\quad + 2\left(1 - \left|c_1\right|^2 - \left|c_2\right|^2\right) + 4\left|c_1\right|\left(1 - \left|c_1\right|^2 - \frac{\left|c_2\right|^2}{1 + \left|c_1\right|}\right) + 2\left|c_2\right|^2 + 2\left|c_1\ \right|c_1\right|^2 + 2\left|c_1\right|^4 \\ &= 10 + 20\left|c_1\right| + 6\left|c_2\right| + 16\left|c_1\right|^2 + 8\left|c_1\ \right|c_2\right| + 8\left|c_1\right|^3 - 4\left|c_2\right|^2 + 2\left|c_1\ \right|c_2\right|^2 + 2\left|c_1\right|^4 = \varphi_7(c,d),\end{aligned} \tag{70}$$

where $c = \left|c_1\right|$ and $d = \left|c_2\right|$.

We need to find the maximum value of $\varphi_7(c,d)$ on $\Omega = \left\{(c,d)\colon 0 \leq c \leq 1, 0 \leq d \leq 1 - c^2\right\}$. First, assume that there is a maximum at an interior point (c_0, d_0) of Ω. Differentiating $\varphi_7(c,d)$ with respect to c and d, we get

$$\frac{\partial \varphi_7}{\partial d} = -8d + 6 + 8c + 2c^2, \tag{71}$$

and

$$\frac{\partial \varphi_7}{\partial c} = 20 + 32c + 24c^2 + 8c^3 + (8 + 4c)d. \tag{72}$$

Setting $\partial \varphi_7/\partial d = 0, \partial \varphi_7/\partial c = 0$ and simplifying, we obtain

$$9c^3 + 30c^2 + 43c + 26 = 0. \tag{73}$$

A calculation shows that there is no solution of (73) in (0, 1) as $c \approx -1.3919$. On the boundary of Ω, we get

$$\begin{aligned}\varphi_7(c,0) &= 10 + 20c + 16c^2 + 8c^3 + 2c^4 \leq \varphi_7(1,0) = 56, \\ \varphi_7(0,d) &= 10 + 6d - 4d^2 \leq \varphi_7\left(0, \frac{3}{4}\right) = \frac{49}{4}, \\ \varphi_7\left(c, 1 - c^2\right) &= 12 + 28c + 20c^2 - 4c^4 \leq \varphi_7(1,0) = 56.\end{aligned} \tag{74}$$

Observe that the equalities in Theorem 9 hold for the functions f given by (16) with $\omega(z) = z$. We will study the estimate of Zalcman functional $J_{2,3}(f)$ of the logarithmic coefficients for the family $\mathcal{ST}(1)$. □

Theorem 10. *Let* $(c_0, d_0) \approx (0.030175, 0.337301)$ *be the approximate root of the system of linear equations*

$$\begin{cases} -36d^2 - (24c + 36)d + 4c^4 + 4c^3 + 16c + 16 = 0, \\ 12d^3 + 6d^2 + \left(12c^4 + 24c^3 + 12c^2\right)d + 6c^4 - 24c^3 - 66c^2 - 36c = 0. \end{cases} \tag{75}$$

If $f \in \mathcal{ST}$ (1), then

$$\left|\gamma_2\gamma_3-\gamma_4\right|\leq\frac{1}{12}\left[-\frac{12}{1+c_0}d_0^3-\frac{18+12c_0}{1+c_0}d_0^2+\left(16+4c_0^3\right)d_0+19-18c_0^2+2c_0^3\right]=1.825495\ldots. \quad (76)$$

Proof. From (10), we obtain

$$\left|\gamma_2\gamma_3-\gamma_4\right|=\frac{1}{12}\left|-1+6c_3+2c_1^3+4c_2+12c_2c_3+4c_2c_1^3-12c_4-12c_2c_1^2\right|. \quad (77)$$

Applying the triangle inequality and Lemma 3 to the equation above, we obtain

$$\begin{aligned}\left|\gamma_2\gamma_3-\gamma_4\right|&\leq\frac{1}{12}\left[1+6\left(1-\left|c_1\right|^2-\frac{\left|c_2\right|^2}{1+\left|c_1\right|}\right)+2\left|c_1\right|^3+4\left|c_2\right|+12\left|c_2\right|\left(1-\left|c_1\right|^2-\frac{\left|c_2\right|^2}{1+\left|c_1\right|}\right)\right.\\&\quad\left.+4\left|c_1\right|^3\left|c_2\right|+12\left(1-\left|c_1\right|^2-\left|c_2\right|^2\right)+12\left|c_1\right|^2\left|c_2\right|\right]\\&=\frac{1}{12}\left[-\frac{12}{1+\left|c_1\right|}\left|c_2\right|^3-\frac{12\left|c_1\right|+18}{1+\left|c_1\right|}\left|c_2\right|^2+\left(16+4\left|c_1\right|^3\right)\left|c_2\right|+19-18\left|c_1\right|^2+2\left|c_1\right|^3\right].\end{aligned} \quad (78)$$

Setting $c=\left|c_1\right|$ and $d=\left|c_2\right|$, we get

$$\begin{aligned}12\left|\gamma_2\gamma_3-\gamma_4\right|&\leq-\frac{12}{1+c}d^3-\frac{12c+18}{1+c}d^2+\left(16+4c^3\right)d\\&\quad+19-18c^2+2c^3=\varphi_8(c,d).\end{aligned} \quad (79)$$

We need to find the maximum value of $\varphi_8(c,d)$ on $\Omega=\left\{(c,d)\colon 0\leq c\leq 1, 0\leq d\leq 1-c^2\right\}$. First, assume that there is a maximum at an interior point (c_0,d_0) of Ω. Differentiating $\varphi_8(c,d)$ with respect to c and d, we get

$$\frac{\partial\varphi_8}{\partial d}=-\frac{36}{1+c}d^2-\frac{24c+36}{1+c}d+4c^3+16, \quad (80)$$

and

$$\frac{\partial\varphi_8}{\partial c}=\frac{12}{(1+c)^2}d^3+\frac{6}{(1+c)^2}d^2+12c^2d-36c+6c^2. \quad (81)$$

Setting $\partial\varphi_8/\partial d=0, \partial\varphi_8/\partial c=0$ and simplifying, we obtain

$$\begin{cases}-36d^2-(24c+36)d+4c^4+4c^3+16c+16=0,\\12d^3+6d^2+\left(12c^4+24c^3+12c^2\right)d+6c^4-24c^3-66c^2-36c=0.\end{cases} \quad (82)$$

By a numerical computation,

$$\begin{cases} c_0 \approx 0.030175, \\ d_0 \approx 0.337301, \end{cases}$$
$$\begin{cases} c_1 = -1, \\ d_1 = 0, \end{cases}$$
$$\begin{cases} c_2 \approx 1.882189, \\ d_2 \approx 1.037633 \end{cases} \tag{83}$$
$$\begin{cases} c_3 \approx -0.477731, \\ d_3 \approx -0.925485, \end{cases}$$
$$\begin{cases} c_4 \approx -2.785022, \\ d_4 \approx -1.488559. \end{cases}$$

Thus, in Ω, there is a critical point $(c_0, d_0) \approx (0.030175, 0.337301)$. For the point, we have $\varphi_5(c_0, d_0) = 21.905936\ldots$

On the boundary of Ω, we get

(1) $c = 0$. Then,

$$\begin{aligned}\varphi_8(0, d) &= -12d^3 - 18d^2 + 16d + 19 \leq \varphi_8\left(0, \frac{1}{3}\right) \\ &= \frac{591}{27} = 21.888889\ldots.\end{aligned} \tag{84}$$

(2) $d = 0$. Then,

$$\varphi_8(c, 0) = 2c^3 - 18c^2 + 19 \leq \varphi_8(0, 0) = 19. \tag{85}$$

(3) $d = 1 - c^2$. Then,

$$\begin{aligned}\varphi_8\left(c, 1 - c^2\right) &= 8c^5 - 24c^4 - 24c^3 + 20c^2 + 18c + 5 \\ &= \tau_3(c) \leq \tau_3(0.6045) = 15.3289\ldots.\end{aligned} \tag{86}$$

Thus, we get

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \leq \frac{\varphi_8(c_0, d_0)}{12} = \frac{21.905936\ldots}{12} = 1.825495\ldots. \tag{87}$$

Hence, the proof is completed. □

5. Conclusions

Due to the great importance of logarithmic coefficients, Kowalczyk et al. [3, 4] proposed the topic of studying the Hankel determinant with the entry of logarithmic coefficients. In our present investigation, we have successfully examined and studied a subclass of close-to-star functions. We have obtained estimate for some initial logarithmic coefficients and some related inequalities problems on logarithmic coefficients. The third-order Hankel determinant bound for the logarithmic coefficients as the entry for this class were determined.

The bounds of various coefficient functionals in the class $\mathcal{ST}(1)$ presented in this paper were obtained due to connecting this class with the class $\mathcal{B}_0$ of Schwarz functions. It is worth noting that knowing everything about $\mathcal{B}_0$, including estimates of coefficient functionals, is a good tool in studies of other classes of analytic functions. Using these results, one can easily obtain the fourth and fifth Hankel determinants as studied in the articles [20–23].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] C. Pommerenke, “On the coefficients and Hankel determinants of univalent functions,” *Journal of the London Mathematical Society*, vol. 41, no. 1, pp. 111–122, 1966.
[2] C. Pommerenke, “On the Hankel determinants of univalent functions,” *Mathematika*, vol. 14, no. 1, pp. 108–112, 1967.
[3] B. Kowalczyk, A. Lecko, and Y. J. Sim, “The sharp bound of the Hankel determinant for starlike functions,” *Forum Mathematicum*, vol. 34, no. 5, pp. 1249–1254, 2022.
[4] B. Kowalczyk, A. Lecko, and Y. J. Sim, “The sharp bound of the Hankel determinant of the third kind for convex functions,” *Bulletin of the Australian Mathematical Society*, vol. 97, no. 3, pp. 435–445, 2018.
[5] M. S. Robertson, “Analytic functions starlike in one direction,” *American Journal of Mathematics*, vol. 58, no. 3, pp. 465–472, 1936.
[6] I. M. Milin, in *Univalent Functions And Orthonormal Systems Izdat. Nauka*, American Mathematical Society, Providence, Moscow, 1971.
[7] P. T. Duren, *Univalent Functions*, Springer, New York, 1983.
[8] A. Lecko and Y. J. Sim, “Coefficient problems in the subclasses of close-to-star functions,” *Results in Mathematics*, vol. 74, no. 3, p. 104, 2019.
[9] P. Jastrzębski, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, “Hermitian Toeplitz determinants of the second and third-order for classes of close-to-star functions,” *RACSAM*, vol. 114, no. 4, p. 166, 2020.
[10] M. Obradović, S. Ponnusamy, and K. J. Wirths, “Logarithmic coefficients and a coefficient conjecture for univalent functions,” *Monatshefte für Mathematik*, vol. 185, no. 3, pp. 489–501, 2018.
[11] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, “On the third logarithmic coefficient in some subclasses of close-to-convex functions,” *RACSAM*, vol. 114, no. 2, pp. 52–14, 2020.
[12] S. Ponnusamy, N. L. Sharma, and K. J. Wirths, “Logarithmic coefficients problems in families related to starlike and convex functions,” *J.Aust.Math. Soc*, vol. 109, no. 2, pp. 230–249, 2019.
[13] P. Zaprawa, “Initial logarithmic coefficients for functions starlike with respect to symmetric points,” *Bol. Soc. Mat. Mex.*, vol. 27, no. 3, pp. 62–13, 2021.

[14] B. Kowalczyk and A. Lecko, "Second Hankel determinant of logarithmic coefficients of convex and starlike functions," *Bulletin of the Australian Mathematical Society*, vol. 105, no. 3, pp. 458–467, 2022.

[15] B. Kowalczyk and A. Lecko, "Second hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha," *Bull. Malays. Math. Sci. Soc.*, vol. 45, no. 2, pp. 727–740, 2022.

[16] L. Shi, M. Arif, A. Rafiq, M. Abbas, and J. Iqbal, "Sharp bounds of hankel determinant on logarithmic coefficients for functions of bounded turning associated with petal-shaped domain," *Mathematics*, vol. 10, no. 11, p. 2924, 2022.

[17] P. L. Duren, *Univalent Funtions*, Springer, New York, NY USA, 1983.

[18] D. V. Prokhorov and J. Szynal, "*Inverse coefficients for α,β-convex functions*," *Ann. Univ. Mariae. Curie-Sklodowska*, vol. 35, no. A, pp. 125–143, 1981.

[19] F. Carlson, "Sur les coeffcients d´une fonction bornée dans le cercle unité," *Ark. Mat. Astr. Fys.*, vol. 27, p. 8, 1940.

[20] M. Arif, I. Ullah, M. Raza, and P. Zaprawa, "Investigation of the fifth Hankel determinant for a family of functions with bounded turnings," *Mathematica Slovaca*, vol. 70, no. 2, pp. 319–328, 2020.

[21] Z. G. Wang, M. Raza, M. Arif, and K. Ahmad, "On the third and fourth hankel determinants for a subclass of analytic functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 45, no. 1, pp. 323–359, 2022.

[22] M. Arif, L. Rant, M. Raza, and P. Zaprawa, "Fourth Hankel determinant for the family of functions with bounded turning," *Bulletin of the Korean Mathematical Society*, vol. 55, no. 6, pp. 1703–1711, 2018.

[23] K. Trąbka-Więcław, "On coefficient problems for functions connected with the sine function," *Symmetry*, vol. 13, no. 7, p. 1179, 2021.