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Research Article

Geometric Characterization of the Numerical Range of Parallel Sum of Two Orthogonal Projections

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Let $\mathscr{H}$ be a complex separable Hilbert space and $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$. Our goal in this article is to describe the closure of numerical range of parallel sum operator $P:$ PQ for two orthogonal projections P and Q in $\mathscr{B}(\mathscr{H})$ as a closed convex hull of some explicit ellipses parameterized by points in the spectrum.

1. Introduction

Let $\mathscr{H}$ be a complex separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathscr{B}(\mathscr{H})$ be the algebra of bounded linear operators on $\mathscr{H}$. The numerical range $W(T)$ of an operator $T \in \mathscr{B}(\mathscr{H})$ is defined as

$$W(T) = \{\langle Tx, x \rangle : x \in \mathscr{H}, \|x\| = 1\}. \tag{1}$$

It is known that $W(T)$ is a nonempty bounded convex set in the complex plane $\mathbb{C}$ and its closure, denoted by $\overline{W(T)}$, always contains the spectrum $\sigma(T)$ of T (see [1, 2]). In addition, for $T_1, T_2 \in \mathscr{B}(\mathscr{H})$, we have $W(T_1 \oplus T_2) = \mathrm{conv}(W(T_1) \cup W(T_2))$, where $\mathrm{conv}(S)$ stands for the convex hull of the set S. For references on the numerical range and its generalizations, see, for instance, [3–8].

This paper arose from an attempt to gain a geometric characterization of the numerical range of parallel sum with a view of operator block. In what follows we always suppose $A, B \in \mathscr{B}(\mathscr{H})$ and $A + B$ has closed range. The parallel sum of A and B is defined as

$$A : B = A(A + B)^{\dagger} B, \tag{2}$$

where $T^{\dagger}$ is the Moore–Penrose generalized inverse of T (see [9, 10]). The study of parallel sum is motivated by the fact that if A and B are impedance operators of resistive n-port electrical networks, then $A : B$ is the impedance operator of the parallel connection [11]. Several authors, in particular Anderson and Trapp [11], Anderson and Duffin [12], Ando [13], and Wang et al. [10], extended this result and established many different equivalent definitions and properties on parallel sum (see also [9, 10]). Recently, Klaja [14] applied Halmos' two projections theorem to describe the numerical range of a product of two orthogonal projections P and Q. He showed that the closure of its numerical range is equal to a closed convex hull of some ellipses parametrized by points in the spectrum. In [8], Wang et al. also used Halmos' two projections theorem to study the containment region of the numerical range of the product of a pair of positive contractions. Zhang and Yu [15] described the numerical range of the operator $P + QP$. Motivated by these, we consider the numerical range of the parallel sum $P:$ PQ for orthogonal projections P and Q. The investigation uses in an essential way Halmos' two projections theorem, which is introduced as follows.

Let P and Q be two orthogonal projections on $\mathscr{H}$. Thus, $P = P^2 = P^*$ and $Q = Q^2 = Q^*$. The ranges of P and Q are denoted by $\mathscr{L}$ and $\mathscr{N}$, respectively. According to Halmos' two projections theorem (see [16] and consult [17] for the history and more on the subject), there is a representation of $\mathscr{H}$ as an orthogonal sum:

$$\mathscr{H} = (\mathscr{L}\cap\mathscr{N})\oplus(\mathscr{L}\cap\mathscr{N}^{\perp})\oplus(\mathscr{L}^{\perp}\cap\mathscr{N})\oplus(\mathscr{L}^{\perp}\cap\mathscr{N}^{\perp})\oplus\widetilde{\mathscr{H}}, \tag{3}$$

where $\widetilde{\mathscr{H}} = \mathscr{M}_0\oplus\mathscr{M}_1$, $\mathscr{M}_0 = \mathscr{L}\ominus((\mathscr{L}\cap\mathscr{N})\oplus(\mathscr{L}\cap\mathscr{N}^{\perp}))$, $\mathscr{M}_1 = \mathscr{L}^{\perp}\ominus((\mathscr{L}^{\perp}\cap\mathscr{N})\oplus(\mathscr{L}^{\perp}\cap\mathscr{N}^{\perp}))$. If one of the spaces $\mathscr{M}_0$ and $\mathscr{M}_1$ is nontrivial, then these two spaces have the same dimension and there exist two self-adjoint operators S and C of $\mathscr{M}_0$ into itself such that $0\le S\le I, 0\le C\le I$, $S^2+C^2=I$, $\mathrm{Ker}(S)=\mathrm{Ker}(C)=\{0\}$, and such that P and Q are simultaneously unitary equivalent to the following operator matrices:

$$P \sim I\oplus I\oplus 0\oplus 0\oplus\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}, Q\sim I\oplus 0\oplus I\oplus 0\oplus\begin{pmatrix} C^2 & CS\\ CS & S^2\end{pmatrix}. \tag{4}$$

Moreover, there exists a self-adjoint operator T verifying $0\le T\le \pi/2I$ such that $\cos(T)=C$ and $\sin(T)=S$.

In [18], Deng and Du introduced the pair (P,Q) in generic position, if $\mathscr{L}\cap\mathscr{N}=\mathscr{L}\cap\mathscr{N}^{\perp}=\mathscr{L}^{\perp}\cap\mathscr{N}=\mathscr{L}^{\perp}\cap\mathscr{N}^{\perp}=\{0\}$. If two orthogonal projections P and Q are in generic position, then $\mathscr{H}=\mathscr{M}_0\oplus\mathscr{M}_1$ and the operator matrices in (4) can be simplified to

$$P\sim\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}, Q\sim\begin{pmatrix}\cos^2(T) & \cos(T)\sin(T)\\ \cos(T)\sin(T) & \sin^2(T)\end{pmatrix}. \tag{5}$$

Tian et al. [9] gave a specific matrix representation of the operator $P:\ PQ$ with respect to the decomposition (1). Let P and Q have the operator matrices in (4), and we can get

$$P:\ PQ\sim\frac{1}{2}I\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix}\dfrac{\cos^2(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)} & \dfrac{\cos(T)\sin(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)}\\ 0 & 0\end{pmatrix}. \tag{6}$$

If P and Q are in generic position, the above operator matrix in turn will be

$$P:\ PQ\sim\begin{pmatrix}\dfrac{\cos^2(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)} & \dfrac{\cos(T)\sin(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)}\\ 0 & 0\end{pmatrix}, \tag{7}$$

which will be very useful in the next section.

2. Main Results

The following theorems are the main results of this article. Let $\lambda\in[0,1/2]$, $\mathscr{E}(\lambda)$. We denote the domain delimited by the ellipse with foci 0 and λ, and minor axis length

$$\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\,\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\,\right]}-\lambda^2}. \tag{8}$$

Theorem 1. *Let P, Q be two orthogonal projections; then, for $\lambda\in\sigma(P:\ PQ)$, the closure of the numerical range of operator $P:\ PQ$ is the closed convex hull of the elliptical disk $\mathscr{E}(\lambda)$:*

$$\overline{W(P:\ PQ)}=\overline{\mathrm{conv}\left\{\bigcup_{\lambda\in\sigma(P:PQ)}\mathscr{E}(\lambda)\right\}}. \tag{9}$$

If (P,Q) are in generic position, we will first prove the following theorem.

Theorem 2. *Let P, Q be two orthogonal projections in generic position; then, for $\lambda\in\sigma(P:\ PQ)$, the closure of the numerical range of operator $P:\ PQ$ is the closed convex hull of the elliptical disk $\mathscr{E}(\lambda)$:*

$$\overline{W(P:\ PQ)}=\overline{\mathrm{conv}\left\{\bigcup_{\lambda\in\sigma(P:PQ)}\mathscr{E}(\lambda)\right\}}. \tag{10}$$

In order to prove Theorem 2, we need the following definition and lemmas.

Definition 3 (see [14]). Let $\mathcal{S}$ be a bounded convex set in $\mathbb{C}$. Let $\alpha\in\mathbb{R}$. The support function of $\mathcal{S}$, of angle α, is defined by the following formula:

$$\rho_{\mathcal{S}}(\alpha)=\sup\{\mathrm{Re}(z\exp(-i\alpha)), z\in\mathcal{S}\}. \tag{11}$$

Lemma 4 (see [14, 19]). *We denote by $\overline{\mathcal{S}}$ the closure of $\mathcal{S}$. We have*

$$\overline{\mathcal{S}} = \{z \in \mathbb{C}, \forall \alpha, \operatorname{Re}(z \exp(-i\alpha)) \leq \rho_{\mathcal{S}}(\alpha)\}. \tag{12}$$

Lemma 5 (see [14, 19]). *Let $\mathcal{S}_1, \mathcal{S}_2$ be two bounded convex sets of the plane $\mathbb{C}$ with support functions $\rho_{\mathcal{S}_1}(\alpha)$ and $\rho_{\mathcal{S}_2}(\alpha)$, respectively. Let $\mathcal{S}$ be such that $\rho_{\mathcal{S}}(\alpha) = \max_{i=1,2} \rho_{\mathcal{S}_i}(\alpha)$. Then, we have*

$$\overline{\mathcal{S}} = \overline{\operatorname{conv}\{\mathcal{S}_1 \cup \mathcal{S}_2\}}. \tag{13}$$

Lemma 6. *Let (P, Q) be in generic position. Then, the support function of the numerical range of operator $P : PQ$ is*

$$\rho_{W(P:PQ)}(\alpha) = \sup_{\lambda \in \sigma(P:PQ)} \frac{1}{2}\left[\lambda \cos(\alpha) + \sqrt{\lambda \frac{1 + (1/2)\left[(3\lambda - 1) + \sqrt{(1-3\lambda)^2 + 4\lambda}\right]}{1 + (3/2)\left[(3\lambda - 1) + \sqrt{(1-3\lambda)^2 + 4\lambda}\right]} - \lambda^2 \sin^2(\alpha)}\right]. \tag{14}$$

Proof. We fix $\alpha \in \mathbb{R}$. From Definition 3, we can get

$$\begin{aligned} \rho_{W(P:PQ)}(\alpha) &= \sup\{\operatorname{Re}(\langle (P : PQ)h, h\rangle \exp(-i\alpha)), h \in \mathcal{H}, \|h\| = 1\} \\ &= \sup\{\operatorname{Re}(\langle \exp(-i\alpha)(P : PQ)h, h\rangle), h \in \mathcal{H}, \|h\| = 1\} \\ &= \sup\{\langle \operatorname{Re}(\exp(-i\alpha)(P : PQ))h, h\rangle, h \in \mathcal{H}, \|h\| = 1\}. \end{aligned} \tag{15}$$

It follows that

$$\begin{aligned} P : PQ &\sim \begin{pmatrix} \dfrac{\cos^2(T)\left(I + \cos^2(T)\right)}{I + 3\cos^2(T)} & \dfrac{\cos(T)\sin(T)\left(I + \cos^2(T)\right)}{I + 3\cos^2(T)} \\ 0 & 0 \end{pmatrix}, \\ (P : PQ)^* &\sim \begin{pmatrix} \dfrac{\cos^2(T)\left(I + \cos^2(T)\right)}{I + 3\cos^2(T)} & 0 \\ \dfrac{\cos(T)\sin(T)\left(I + \cos^2(T)\right)}{I + 3\cos^2(T)} & 0 \end{pmatrix}, \end{aligned} \tag{16}$$

and we have

$$\begin{aligned} \operatorname{Re}(\exp(-i\alpha)(P : PQ)) &= \frac{1}{2}\left[\exp(-i\alpha)(P : PQ) + \exp(i\alpha)(P : PQ)^*\right] \\ &\sim \begin{pmatrix} \dfrac{\cos(\alpha)\cos^2(T)\left(I + \cos^2(T)\right)}{I + 3\cos^2(T)} & \dfrac{\exp(-i\alpha)\cos(T)\sin(T)\left(I + \cos^2(T)\right)}{2\left(I + 3\cos^2(T)\right)} \\ \dfrac{\exp(i\alpha)\cos(T)\sin(T)\left(I + \cos^2(T)\right)}{2\left(I + 3\cos^2(T)\right)} & 0 \end{pmatrix}. \end{aligned} \tag{17}$$

From

$$A(t,\alpha) = \begin{pmatrix} \dfrac{\cos(\alpha)\cos^2(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)} & \dfrac{\exp(-i\alpha)\cos(t)\sin(t)\left(1+\cos^2(t)\right)}{2\left(1+3\cos^2(t)\right)} \\ \dfrac{\exp(i\alpha)\cos(t)\sin(t)\left(1+\cos^2(t)\right)}{2\left(1+3\cos^2(t)\right)} & 0 \end{pmatrix}, \tag{18}$$

where $t \in [0, \pi/2]$, it follows that $\mathrm{Re}(e^{-i\alpha}(P:\mathrm{PQ})) \sim A(T,\alpha)$. After some computations, we can get $A(t,\alpha) = U^*(t,\alpha)B(t,\alpha)U(t,\alpha)$ with

$$B(t,\alpha) = \begin{pmatrix} v_1(t,\alpha) & 0 \\ 0 & v_2(t,\alpha) \end{pmatrix}, \tag{19}$$

and

$$U(t,\alpha) = \begin{pmatrix} \dfrac{2v_1(t,\alpha)}{u_1(t,\alpha)} & \dfrac{2v_2(t,\alpha)}{u_2(t,\alpha)} \\ \dfrac{\exp(i\alpha)\cos(t)\sin(t)\left(1+\cos^2(t)\right)}{u_1(t,\alpha)\left(1+3\cos^2(t)\right)} & \dfrac{\exp(i\alpha)\cos(t)\sin(t)\left(1+\cos^2(t)\right)}{u_2(t,\alpha)\left(1+3\cos^2(t)\right)} \end{pmatrix}, \tag{20}$$

where

$$v_1(t,\alpha) = \frac{1}{2}\left[\frac{\cos(\alpha)\cos^2(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)} + \sqrt{\left(\frac{\cos(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)}\right)^2 - \left(\frac{\cos^2(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)}\right)^2\sin^2(\alpha)}\right],$$

$$v_2(t,\alpha) = \frac{1}{2}\left[\frac{\cos(\alpha)\cos^2(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)} - \sqrt{\left(\frac{\cos(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)}\right)^2 - \left(\frac{\cos^2(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)}\right)^2\sin^2(\alpha)}\right], \tag{21}$$

$$u_i(t,\alpha) = \sqrt{4v_i^2(t,\alpha) + \left(\frac{\cos(t)\sin(t)\left(1+\cos^2(t)\right)}{1+3\cos^2(t)}\right)^2}, \quad i = 1, 2.$$

It is easy to verify passing to the limit when t goes to $\pi/2$ that

$$U\left(\frac{\pi}{2},\alpha\right) = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{\exp(i\alpha)}{\sqrt{2}} & \dfrac{\exp(i\alpha)}{\sqrt{2}} \end{pmatrix}. \tag{22}$$

We also have that $U^*(T,\alpha)U(T,\alpha) = U(T,\alpha)U^*(T,\alpha) = I$. As all entries of $U(t,\alpha)$ are Borelians functions and T is a self-adjoint operator, according to Borelians functional calculus (see [20]), we can define

$$\frac{2v_1(T,\alpha)}{u_1(T,\alpha)}, \frac{\exp(i\alpha)\cos(T)\sin(T)\left(I+\cos^2(T)\right)}{u_1(T,\alpha)\left(I+3\cos^2(T)\right)}, \frac{2v_2(T,\alpha)}{u_2(T,\alpha)}, \frac{\exp(i\alpha)\cos(T)\sin(T)\left(I+\cos^2(T)\right)}{u_2(T,\alpha)\left(I+3\cos^2(T)\right)}. \tag{23}$$

Then, we also can define $B(T,\alpha)$ and $U(T,\alpha)$, and we have that

$$\begin{aligned} A(T,\alpha) &= U^*(T,\alpha)B(T,\alpha)U(T,\alpha) \quad \text{and} \quad U^*(T,\alpha)U(T,\alpha) \\ &= U(T,\alpha)U^*(T,\alpha) \\ &= I. \end{aligned} \tag{24}$$

So, we obtain

$$\operatorname{Re}(\exp(-i\alpha)(P:PQ)) \sim A(T,\alpha) \sim B(T,\alpha) = v_1(T,\alpha)\oplus v_2(T,\alpha). \tag{25}$$

Note that $v_2(t,\alpha)\leq 0\leq v_1(t,\alpha)$ for every $t\in[0,\pi/2]$ and $\alpha\in[0,2\pi]$. Since $\sigma(T)\subset[0,\pi/2]$, we also have that $v_2(T,\alpha)\leq 0\leq v_1(T,\alpha)$, and then we obtain

$$\begin{aligned} \rho_{W(P:\mathrm{PQ})}(\alpha) &= \sup\left\{\langle \operatorname{Re}(\exp(-i\alpha)(P:\mathrm{PQ}))h,h\rangle, h\in\mathscr{H}, \|h\|=1\right\} \\ &= \sup\left\{\langle (v_1(T,\alpha)\oplus v_2(T,\alpha))h',h'\rangle, h'\in\mathscr{H}, \left\|h'\right\|=1\right\} \\ &= \left\|v_1(T,\alpha)\oplus v_2(T,\alpha)\right\| \\ &= \left\|v_1(T,\alpha)\right\| = \sup_{t_0\in\sigma(T)} v_1(t_0,\alpha). \end{aligned} \tag{26}$$

From (5) and (7), we can get

$$(P:\mathrm{PQ})P \sim \begin{pmatrix} \dfrac{\cos^2(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{27}$$

It follows that

$$\begin{aligned} \frac{\sigma((P:PQ)P)}{\{0\}} &= \frac{\sigma(P(P:PQ))}{\{0\}} \\ &= \frac{\sigma(P(P:PQ))}{\{0\}}, \end{aligned} \tag{28}$$

and we have

$$\sigma(P:\mathrm{PQ}) = \frac{\cos^2(\sigma(T))\left(1+\cos^2(\sigma(T))\right)}{1+3\cos^2(\sigma(T))}\cup\{0\}. \tag{29}$$

Denoting $\lambda = (\cos^2(t_0)(1+\cos^2(t_0)))/(1+3\cos^2(t_0))$, where $t_0\in\sigma(T)$, then $\lambda\in\sigma(P:\mathrm{PQ})$. We obtain that

$$\begin{aligned} \rho_{W(P:\mathrm{PQ})}(\alpha) &= \sup_{t_0\in\sigma(T)} v_1(t_0,\alpha) \\ &= \sup_{\lambda\in\sigma(P:\mathrm{PQ})} \frac{1}{2}\left[\lambda\cos(\alpha) + \sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]} - \lambda^2\sin^2(\alpha)}\right]. \end{aligned} \tag{30}$$

This completes the proof.

In order to describe $W(P: PQ)$ clearly, we characterize it as the closed convex hull of ellipses $\mathscr{E}(\lambda)$. Several of these ellipses are shown in Figure 1. ☐

Remark 7. The Cartesian equation of the boundary of $\mathscr{E}(\lambda)$ is given by

$$\frac{4(x-(\lambda/2))^2}{\lambda\left(1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]/1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]\right)} + \frac{4y^2}{\lambda\left(1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]/1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]\right)-\lambda^2} = 1, \tag{31}$$

and the parametric equation of the boundary of $\mathscr{E}(\lambda)$ is given by

$$\begin{aligned} x_\lambda(\theta) &= \frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}}\cos(\theta)+\frac{\lambda}{2}, \\ y_\lambda(\theta) &= \frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2}\sin(\theta), \end{aligned} \tag{32}$$

where $\theta \in [0, \pi/2]$.

Lemma 8. *Let $\alpha \in \mathbb{R}$. The support function of the elliptical disk $\mathscr{E}(\lambda)$ is*

$$\rho_{\mathscr{E}(\lambda)}(\alpha) = \frac{1}{2}\left[\lambda\cos(\alpha)+\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2\sin^2(\alpha)}\right]. \tag{33}$$

Proof. Let $\lambda \in [0, 1/2]$. The support function of $\mathscr{E}(\lambda)$ relative to the original point 0 is given by

$$\rho_{\mathscr{E}_\lambda}(\alpha) = \sup_{\theta\in\mathbb{R}}\{x_\lambda(\theta)\cos(\alpha)+y_\lambda(\theta)\sin(\alpha)\}, \tag{34}$$

where $x_\lambda(\theta)$, $y_\lambda(\theta)$ represent the parametric equation of the boundary of $\mathscr{E}_\lambda$. Let $f = f_{\lambda,\alpha}$ be the function defined by the following formula:

$$\begin{aligned} f_{\lambda,\alpha}(\theta) &= x_\lambda(\theta)\cos(\alpha)+y_\lambda(\theta)\sin(\alpha) \\ &= \frac{\lambda}{2}\cos(\alpha)+\frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}}\cos(\theta)\cos(\alpha) \\ &\quad+\frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2}\sin(\theta)\sin(\alpha). \end{aligned} \tag{35}$$

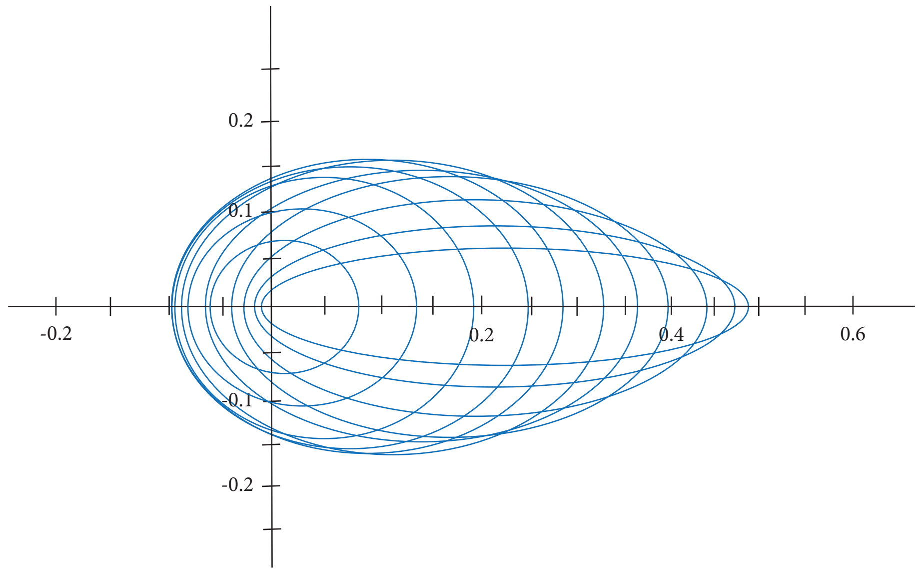

Figure 1: Ellipse $\mathscr{E}(\lambda)$ for $\lambda = 0.02, 0.05, 0.1, 0.15\ldots, 0.45, 0.48, 0.5$.

Since $\mathscr{E}(\lambda)$ is symmetric about $y=0$, only $\alpha \in [0,\pi]$ needs to be considered, and the proof will be divided into two cases.

Case One. Suppose that $\cos(\alpha) \neq 0$. It follows from

$$f_{\lambda,\alpha}'(\theta) = -\frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}}\sin(\theta)\cos(\alpha) + \frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}} - \lambda^2\cos(\theta)\sin(\alpha), \tag{36}$$

that we have $f_{\lambda,\alpha}'(\theta)=0$ if and only if $\tan(\theta) = \sqrt{1-\lambda(1+(3/2)[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])/(1+(1/2)[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])}\tan(\alpha)$. So, the critical points are $\theta_1 = \arctan\left(\sqrt{1-\lambda(1+(3/2)[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])/(1+(1/2)[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])}\tan(\alpha)\right)$ and $\theta_2 = \theta_1 + \pi$. We denoted $\epsilon_1 = 1, \epsilon_2 = -1$; then,

$$\cos(\theta_i) = \epsilon_i\frac{1}{\sqrt{2-\lambda\left(1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]/1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]\right)\tan^2(\alpha)}},$$
$$\sin(\theta_i) = \epsilon_i\frac{\sqrt{1-\lambda\left(1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]/1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]\right)}\tan(\alpha)}{\sqrt{2-\lambda\left(1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]/1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]\right)\tan^2(\alpha)}}, \tag{37}$$

where $i = 1, 2$. Substituting the above formula into $f_{\lambda,\alpha}(\theta_i)$, we can get

$$\begin{aligned} f_{\lambda,\alpha}(\theta_i) &= x_\lambda(\theta_i)\cos(\alpha) + y_\lambda(\theta_i)\sin(\alpha) \\ &= \frac{\lambda}{2}\cos(\alpha) + \frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}}\cos(\theta_i)\cos(\alpha) \\ &\quad + \frac{1}{2}\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2}\sin(\theta_i)\sin(\alpha). \end{aligned} \tag{38}$$

By simple calculation, we have

$$f_{\lambda,\alpha}(\theta_i) = \frac{1}{2}\left[\lambda\cos(\alpha) \pm \varepsilon_i\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2\sin^2(\alpha)}\right]. \tag{39}$$

Then, we finally get that

$$\rho_{\mathscr{E}(\lambda)}(\alpha) = \sup_{i=1,2} f_{\lambda,\alpha}(\theta_i) = \frac{1}{2}\left[\lambda\cos(\alpha) + \sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2\sin^2(\alpha)}\right]. \tag{40}$$

Case Two. If $\cos(\alpha) = 0$, then $f_{\lambda,\alpha}(\theta) = 1/2\sqrt{\lambda(1+1/2[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])/(1+3/2[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])-\lambda^2}\sin(\theta)$. The function $f_{\lambda,\alpha}(\theta)$ reaches its maximum value $1/2\sqrt{\lambda(1+1/2[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])/(1+3/2[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}])-\lambda^2}$ while $\sin(\theta) = 1$ and also satisfies

$$\rho_{\mathscr{E}(\lambda)}(\alpha) = \frac{1}{2}\left[\lambda\cos(\alpha) + \sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2\sin^2(\alpha)}\right]. \tag{41}$$

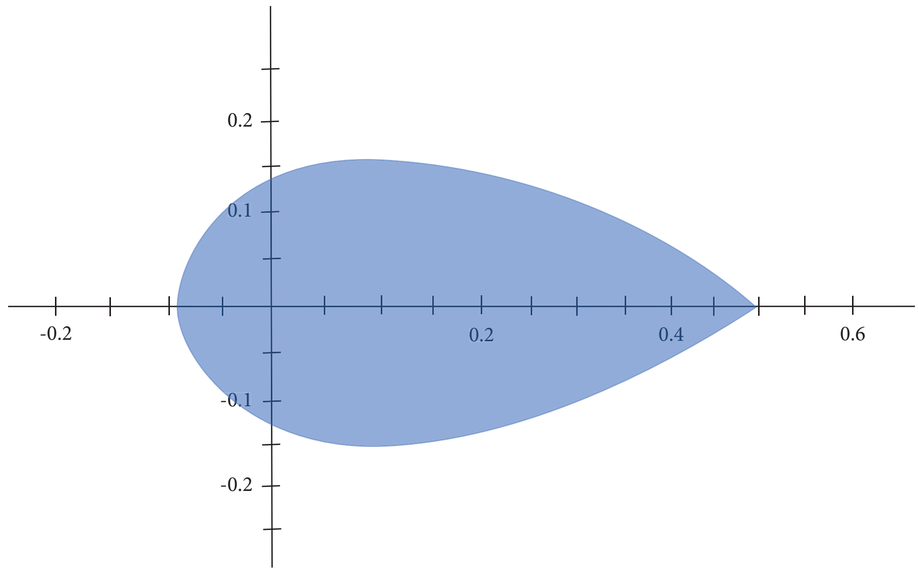

Figure 2: $\overline{\operatorname{conv}\left\{\cup_{\lambda\in[0,1/2]}\mathscr{E}(\lambda)\right\}}$.

The proof is completed. □

Then, we can prove Theorems 1 and 2.

Proof of Theorem 1. From Lemmas 6 and 8, we have

$$\begin{aligned}\rho_{W(P:\mathrm{PQ})}(\alpha) &= \sup_{\lambda\in\sigma(P:PQ)}\frac{1}{2}\left[\lambda\cos(\alpha)+\sqrt{\lambda\frac{1+(1/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}{1+(3/2)\left[(3\lambda-1)+\sqrt{(1-3\lambda)^2+4\lambda}\right]}-\lambda^2\sin^2(\alpha)}\right]\\ &= \sup_{\lambda\in\sigma(P:\mathrm{PQ})}\rho_{\mathscr{E}(\lambda)}(\alpha).\end{aligned} \tag{42}$$

It follows from Lemma 6 that

$$\overline{W(P:\mathrm{PQ})}=\overline{\operatorname{conv}\left\{\bigcup_{\lambda\in\sigma(P:\mathrm{PQ})}\mathscr{E}(\lambda)\right\}}. \tag{43}$$

The proof is completed. □

Proof of Theorem 2. From the matrix form in (2), we have

$$P:\mathrm{PQ}\sim\frac{1}{2}I\oplus 0\oplus 0\oplus 0\oplus\begin{pmatrix}\dfrac{\cos^2(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)} & \dfrac{\cos(T)\sin(T)\left(I+\cos^2(T)\right)}{I+3\cos^2(T)}\\ 0 & 0\end{pmatrix}. \tag{44}$$

Suppose $\widetilde{\mathscr{H}}=\{0\}$. The following proof will be divided into two cases.

(1) If $\mathscr{L}\cap\mathscr{N}\neq\{0\}$, then $W(P:\mathrm{PQ})=W(1/2I\oplus 0\oplus 0\oplus 0)=\operatorname{conv}\{\{0\}\cup\{1/2\}\}=[0,1/2]$. In this case, $\lambda\in\sigma(P:\mathrm{PQ})=\{0,1/2\}$ and $\mathscr{E}(0)=\{0\}$, $\mathscr{E}(1/2)=[0,1/2]$. Thus, $W(P:\mathrm{PQ})=[0,1/2]=\operatorname{conv}\{\mathscr{E}(0)\cup\mathscr{E}(1/2)\}=\operatorname{conv}\left\{\cup_{\lambda\in\sigma(P:PQ)}\mathscr{E}(\lambda)\right\}$.

(2) If $\mathscr{L}\cap\mathscr{N}=\{0\}$, we have $W(P:\mathrm{PQ})=\{0\}=\mathscr{E}(0)$.

Suppose $\widetilde{\mathscr{H}}\neq\{0\}$. The following proof will be divided into two cases.

(3) If $\mathscr{L}\cap\mathscr{N}=\{0\}$, we have $P:\mathrm{PQ}=0$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$. Thus, the closure of the numerical range of $P:PQ$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$ is $\operatorname{conv}\left\{\{0\}\cup\overline{\operatorname{conv}\left\{\cup_{\lambda\in\sigma(P:\mathrm{PQ})}\mathscr{E}(\lambda)\right\}}\right\}$. As $\{0\}\subset\mathscr{E}(\lambda)$ for all $\lambda\in[0,1/2]$, we can have

$\overline{W(P:\mathrm{PQ})}=\overline{\mathrm{conv}\{\cup_{\lambda\in\sigma(P:\mathrm{PQ})}\mathscr{E}(\lambda)\}}$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$.

(4) If $\mathscr{L}\cap\mathscr{N}\neq\{0\}$, we have $P:\mathrm{PQ}=1/2I$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$. Thus, the closure of the numerical range of $P:PQ$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$ is $\mathrm{conv}\left\{\{1/2\}\cup\overline{\mathrm{conv}\{\cup_{\lambda\in\sigma(P:PQ)}\mathscr{E}(\lambda)\}}\right\}$. As $\{0\}\subset\mathscr{E}(\lambda)$ for all $\lambda\in[0,1/2]$ and $\widetilde{\mathscr{H}}\neq\{0\}$, the convex combination of 0 and 1/2 is contained in the closure of the numerical range of $P:PQ$. Thus, the closure of the numerical range of $P:\mathrm{PQ}$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$ is $\mathrm{conv}\{[0,1/2]\cup\overline{\mathrm{conv}\{\cup_{\lambda\in\sigma(P:PQ)}\mathscr{E}(\lambda)\}}\}$. But $\mathscr{E}(1/2)=[0,1/2]$. So, we have $\overline{W(P:\mathrm{PQ})}=\overline{\mathrm{conv}\{\cup_{\lambda\in\sigma(P:\mathrm{PQ})}\mathscr{E}(\lambda)\}}$ on the space $(\mathscr{L}\cap\mathscr{N})\oplus\widetilde{\mathscr{H}}$. The proof is completed. □

Corollary 9. *Let P and Q be orthogonal projections. Then, for $\lambda\in\sigma(P:PQ)$, we can get*

$$\overline{W(P:\mathrm{PQ})}=\overline{\mathrm{conv}\left\{\bigcup_{\lambda\in\sigma(P:\mathrm{PQ})}\mathscr{E}(\lambda)\right\}}\subset\overline{\mathrm{conv}\left\{\bigcup_{\lambda\in[0,1/2]}\mathscr{E}(\lambda)\right\}}. \quad (45)$$

In particular, we have $\overline{W(P:\mathrm{PQ})}=\overline{\mathrm{conv}\{\cup_{\lambda\in[0,1/2]}\mathscr{E}(\lambda)\}}$ when $\sigma(P:PQ)=[0,1/2]$, as shown in Figure 2 [21–24].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in this paper. All authors have read and approved the final manuscript.

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