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Research Article

Stabilization of a Rao–Nakra Sandwich Beam System by Coleman–Gurtin's Thermal Law and Nonlinear Damping of Variable-Exponent Type

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In this paper, we explore the asymptotic behavior of solutions in a thermoplastic Rao–Nakra (sandwich beam) beam equation featuring nonlinear damping with a variable exponent. The heat conduction in this context adheres to Coleman–Gurtin's thermal law, encompassing linear damping, Fourier, and Gurtin–Pipkin's laws as specific instances. By employing the multiplier approach, we establish general energy decay results, with exponential decay as a particular manifestation. These findings extend and generalize previous decay results concerning the Rao–Nakra sandwich beam equations.

1. Introduction

Partial differential equations (PDEs) featuring variable exponents have garnered considerable attention from researchers in recent times. Unlike conventional PDEs with constant exponents, equations with variable exponents entail power-law dependencies on spatial variables, with the exponents subject to variation along the spatial coordinates. This variation introduces added intricacy and complexities in both the analysis and solution of these equations.

The importance of PDEs with variable exponents is due to several reasons. They offer a more adaptable framework for modeling physical phenomena, enabling the representation of diverse behaviors in different regions of the domain by allowing spatial variation in the exponents. This type of nonlinearity within PDEs allows for a more precise representation of various physical phenomena, particularly in instances where the system's behavior cannot be accurately depicted by linear models. The variable exponents enable a more adaptive and versatile approach to processing, contingent upon the characteristics of the image content. They come up in the research on optimal control problems, where the goal is to find strategies that make certain criteria as good as possible while considering the changes described by the PDE with variable exponents.

Moreover, heat conduction is a fundamental phenomenon with widespread applications in various physical processes, frequently described through partial differential equations (PDEs). These equations are instrumental in capturing the intricacies of heat distribution, as evidenced by their application in diverse scenarios. For instance, the temperature variation along a solid rod or bar finds representation in a 1D heat conduction equation, commonly formulated as the one-dimensional heat equation. This PDE correlates temporal changes with the distribution of temperature, offering a comprehensive understanding of the thermal dynamics. In the realm of electronics, especially for components such as computer chips and integrated circuits, a profound comprehension of heat conduction is indispensable. PDEs prove invaluable in modeling the temperature distribution within these devices, thereby contributing to the design of efficient cooling systems.

Furthermore, engineering simulations leverage PDEs to analyze the thermal behavior of structures and systems, extending their application to fields such as civil engineering. The study of how buildings respond to environmental temperature changes relies on the insights gained from understanding heat conduction, showcasing the pervasive significance of PDEs in elucidating complex thermal phenomena.

For additional insights into the hierarchy of heat conduction laws, the authors refer to [1]. Motivated by the mathematical complexities presented by these equations and their efficacy in modeling intricate physical phenomena, we examine the following Rao–Nakra with a sandwich beam system:

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + \delta_1 \theta_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - \delta_1 \theta + \delta_2 \vartheta_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x + \delta_3(t)\left|w_t\right|^{m(x)-2} w_t = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho_4 \theta_t + (\beta_1 - 1)\theta_{xx} - \beta_1 \int_0^{+\infty} g_1(s)\theta_{xx}(x,t-s)ds + \delta_1(u_{xt} + v_t) = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho_5 \vartheta_t + (\beta_2 - 1)\theta_{xx} - \beta_2 \int_0^{+\infty} g_2(s)\vartheta_{xx}(x,t-s)ds + \delta_2 v_{xt} = 0, & \text{in } (0,L)\times\mathbb{R}_+, \end{cases} \tag{1}$$

in which u and v denote the longitudinal displacement of the top layer and shear angle of the bottom layer, respectively, and w represents the transverse displacement of the beam. The positive constants ρ_i, h_i, E_i, and I_i $(i = 1; 3)$ are physical parameters representing, respectively, density, thickness, Young's modulus, and the moments of inertia of the i-th layer for $i = 1, 2, 3$ and $\rho h = \Sigma_{i=1}^{3}\rho_i h_i$. Here, $EI = \Sigma_{i=1}^{3} E_i I_i$, $\alpha = h_2 + (h_1 + h_3/2)$, and $k = (E_2/2h_2(1+\mu))$ where $-1 < \mu < 1/2$ is the Poisson ratio. $\delta_1 > 0$ and $\delta_2 > 0$ are coupling constants. δ_3 is a time-dependent coefficient and $\beta_1, \beta_2 \in (0, 1)$. θ is the temperature supposed to be known for negative times. ρ_4 and ρ_5 are the ratio between the relaxation time and the thermal conductivity. $m(.)$ is the variable exponent function and satisfies some conditions that will be mentioned later. The functions g_1 and g_2 represent the convolution thermal kernel, nonnegative bounded convex summable function on $[0, +\infty)$, belonging to a wide class of relaxation functions that satisfy the unitary total mass and additional properties specified in the paper. The system (1) consists of one Euler–Bernoulli beam equation for the transverse displacement and two wave equations for the longitudinal displacement of the top layer and the shear angle of the bottom layer. The top and bottom layers of the beam are subjected to Coleman–Gurtin's thermal law [2], where the Fourier, Maxwell–Cattaneo's, and Gurtin–Pipkin's laws [3] are special cases. We subject the system (1) to the following boundary conditions:

$$\begin{cases} u_x(0,t) = v_x(0,t) = w(0,t) = w_{xx}(0,t) = \theta(0,t) = \vartheta(0,t) = 0, & t \geq 0, \\ u(L,t) = v(L,t) = w(L,t) = w_{xx}(L,t) = \theta_x(L,t) = \vartheta_x(L,t) = 0, & t \geq 0, \end{cases} \tag{2}$$

and the initial data are given by

$$\begin{cases} u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in (0,L), \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), w_t(x,0) = w_1(x), & x \in (0,L), \\ \theta(x,-t) = \theta_0(x,t), \vartheta(x,-t) = \vartheta_0(x,t), & x \in (0,L), t \geq 0. \end{cases} \tag{3}$$

The stabilization of Rao–Nakra beam systems has recently captured significant attention among researchers, leading to the establishment of numerous findings. The Rao–Nakra beam model involves the dynamics of two outer face plates, presumed to be relatively rigid, along with a compliant inner core layer sandwiched between them. The authors of [4–7] provide insights into Rao–Nakra, Mead–Markus, and multilayer plate or sandwich models. The fundamental equations of motion for the Rao–Nakra model are derived based on Euler–Bernoulli beam assumptions for the outer face plate layers, Timoshenko beam assumptions for the Sandwich layer, and a no-slip assumption for motion along the interface.

Let us begin by revisiting some prior works concerning multilayered sandwich beam models. Wang et al. [8] explored a sandwich beam system with boundary control using the Riesz basis approach, establishing exponential stability, exact controllability, and observability. In a different approach, Rajaram [9] utilized the multiplier approach to determine the precise controllability of a Rao–Nakra sandwich beam with boundary controls. Hansen and Imanuvilov [10, 11] investigated a multilayer plate system with locally distributed control in the boundary, employing Carleman estimations to establish precise controllability. Özer and Hansen [12, 13] achieved boundary feedback stabilization and perfect controllability for a multilayer Rao–Nakra Sandwich beam. Liu et al. [14] considered viscous damping effects on either the beam equation or one of the wave equations, establishing a polynomial decay rate using the frequency domain technique. Wang [15] analyzed a Rao–Nakra beam with boundary damping on one end, finding that the semigroup created by the system is polynomially stable of order 1/2. Mukiawa [16] recently studied system (1) with linear damping and Gurtin–Pipkin's thermal law for heat conduction, proving the existence and establishing an exponential decay rate. Additional results on multilayer beams can be found in [17–30].

Our objective is to explore the asymptotic behavior of solutions in the context of the system (1)–(3). We aim to investigate how the thermal damping and the nonlinear damping with a variable exponent power, introduced in equation $(1)_3$, impact the asymptotic behavior of the energy function. Without imposing restrictions on the wave propagation speeds in the system, we employ the multiplier approach to establish both exponential and general energy decay rates for this system. These results extend and generalize previous decay findings related to the Rao–Nakra sandwich beam equation. The primary objectives are as follows:

(i) Establishing an exponential decay of the system when the relaxation functions converge exponentially, and the variable exponent is set to $m(x) = 2$.

(ii) Establishing more generalized decay results for the system applies when the relaxation functions do not converge exponentially, and the variable exponent $m(x) \neq 2$. In this scenario, various cases will be discussed based on the range of variable exponents and the convergence type of the relaxation functions.

To the best of our knowledge, stability results for the Rao–Nakra sandwich beam with nonlinear damping of variable exponent type have not been explored.

The rest of the paper is structured as follows: Section 2 introduces preliminary results and notations. Section 3 presents and proves technical lemmas, and Section 4 outlines and proves the stability theorem, offering a detailed proof.

2. Notations, Assumptions, and Transformations

This section is dedicated to the assumptions and specific transformations required for our problem. We make the following assumptions:

(A_1). g_1, g_2: $[0,+\infty) \longrightarrow (0,+\infty)$ are nonincreasing C^2 $([0,+\infty))$ and convex summable functions satisfying the following:

$$\lim_{s\longrightarrow+\infty} g_i(s) = 0, \qquad \int_0^{+\infty} g_i(s)ds = 1, \quad i = 1, 2. \tag{4}$$

Furthermore, there exists $\xi_i > 0, i = 1, 2$ such that

$$-g_i''(s) \le \xi_i(s)\left(g_i'(s)\right), \quad \forall s \ge 0\ i = 1, 2. \tag{5}$$

By setting:

$$\mu_1(s) = -g_1'(s), \qquad \mu_2(s) = -g_2'(s), \tag{6}$$

we obtain the following:

(A_2). μ_1, μ_2: $[0,+\infty) \longrightarrow (0,+\infty)$ are nonincreasing C^1 $([0,+\infty))$ and convex summable functions satisfying

$$\mu_{0i} = \int_0^{+\infty} \mu_i(s)ds = g_i(0) > 0, \qquad \int_0^{+\infty} s\mu_i(s)ds = 1, \quad i = 1, 2, \tag{7}$$

and there exists $\xi_i > 0, i = 1{,}2$ such that

$$\mu_i'(s) \le -\xi_i(s)\mu_i(s), \forall s \ge 0, \quad i = 1, 2, \tag{8}$$

(A_3). We assume the existence of two positive constants ζ, ζ such that

$$\left\|\eta_{0x}(s)\right\|^2 \le \zeta, \quad \forall s > 0, \qquad \left\|\tilde{\eta}_{0x}(s)\right\|^2 \le \tilde{\zeta}, \quad \forall s > 0, \tag{9}$$

where η_{0x} and $\tilde{\eta}_{0x}$ are defined (below).

We define $L_{\mu_1} = \{\eta: \mathbb{R}_+ \longrightarrow H_0^1(0,L): \|\eta\|_{L_{\mu_1}}^2 := \int_0^\infty \mu_1(s)\|\eta_x(s)\|^2 ds < \infty\}$, and $L_{\mu_2} = \left\{\tilde{\eta}: \mathbb{R}_+ \longrightarrow H_0^1(0,L): \|\tilde{\eta}\|_{L_{\mu_2}}^2 := \int_0^\infty \mu_2(s)\|\tilde{\eta}_x(s)\|^2 ds < \infty\right\}$, which defines a Hilbert space.

(A_4). The time-dependent coefficient δ_3: $[0,\infty) \longrightarrow (0,\infty)$ is a nonincreasing C^1 function satisfying $\int_0^\infty \delta_3(s)ds = \infty$.

(A_5). The variable exponent $m(x)\colon [0,L] \longrightarrow [1,\infty)$ is a continuous function such that

$$m_1 \coloneqq \operatorname{essinf}_{x\in[0,L]} m(x), m_2 \coloneqq \operatorname{esssup}_{x\in[0,L]} m(x), \quad (10)$$

and $1 < m_1 \leq m(x) \leq m_2 < \infty$. Moreover, the variable function m satisfies the log-Hölder continuity condition; that is, for any δ with $0 < \delta < 1$, there exists a constant $A > 0$ such that

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \text{for all } x, y \in \Omega, \text{with } |x-y| < \delta. \quad (11)$$

For further details on the memory kernels, refer to [31, 32].

Due to Dafermos [33], we define new functions for the relative past history of θ and ϑ as follows:

$$\eta, \tilde{\eta}\colon (0,L) \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad (12)$$

define by

$$\begin{aligned} \eta(x,t,s) &\coloneqq \int_{t-s}^{t} \theta(x,r)dr, \\ \tilde{\eta}(x,t,s) &\coloneqq \int_{t-s}^{t} \vartheta(x,r)dr. \end{aligned} \quad (13)$$

On account of the boundary conditions (2), we have

$$\eta(0,t,s) = \eta_x(L,t,s) = \tilde{\eta}(0,t,s) = \tilde{\eta}_x(L,t,s) = 0, \quad (14)$$

and routine calculation gives

$$\begin{cases} \eta_t + \eta_s - \theta = 0, & \text{in } (0,L) \times (\mathbb{R}_+ \times \mathbb{R}_+, \\ \tilde{\eta}_t + \tilde{\eta}_s - \vartheta = 0, & \text{in } (0,L) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta(x,t,0) = \tilde{\eta}(x,t,0) = 0, & \text{in } (0,L) \times \mathbb{R}_+, \\ \eta(x,0,s) = \int_0^s \theta_0(x,r)dr \coloneqq \eta_0(x,s), & \text{in } (0,L) \times \mathbb{R}_+, \\ \tilde{\eta}(x,0,s) = \int_0^s \vartheta_0(x,r)dr \coloneqq \tilde{\eta}_0(x,s), & \text{in } (0,L) \times \mathbb{R}_+, \end{cases} \quad (15)$$

where η_0 and $\tilde{\eta}_0$ represent the history of θ and ϑ, respectively. Also, using integration by parts and change of variables, we have

$$\begin{aligned} &\int_0^{+\infty} g_1(s)\theta_{xx}(x,t-s)ds \\ &= \lim_{a\longrightarrow+\infty} g_1(s) \int_{t-s}^{t} \theta_{xx}(x,r)dr\|_{s=0}^{s=a} - \int_0^{+\infty} g_1'(s) \int_{t-s}^{t} \theta_{xx}(x,r)drds \\ &= \int_0^{+\infty} \mu_1(s)\eta_{xx}(x,t,s)ds. \end{aligned} \quad (16)$$

Similarly, we get

$$\int_0^{+\infty} g_2(s)\vartheta_{xx}(x,t-s)ds = \int_0^{+\infty} \mu_2(s)\tilde{\eta}_{xx}(x,t,s)ds. \tag{17}$$

Using (13)–(17), system (1)–(3) becomes

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + \delta_1 \theta_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - \delta_1 \theta + \delta_2 \vartheta_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x + \delta_3(t)|w_t|^{m(x)-2} w_t = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho_4 \theta_t + \beta\theta_{xx} - \beta_1 \int_0^{+\infty} \mu_1(s)\eta_{xx}(x,t,s)ds + \delta_1(u_{xt} + v_t) = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \eta_t + \eta_s - \theta = 0, & \text{in } (0,L)\times\mathbb{R}_+\times\mathbb{R}_+, \\ \rho_5 \vartheta_t + \tilde{\beta}\theta_{xx} - \beta_2 \int_0^{+\infty} \mu_2(s)\tilde{\eta}_{xx}(x,t,s)ds + \delta_2 v_{xt} = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \tilde{\eta}_t + \tilde{\eta}_s - \vartheta = 0, & \text{in } (0,L)\times\mathbb{R}_+\times\mathbb{R}_+, \end{cases} \tag{18}$$

with the boundary conditions

$$\begin{cases} u_x(0,t) = v_x(0,t) = w(0,t) = w_{xx}(0,t) = \theta(0,t) = \vartheta(0,t), & t \geq 0, \\ u(L,t) = v(L,t) = w(L,t) = w_{xx}(L,t) = \theta_x(L,t) = \vartheta_x(L,t) = 0, & t \geq 0, \\ \eta(0,t,s) = \eta_x(L,t,s) = \tilde{\eta}(0,t,s) = \tilde{\eta}_x(L,t,s) = 0, & s,t \in \mathbb{R}_+, \\ \eta(x,t,0) = \tilde{\eta}(x,t,0) = 0, & x \in (0,L), t \in \mathbb{R}_+, \end{cases} \tag{19}$$

and the initial data

$$\begin{cases} u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in (0,L), \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), w_t(x,0) = w_1(x), & x \in (0,L), \\ \theta(x,-t) = \theta_0(x,t), \vartheta(x,-t) = \vartheta_0(x,t), & x \in (0,L), t > 0, \\ \eta(x,0,s) = \eta_0(x,s), \tilde{\eta}(x,0,s) = \tilde{\eta}_0(x,s), & x \in (0,L), s > 0, \end{cases} \tag{20}$$

where $\beta = \beta_1 - 1$ and $\tilde{\beta} = \beta_2 - 1$.

3. Essential Lemmas

In this section, we establish essential lemmas necessary for proving the stability of the system (18)–(20).

The energy function associated with the solution $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \eta, \vartheta, \widetilde{\eta})$ of the aforementioned system is precisely defined as follows:

$$\begin{aligned} E(t) = {} & \frac{1}{2}\left[\rho_1 h_1\|u_t\|^2 + \rho_3 h_3\|v_t\|^2 + \rho h\|w_t\|^2 + E_1 h_1\|u_x\|^2 + E_3 h_3\|v_x\|^2 + EI\|w_{xx}\|^2\right] \\ & + \frac{1}{2}\left[k\|(-u + v + \alpha w_x)\|^2 + \rho_4\|\theta\|^2\right] \\ & + \frac{1}{2}\left[\beta_1\int_0^\infty \mu_1(s)|\eta_x|^2 ds + \rho_5\|\vartheta\|^2 + \beta_2\int_0^\infty \mu_2(s)\left|\widetilde{\eta}_x\right|^2 ds\right], \quad \forall\, t \ge 0. \end{aligned} \tag{21}$$

Lemma 1. *The energy functional (21) satisfies*

$$\begin{aligned} E'(t) = {} & -\beta\int_0^L \theta_x^2 dx - \widetilde{\beta}\int_0^L \vartheta_x^2 dx - \delta_3(t)\int_0^L |w_t|^{m(x)} dx \\ & + \frac{\beta_1}{2}\int_0^{+\infty} \mu_1'(s)\|\eta_x(s)\|^2 ds + \frac{\beta_2}{2}\int_0^{+\infty} \mu_2'(s)\left\|\widetilde{\eta}_x(s)\right\|^2 ds \le 0, \quad \forall t \ge 0. \end{aligned} \tag{22}$$

Proof. Multiplying the equations $(18)_1$, $(18)_2$, $(18)_3$, $(18)_4$, and $(18)_6$ by u_t, v_t, w_t, θ, and ϑ, respectively, followed by the multiplication of $(18)_5$ and $(18)_7$ by η and $\widetilde{\eta}$, respectively. Subsequently, utilizing integration by parts and incorporating the boundary conditions (19), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left[\rho_1 h_1\|u_t\|^2 + E_1 h_1\|u_x\|^2\right] - k\int_0^L (-u + v + \alpha w_x)u_t dx - \delta_1\int_0^L \theta u_{xt} dx = 0, \\ & \frac{1}{2}\frac{d}{dt}\left[\rho_3 h_3\|v_t\|^2 + E_3 h_3\|v_x\|^2\right] + k\int_0^L (-u + v + \alpha w_x)v_t dx - \delta_1\int_0^L \theta v_t dx - \delta_2\int_0^L \vartheta v_{xt} dx = 0, \\ & \frac{1}{2}\frac{d}{dt}\left[\rho h\|w_t\|^2 + EI\|w_{xx}\|^2\right] + \alpha k\int_0^L (-u + v + \alpha w_x)w_{xt} dx + \delta_3(t)\int_0^L |w_t|^{m(x)} dx = 0, \\ & \frac{1}{2}\frac{d}{dt}\left[\rho_4\|\theta\|^2\right] - \beta\int_0^L \theta_x^2 dx + \beta_1\int_0^{+\infty} \mu_1(s)\int_0^L \eta_x(s)\theta_x(t) dx ds + \delta_1\int_0^L \theta(u_{xt} + v_t) dx = 0, \\ & \frac{1}{2}\frac{d}{dt}\left[\beta_1\int_0^L\int_0^\infty \mu_1(s)|\eta_x|^2 ds dx\right] - \frac{\beta_1}{2}\int_0^{+\infty} \mu_1'(s)\|\eta_x(s)\|^2 ds - \beta_1\int_0^{+\infty} \mu_1(s)\int_0^L \eta_x(s)\theta_x(t) dx ds = 0, \\ & \frac{1}{2}\frac{d}{dt}\left[\rho_5\|\vartheta\|^2\right] - \widetilde{\beta}\int_0^L \vartheta_x^2 dx + \beta_2\int_0^{+\infty} \mu_2(s)\int_0^L \widetilde{\eta}_x(s)\vartheta_x(t) dx ds + \delta_2\int_0^L \vartheta v_{xt} dx = 0, \\ & \frac{1}{2}\frac{d}{dt}\left[\beta_2\int_0^L\int_0^\infty \mu_2(s)\left|\widetilde{\eta}_x\right|^2 ds dx\right] - \frac{\beta_2}{2}\int_0^{+\infty} \mu_2'(s)\left\|\widetilde{\eta}_x(s)\right\|^2 ds - \beta_1\int_0^{+\infty} \mu_2(s)\int_0^L \widetilde{\eta}_x(s)\vartheta_x(t) dx ds = 0. \end{aligned} \tag{23}$$

The summation of the aforementioned equations results in

$$E'(t) = -\beta\int_0^L \theta_x^2 dx - \widetilde{\beta}\int_0^L \vartheta_x^2 dx - \delta_3(t)\int_0^L |w_t|^{m(x)} dx + \frac{\beta_1}{2}\int_0^{+\infty} \mu_1'(s)\|\eta_x(s)\|^2 ds + \frac{\beta_2}{2}\int_0^{+\infty} \mu_2'(s)\left\|\widetilde{\eta}_x(s)\right\|^2 ds \le 0. \tag{24}$$

This completes the proof. □

Lemma 2 (see [34]). *Assume that $(A_1)-(A_3)$ hold. Then, for all $t\in\mathbb{R}^+$ and $i = 1, 2$, there exists a positive constant c_0 such that*

$$\int_t^{+\infty} \mu_i(s)\|\eta_x\|_2^2 ds \le c_0\int_t^{+\infty} \mu_i(s) ds. \tag{25}$$

Now, we present three lemmas without providing proofs; the methodology aligns with that employed in [35].

Lemma 3. *Given the assumptions (A_4) and (A_5), the subsequent approximations are as follows:*

$$\begin{aligned}
&c\delta_3(t)\int_0^L w_t^2 dx \le -cE'(t), \quad \text{if } m_1 = m_2 = 2.\\
&c\delta_3(t)\int_0^L w_t^2 dx \le c\varepsilon\delta_3 E(t) - C_\varepsilon E^{-\hat{\kappa}}(E'(t)), \quad \text{if } m_1 \ge 2, m_2 > 2.\\
&c\delta_3(t)\int_0^L w_t^2 dx \le c\varepsilon_1\delta_3 E + c\varepsilon_2\delta_3 E - C_{\varepsilon_1}(E'(t))E^{-\hat{\kappa}} - C_{\varepsilon_2}(E'(t))E^{-\hat{\kappa}}, \quad \text{if } 1 < m_1 < 2, m_2 \ne 2,
\end{aligned} \tag{26}$$

where $\hat{\kappa} = (m_2/2) - 1 > 0$.

Lemma 4. *Assuming that (A_4) and (A_5) hold, then for any $\lambda > 0$, we have*

$$-\rho h\delta_3(t)\int_0^L w|w_t|^{m(x)-2} w_t dx \le c\lambda\int_0^L |w_{xx}|^2 dx + \delta_3(t)\int_{\Omega*} C_\lambda(x)|w_t|^{2m(x)-2} dx + \delta_3(t)\int_{\Omega**} C_\lambda(x)|w_t|^{m(x)} dx, \tag{27}$$

where

$$\Omega_* = \{x\in[0,L]\colon m(v) < 2\}, \Omega_{**} = \{x\in[0,L]\colon m(x)\ge 2\}. \tag{28}$$

Lemma 5. *If the assumptions (A_1)–(A_5) hold, then we have the following estimates:*

$$c\delta_3(t)\int_{\Omega_*} |w_t|^{2m(x)-2} dx = \begin{cases} 0, & m_1 \ge 2,\\ c\varepsilon E(t) + c\delta_3(t)E^{(2m_1-2/2-m_1)}\int_{\Omega_*} C_\varepsilon|w_t|^{m(x)}, & 1 < m_1 < 2. \end{cases} \tag{29}$$

Lemma 6. *The functional Δ_1 defined by*

$$\Delta_1(t) = \rho_1 h_1 \int_0^L u u_t dx + \rho_3 h_3 \int_0^L v v_t dx + \rho h \int_0^L w w_t dx, \tag{30}$$

satisfies the estimate

$$\begin{aligned}\Delta_1'(t) \le & -\frac{E_1 h_1}{2}\|u_x\|^2 - \frac{E_3 h_3}{2}\|v_x\|^2 - EI\|w_{xx}\|^2 - k\|(-u + v + \alpha w_x)\|^2 \\ & + \rho_1 h_1 \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h \|w_t\|^2 + C\|\theta\|^2 + C\|\vartheta\|^2 \\ & - \rho h \delta_3(t) \int_0^L w |w_t|^{m(x)-2} w_t dx, \quad \forall\, t \ge 0.\end{aligned} \tag{31}$$

Proof. Differentiation of Δ_1 gives

$$\begin{aligned}\Delta_1'(t) = & \rho_1 h_1 \int_0^L u u_{tt} dx + \rho_3 h_3 \int_0^L v v_{tt} dx + \rho h \int_0^L w w_{tt} dx \\ & \cdot \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h \|w_t\|^2.\end{aligned} \tag{32}$$

Using equations $(18)_1$, $(18)_2$, and $(18)_3$, we obtain

$$\begin{aligned}\Delta_1'(t) = & \int_0^L u\left[E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - \delta_1 \theta_x\right] dx \\ & + \int_0^L v\left[E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) + \delta_1 \theta - \delta_2 \vartheta_x\right] dx \\ & + \int_0^L w\left[-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x - \delta_3(t)|w_t|^{m(x)-2} w_t\right] dx \\ & \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h \|w_t\|^2.\end{aligned} \tag{33}$$

Subsequently employing integration by parts across the interval $(0, L)$ and incorporating the boundary conditions (19) results in

$$\begin{aligned}\Delta_1'(t) = & -E_1 h_1 \|u_x\|^2 - E_3 h_3 \|v_x\|^2 - EI\|w_{xx}\|^2 - k\|(-u + v + \alpha w_x)\|^2 \\ & \cdot \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h \|w_t\|^2 \\ & - \rho h \delta_3(t) \int_0^L w |w_t|^{m(x)-2} w_t dx.\end{aligned} \tag{34}$$

Utilizing Young's and Poincaré's inequalities, we acquire

$$\begin{aligned}\Delta_1'(t) \leq & -\frac{E_1 h_1}{2}\|u_x\|^2 - \frac{E_3 h_3}{2}\|v_x\|^2 - EI\|w_{xx}\|^2 - k\|(-u+v+\alpha w_x)\|^2 \\ & \cdot \|u_t\|^2 + \rho_3 h_3\|v_t\|^2 + \rho h\|w_t\|^2 + C\|\theta\|^2 + C\|\vartheta\|^2 \\ & - \rho h \delta_3(t)\int_0^L w|w_t|^{m(x)-2} w_t dx.\end{aligned} \tag{35}$$

This completes the proof. □

Lemma 7. *The functional Δ_2 defined by*

$$\Delta_2(t) = -\rho_1 h_1 \rho_4 \int_0^L \theta \int_0^x u_t(y,t)dydx, \tag{36}$$

satisfies, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, the estimate

$$\begin{aligned}\Delta_2'(t)\|u_t\|^2 + \epsilon_1\|u_x\|^2 + \epsilon_2\|(-u+v+\alpha w_x)\|^2 \\ + C\|v_t\|^2 + C\int_0^L\int_0^\infty \mu_1(s)|\eta_x|^2 dsdx + C\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\|\theta\|^2, \quad \forall\, t \geq 0.\end{aligned} \tag{37}$$

Proof. Differentiation of Δ_2, using $(18)_1$ and $(18)_4$, we get

$$\begin{aligned}\Delta_2'(t) &= -\rho_1 h_1 \rho_4 \int_0^L \theta \int_0^x u_{tt} dx - \rho_1 h_1 \rho_4 \int_0^L \theta_t \int_0^x u_t dx \\ &= -\rho_4 \int_0^L \theta \int_0^x \left[E_1 h_1 u_{xx} + k(-u+v+\alpha w_x) - \delta_1 \theta_x\right] dx \\ &\quad + \rho_1 h_1 \int_0^L \left[(\beta_1 - 1)\theta_{xx} + \beta_1 \int_0^{+\infty} g_1(s)\theta_{xx}(x,t-s)ds - \delta_1(u_{xt} + v_t)\right] \int_0^x u_t dx.\end{aligned} \tag{38}$$

Now, through the application of integration by parts and considering the boundary conditions (19), we reach the following:

$$\begin{aligned}\Delta_2'(t) = & -\rho_1 h_1 \delta_1\|u_t\|^2 - \rho_4 E_1 h_1 \int_0^L \theta u_x dx + \rho_1 h_1 \delta_1 \int_0^L v_t \int_0^x u_t dx \\ & - \rho_4 k \int_0^L \theta \int_0^x (-u+v+\alpha w_x) dydx + \rho_3 \delta_1\|\theta\|^2 \\ & + \rho_1 h_1 \beta_1 \int_0^L u_t \int_0^{+\infty} \mu_1(s)\eta_x(.,t,s)dsdx.\end{aligned} \tag{39}$$

Employing Cauchy–Schwarz, Young's, and Poincaré's inequalities leads to

$$\Delta_2'(t) \leq -\rho_1 h_1 \delta_1 \|u_t\|^2 + \epsilon_1 \|u_x\|^2 + \frac{(\rho_4 E_1 h_1)^2}{4\epsilon_1}\|\theta\|^2 + \frac{3\rho_1 h_1 \delta_1}{4}\|v_t\|^2 \cdot \|u_t\|^2 + \epsilon_2 \|(-u + v + \alpha w_x)\|^2 + \frac{(\rho_4 k)^2}{4\epsilon_2}\|\theta\|^2 + \rho_3 \delta_1 \|\theta\|^2 + \frac{\rho_1 h_1 \delta_1}{4}\|u_t\|^2 + \frac{3\rho_1 h_1 \beta_1^2}{4\delta_1}\int_0^L \int_0^\infty \mu_1(s) |\eta_x|^2 ds dx. \tag{40}$$

Thus, we establish (37). □

Lemma 8. *The functional Δ_3 defined by*

$$\Delta_3(t) = -\rho_3 h_3 \rho_5 \int_0^L \vartheta \int_0^x v_t(y,t) dy, \tag{41}$$

satisfies, for any $\epsilon_3 > 0$ and $\epsilon_4 > 0$, the estimate

$$\Delta_3'(t)\|v_t\|^2 + \epsilon_3 \|v_x\|^2 + \epsilon_4 \|(-u + v + \alpha w_x)\|^2 + C\|\theta\|^2 + C\int_0^L \int_0^\infty \mu_2(s) |\tilde{\eta}_x|^2 ds dx + C\left(1 + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4}\right)\|\vartheta\|^2, \quad \forall\, t \geq 0. \tag{42}$$

Proof. Differentiation of Δ_3, using $(18)_2$ and $(18)_5$, we obtain

$$\begin{aligned}\Delta_3'(t) &= -\rho_3 h_3 \rho_5 \int_0^L \vartheta \int_0^x v_{tt} dy dx - \rho_3 h_3 \rho_5 \int_0^L \vartheta_t \int_0^x v_t dy dx \\ &= -\rho_5 \int_0^L \vartheta \int_0^x \left[E_3 h_3 v_{yy} - k(-u + v + \alpha w_y) + \delta_1 \theta - \delta_2 \vartheta_y\right] dy dx \\ &\quad - \rho_3 h_3 \int_0^L \left[(\beta_2 - 1)\theta_{xx} + \beta_2 \int_0^{+\infty} g_2(s) \vartheta_{xx}(x, t-s) ds + \delta_2 v_{xt}\right] \int_0^x v_t dy dx.\end{aligned} \tag{43}$$

Subsequently, through integration by parts and considering the boundary conditions (19), we arrive at

$$\begin{aligned}\Delta_3'(t) &= -\rho_3 h_3 \delta_2 \|v_t\|^2 - \rho_5 E_3 h_3 \int_0^L \vartheta v_x dx - \rho_5 \delta_1 \int_0^L \vartheta \int_0^x \theta dy dx + \rho_5 \delta_2 \|\vartheta\|^2 \\ &\quad + \rho_5 k \int_0^L \vartheta \int_0^x (-u + v + \alpha w_x) dy dx + \rho_3 h_3 \beta_2 \int_0^L v_t \int_0^{+\infty} \mu_2(s) \tilde{\eta}_x(., t, s) ds dx.\end{aligned} \tag{44}$$

Applying Cauchy–Schwarz' Young's, and Poincaré's inequalities, we have

$$\Delta_3'(t) \leq -\rho_3 h_3 \delta_2 \|v_t\|^2 + \epsilon_3 \|v_x\|^2 + \frac{(\rho_5 E_3 h_3)^2}{4\epsilon_3}\|\vartheta\|^2 + \frac{\rho_5 \delta_1}{2}\|\theta\|^2 \cdot \|\vartheta\|^2 + \epsilon_4 \|(-u + v + \alpha w_x)\|^2 + \frac{(\rho_5 k)^2}{4\epsilon_4}\|\vartheta\|^2 + \rho_5 \delta_2 \|\vartheta\|^2 + \frac{\rho_3 h_3 \delta_2}{4}\|v_t\|^2 + \frac{3\rho_3 h_3 \beta_2^2}{4\delta_2}\int_0^L \int_0^\infty \mu_2(s)\left|\tilde{\eta}_x\right|^2 ds dx. \quad (45)$$

Hence, we get (42). □

Lemma 9. *Assuming that* (A_1)–(A_5) *are satisfied, the function defined by*

$$L(t) := NE(t) + N_1 \Delta_1(t) + N_2 \Delta_2(t) + N_3 \Delta_3(t), t \geq 0, \quad (46)$$

satisfies for some $N, N, N_2, N_3 > 0$ *of the following equivalence:*

$$L(t) \sim E(t); \text{that is } \exists\, b_1, b_2 \colon b_1 E(t) \leq L(t) \leq b_2 E(t), \quad t \geq 0, \quad (47)$$

and satisfies the following estimate:

$$L'(t) \leq -cE(t) + c\int_0^L w_t^2 dx + \beta_1 \int_0^L \int_0^\infty \mu_1(s)\left|\eta_x\right|^2 ds dx + \beta_2 \int_0^L \int_0^\infty \mu_2(s)\left|\tilde{\eta}_x\right|^2 ds dx + c\delta_3(t)\int_{\Omega *} \left|w_t\right|^{2m(x)-2} dx + \delta_3(t)\int_{\Omega **} C_\lambda(x)\left|w_t\right|^{m(x)} dx, \quad t \geq 0. \quad (48)$$

Proof. Using Lemmas 6–8, and the estimate (27), we get

$$\begin{aligned} L'(t) \leq &-\left[\frac{\rho_1 h_1 \delta_1}{2}N_2 - \rho_1 h_1 N_1\right]\|u_t\|^2 + \rho h N_1 \|w_t\|^2 \\ &-\left[\frac{\rho_3 h_3 \delta_2}{2}N_3 - \rho_3 h_3 N_1 - CN_2\right]\|v_t\|^2 \\ &\cdot \|u_x\|^2 - \left[\frac{E_3 h_3}{2}N_1 - \epsilon_3 N_3\right]\|v_x\|^2 - (EIN_1 - c\lambda)\|w_{xx}\|^2 \\ &-\left[kN_1 - \epsilon_2 N_2 - \epsilon_4 N_3\right]\|(-u + v + \alpha w_x)\|^2 \\ &-\left[N - CN_1 - CN_2\left(1 + \frac{1}{\epsilon 1} + \frac{1}{\epsilon_2}\right) - CN_3\right]\|\theta_x\|^2 \\ &+ CN_2 \int_0^L \int_0^\infty \mu_1(s)\left|\eta_x\right|^2 ds dx + CN_3 \int_0^L \int_0^\infty \mu_2(s)\left|\tilde{\eta}_x\right|^2 ds dx \\ &- N\beta_1 \int_0^{+\infty} \mu_1'(s)\|\eta_x(s)\|^2 ds - N\beta_2 \int_0^{+\infty} \mu_2'(s)\left\|\tilde{\eta}_x(s)\right\|^2 ds \\ &-\left[N - CN_1 - CN_3\left(1 + \frac{1}{\epsilon_3} + \frac{1}{\epsilon 4}\right)\right]\|\vartheta_x\|^2 \\ &+ \delta_3(t)\int_{\Omega *} C_\lambda(x)\left|w_t\right|^{2m(x)-2} dx + \delta_3(t)\int_{\Omega **} C_\lambda(x)\left|w_t\right|^{m(x)} dx. \end{aligned} \quad (49)$$

By choosing

$$N_1 = 1, \lambda = \frac{EI}{2c}, \epsilon_1 = \frac{E_1 h_1}{4N_2}, \epsilon_2 = \frac{k}{4N_2}, \epsilon_3 = \frac{E_3 h_3}{4N_3}, \epsilon_4 = \frac{k}{4N_3}, \tag{50}$$

then (49) takes the form

$$\begin{aligned} L'(t) \leq & -\left[\frac{\rho_1 h_1 \delta_1}{4} N_2 - \rho_1 h_1\right] \|u_t\|^2 - \left[\frac{\rho_3 h_3 \delta_2}{4} N_3 - CN_2 - \rho_3 h_3\right] \|v_t\|^2 \\ & \cdot \|w_t\|^2 - \frac{E_1 h_1}{4} \|u_x\|^2 - \frac{E_3 h_3}{4} \|v_x\|^2 \\ & - \frac{EI}{2} \|w_{xx}\|^2 - \frac{k}{2} \|(-u + v + \alpha w_x)\|^2 \\ & - \left[N - CN_2\left(1 + \frac{4N_2}{E_1 h_1} + \frac{4N_2}{k}\right) - CN_3 - C\right] \|\theta_x\|^2 \\ & + CN_2 \int_0^L \int_0^\infty \mu_1(s) |\eta_x|^2 ds dx + CN_3 \int_0^L \int_0^\infty \mu_2(s) \left|\tilde{\eta}_x\right|^2 ds dx \\ & - \left[N - CN_3\left(1 + \frac{4N_3}{E_3 h_3} + \frac{4N_3}{k}\right) - C\right] \|\vartheta_x\|^2 \\ & + \delta_3(t) \int_{\Omega *} C_\lambda(x) |w_t|^{2m(x)-2} dx + \delta_3(t) \int_{\Omega **} C_\lambda(x) |w_t|^{m(x)} dx. \end{aligned} \tag{51}$$

Next, we determined the remaining parameters. Initially, we chose N_2 to be sufficiently large, ensuring

$$\frac{\rho_1 h_1 \delta_1}{4} N_2 - \rho_1 h_1 > 0. \tag{52}$$

Second, we chose N_3 to be large enough, such that

$$\frac{\rho_3 h_3 \delta_2}{4} N_3 - CN_2 - \rho_3 h_3 > 0. \tag{53}$$

Finally, we determined N to be large enough so that (47) remains valid and

$$\begin{aligned} & N - CN_2\left(1 + \frac{4N_2}{E_1 h_1} + \frac{4N_2}{k}\right) - CN_3 - C > 0, \\ & N - CN_3\left(1 + \frac{4N_3}{E_3 h_3} + \frac{4N_3}{k}\right) - C > 0. \end{aligned} \tag{54}$$

Thus, we obtain

$$\begin{aligned} L'(t) \leq & -\gamma_0\left[\|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \|u_x\|^2 + \|v_x\|^2 + \|w_{xx}\|^2\right] \\ & -\gamma_0\left[\|(-u + v + \alpha w_x)\|^2 + \|\theta_x\|^2 + \|\eta\|^2_{L^2_{\mu_1}} + \|\vartheta_x\|^2 + \|\tilde{\eta}\|^2_{L^2_{\mu_2}}\right] \\ & + c\|w_t\|^2 + \delta_3(t) \int_{\Omega *} C_\lambda(x) |w_t|^{2m(x)-2} dx \\ & + \delta_3(t) \int_{\Omega **} C_\lambda(x) |w_t|^{m(x)} dx, \end{aligned} \tag{55}$$

for some $\gamma_0 > 0$. Recalling (21), it follows from (55) that

$$L'(t) \leq -\gamma_1 E(t) + c\|w_t\|^2 + \delta_3(t)\int_{\Omega*} C_\lambda(x)|w_t|^{2m(x)-2}dx + \delta_3(t)\int_{\Omega**} C_\lambda(x)|w_t|^{m(x)}dx, \quad \forall\, t \geq 0, \tag{56}$$

for some $\gamma_1 > 0$. □

4. Stability Results

Theorem 9. *If the conditions* $(A1 - A5)$ *hold,* $m_1 = m_2 = 2$, *and* $\xi_2(t) = c$. *Then, there exist constants* $\gamma_0 \in (0, 1)$ *and* $\delta_1 > 0$ *such that, for all* $t \in \mathbb{R}^+$ *and for all* $\delta_0 \in (0, \gamma_0]$, *then the energy functional (21) satisfies*

$$E(t) \leq \delta_1\left(1 + \int_0^t (\mu_1(s))^{1-\delta_0}ds\right)e^{-\delta_0\int_0^t(\delta_3\xi_1)(s)ds} + \frac{c\zeta}{\delta_0}\int_t^{+\infty}\mu_1(s)ds, \tag{57}$$

where δ_1 *will be defined in the proof.*

Proof. To prove the energy decay (57), we multiply (48) by $\delta_3(t)$, yielding

$$\delta_3(t)L'(t) \leq -c\delta_3(t)E(t) + c\delta_3(t)\int_0^L w_t^2dx + c\delta_3(t)\int_0^L\int_0^\infty \mu_1(s)|\eta_x|^2dsdx + c\delta_3(t)\int_0^L\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2dsdx + c\delta_3(t)\int_{\Omega*}|w_t|^{2m(x)-2}dx + c\delta_3(t)\int_{\Omega**}|w_t|^{m(x)}dx. \tag{58}$$

By combining (58) with (21) and utilizing the estimate $c\delta_3(t)\int_0^L w_t^2dx$ in $(26)_1$, (58) can be expressed as follows:

$$\mathcal{L}'(t) \leq -c\delta_3(t)E(t) + c\int_0^t \mu_1(s)\|\eta_x\|^2ds + c\int_t^\infty \mu_1(s)\|\eta_x\|^2ds + c\int_0^\infty \mu_2(s)\|\tilde{\eta}_x\|^2ds, \tag{59}$$

where $\mathcal{L} = \delta_3 L + c\delta_3 E \sim E$. Using (21), (21) and the fact that ξ_1 and μ_1 are nonincreasing, we find that

$$c\xi_1(t)\int_0^t \mu_1(s)\|\eta_x\|^2ds \leq -c\int_0^t \mu_1'(s)\|\eta_x\|^2ds \leq -cE'(t), \quad \forall t \in \mathbb{R}^+. \tag{60}$$

Since $\xi_2 \equiv c$, we have

$$c\xi_2\int_0^\infty \mu_2(s)\|\tilde{\eta}_x\|^2ds \leq -c\int_0^\infty \xi_2\mu_2'(s)\|\tilde{\eta}_x\|^2ds \leq -cE'(t), \forall t \in \mathbb{R}^+. \tag{61}$$

Multiplying (59) by $\xi_1\xi_2$ and combining it with (60), (61), and the constraint provided in hypothesis (A_3) yields

$$\mathcal{F}'(t) \leq -c\xi_1(t)\xi_2(t)\delta_3(t)E(t) + c\zeta\xi_1(t)\int_t^\infty \mu_1(s)ds, \tag{62}$$

where $\mathcal{F} = \xi_1\xi_2\mathcal{L} + cE \sim E$.

Let $\gamma(t) = c\zeta\xi_1(t)\int_t^{+\infty}\mu_1(s)ds$. Then, (62) becomes

$$\mathcal{F}'(t) \leq -\gamma_0(\xi_1\xi_2\delta_3)\mathcal{F}(t) + \gamma(t), \tag{63}$$

for some $\gamma_0 > 0$. This last inequality remains true for any $\delta_0 \in (0, \gamma_0]$; that is,

$$\mathcal{F}'(t) \leq -\delta_0(\delta_3\xi_1\xi_2)(t)\mathcal{F}(t) + \gamma(t), \quad \forall t \in \mathbb{R}^+. \tag{64}$$

Therefore, direct integration leads to

$$\mathcal{F}(T) \leq e^{-\delta_0 \int_0^T (\delta_3 \xi_1 \xi_2)(s)ds} \left(\mathcal{F}(0) + \int_0^T e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \gamma(t)dt \right), \tag{65}$$

and the fact that $\mathcal{F} \sim E$ gives

$$E(T) \leq \gamma_1 e^{-\delta_0 \int_0^T (\delta_3 \xi_1 \xi_2)(s)ds} \left(\mathcal{F}(0) + \int_0^T e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \gamma(t)dt \right). \tag{66}$$

We note that

$$e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \gamma(t) = \frac{c\zeta}{\delta_0} \left(e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \right)' \int_t^{+\infty} \mu_1(s)ds, \quad \forall t \in \mathbb{R}^+. \tag{67}$$

Then, integration by parts gives

$$\begin{aligned} &\int_0^T e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \gamma(t)dt \\ &= \frac{c\zeta}{\delta_0} \left(e^{\delta_0 \int_0^T (\delta_3 \xi_1 \xi_2)(s)ds} \int_T^{+\infty} \mu_1(s)ds - \int_0^{+\infty} \mu_1(s)ds + \int_0^T e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} g(t)dt \right). \end{aligned} \tag{68}$$

Combining with (66), we have

$$E(T) \leq \gamma_1 \left(\mathcal{F}(0) + \frac{c\zeta}{\delta_0} \int_0^T e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \mu_1(t)dt \right) e^{-\delta_0 \int_0^T \delta_3 \xi_1 \xi_2(s)ds} + \frac{c\zeta}{\delta_0} \int_T^{+\infty} \mu_1(s)ds. \tag{69}$$

We note that

$$\left(e^{\int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \mu_1(t) \right)' \leq 0, \quad \forall t \in \mathbb{R}^+. \tag{70}$$

We have $e^{\int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \mu_1(t) \leq \mu_1(0)$ and

$$\int_0^T e^{\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} \mu_1(t)dt \leq (\mu_1(0))^{\delta_0} \int_0^T (\mu_1(t))^{1-\delta_0} dt. \tag{71}$$

Finally, combining (69) and (71), we obtain

$$E(t) \leq \delta_1 \left(1 + \int_0^t (\mu_1(s))^{1-\delta_0} ds \right) e^{-\delta_0 \int_0^t (\delta_3 \xi_1 \xi_2)(s)ds} + \frac{c\zeta}{\delta_0} \int_t^{+\infty} \mu_1(s)ds, \tag{72}$$

where $\delta_1 = \max\{\gamma_1 \mathcal{F}(0), (c\zeta/\delta_0)(\mu_1(0))^{\delta_0}\}$. Thus, the proof of $(57)_1$ is completed, and the proof of $(57)_2$ and $(57)_3$ will be similar to taking $\mu = \max\{\mu_1, \mu_2\}$ and $\xi = \max\{\xi_1, \xi_2\}$. □

Theorem 10. *If the conditions (A1 – A5) hold, $1 < m_1 < 2$, $m_2 \neq 2$, and $\xi_2(t) = c$. Then, the energy functional (21) satisfies for a positive constant C*

$$E(t) \leq C(1+t)^{(-1/\kappa)} (\xi_1\xi_2\delta_3)^{-(\kappa+1/\kappa)} \left[1 + \int_0^t (\xi_1\xi_2\delta_3)^{(\kappa+1/\kappa)}(s)\gamma^{\kappa+1}(s)(1+s)^{(1/\kappa)} ds\right], \tag{73}$$

where $\gamma(t) = c(\xi_1\xi_2\delta_3)(t)\int_t^{+\infty}\mu_1(s)ds$ *and* $\kappa = \max\{(2-m_1/2m_1-2), (m_2/2)-1\}$.

Proof. To prove (73), we first multiply (48) by $\delta_3(t)$, to get

$$\begin{aligned}\delta_3(t)L'(t) \leq & -c\delta_3(t)E(t) + c\delta_3(t)\int_0^L w_t^2 dx + c\delta_3(t)\int_0^L\int_0^\infty \mu_1(s)|\eta_x|^2 dsdx \\ & + c\delta_3(t)\int_0^L\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 dsdx + c\delta_3(t)\int_{\Omega_*}|w_t|^{2m(x)-2}dx - c\delta_3(t)E'(t).\end{aligned} \tag{74}$$

Combining (74), (29), and choosing ε small enough, we arrive at

$$\begin{aligned}L_1'(t) \leq & -c\delta_3(t)E(t) + c\delta_3(t)\int_0^L w_t^2 dx + c\delta_3(t)\int_0^L\int_0^\infty \mu_1(s)|\eta_x|^2 dsdx \\ & + c\delta_3(t)\int_0^L\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 dsdx - c\delta_3(t)E'(t)E^{2m_1-2/2-m_1},\end{aligned} \tag{75}$$

where $L_1 = \delta_3 L + c\delta_3 E \sim E$. Using the estimate of $c\delta_3(t)\int_0^L w_t^2 dx$ in $(26)_3$, we have, for $\varepsilon = \varepsilon_1 = \varepsilon_2$, the following equation:

$$\begin{aligned}L_1'(t) \leq & -c\delta_3(t)E(t) - C_\varepsilon E'(t)E^{-\kappa} - c\delta_3(t)E'(t)E^{2m_1-2/2-m_1} \\ & + c\delta_3(t)\int_0^t \mu_1(s)|\eta_x|^2 ds + c\zeta\delta_3(t)\int_t^\infty \mu_1(s)ds + c\delta_3(t)\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 ds.\end{aligned} \tag{76}$$

Multiplying (76) by $E^{\tilde{\kappa}}(t)$, where $\tilde{\kappa} = \max\{(2-m_1/2m_1-2), (m_2/2)-1\}$, we get

$$\begin{aligned}L_2'(t) \leq & -c\delta_3(t)E^{\tilde{\kappa}+1}(t) + c\varepsilon\delta_3(t)E^{\tilde{\kappa}+1}(t) - C_\varepsilon E'(t) \\ & + c\delta_3(t)E^{\tilde{\kappa}}(t)\int_0^t \mu_1(s)|\eta_x|^2 ds + c\zeta\delta_3(t)E^{\tilde{\kappa}}(t)\int_t^\infty \mu_1(s)ds \\ & + c\delta_3(t)E^{\tilde{\kappa}}(t)\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 ds,\end{aligned} \tag{77}$$

where $L_2 = E^{\widetilde{\kappa}} L_1 + c\delta_3 E \sim E$. Choosing ε small enough, then we get

$$L_2'(t) \leq -c\delta_3(t)E^{\widetilde{\kappa+1}}(t) + c\delta_3(t)E^{\widetilde{\kappa}}(t)\int_0^t \mu_1(s)|\eta_x|^2 ds + c\zeta\delta_3(t)\int_t^\infty \mu_1(s)ds + c\delta_3(t)\int_0^\infty \mu_2(s)\left|\widetilde{\eta}_x\right|^2 ds. \quad (78)$$

Multiplying (78) by $\delta_3^{\widetilde{\kappa}}(\xi_2\xi_1)^{\widetilde{\kappa+1}}$, using (21), and using that $\xi_1 E$ is nonincreasing, we get

$$\mathcal{F}'(t) \leq -c(\xi_1\xi_2\delta_3)^{\widetilde{\kappa+1}}(t)\mathcal{F}^{\widetilde{\kappa+1}}(t) + (\xi_1\xi_2\delta_3 E)^{\widetilde{\kappa}}(t)\gamma(t), \quad (79)$$

where $\gamma(t) = c(\xi_1\xi_2\delta_3)(t)\int_t^\infty \mu_1(s)ds$ and $\mathcal{F} = \delta_3^{\widetilde{\kappa}}(\xi_2 \xi_1)^{\widetilde{\kappa+1}} L_2 + c\delta_3 E \sim E$.

Use of Young's inequality, with $q = \widetilde{\kappa}+1$ and $q^* = (\widetilde{\kappa}+1/\widetilde{\kappa})$, gives for some positive constant c_1 and c_2.

$$\mathcal{F}'(t) \leq -c_1(\xi_1\xi_2\delta_3)^{\widetilde{\kappa+1}}(t)\mathcal{F}^{\widetilde{\kappa+1}}(t) + c_2\gamma^{\widetilde{\kappa+1}}(t). \quad (80)$$

Multiply both sides of (109) by $(\xi_1\xi_2\delta_3)^\eta$, $\eta > 1$, thus, we get

$$(\xi_1\xi_2\delta_3)^\eta \mathcal{F}'(t) \leq -c_1(\xi_1\xi_2\delta_3)^{\widetilde{\kappa+1+\eta}}(t)\mathcal{F}^{\widetilde{\kappa+1}}(t) + c_2(\xi_1\xi_2\delta_3)^\eta\gamma^{\widetilde{\kappa+1}}(t). \quad (81)$$

Let $H: = \xi_1\xi_2\delta_3 > 0$ which is nonincreasing, we find that

$$(\chi^\eta\mathcal{F}(t))' \leq -c_1\chi^{\widetilde{\kappa+1+\eta}}(t)\mathcal{F}^{\widetilde{\kappa+1}}(t) + c_2\chi^\eta\gamma^{\widetilde{\kappa+1}}(t). \quad (82)$$

Setting $\phi = \chi^\eta\mathcal{F}$ and noting $\eta = (\widetilde{\kappa}+1/\widetilde{\kappa})$, one finds that

$$\phi'(t) \leq -c_1\phi^{\widetilde{\kappa+1}}(t) + c_2\chi^\eta(t)\gamma^{\widetilde{\kappa+1}}(t). \quad (83)$$

Let

$$\gamma(t) := \phi(t) - \psi(t); \text{where } \psi(t) = c_2(1+t)^{(-1/\widetilde{\kappa})}\int_0^t \chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(s)(1+s)^{(1/\widetilde{\kappa})}ds. \quad (84)$$

From the definition of ψ, we have

$$c_2\chi^\eta(t)\gamma^{\widetilde{\kappa+1}}(t) = \psi'(t) + \frac{c_2}{\widetilde{\kappa}}(1+t)^{(-1/\widetilde{\kappa})-1}\int_0^t \chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(s)(1+s)^{(1/\widetilde{\kappa})}ds, \quad (85)$$

since $\chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(1+s)^{(1/\widetilde{\kappa})} > 0$, then we have for all $t \geq t_0 > 0$

$$\nu := \int_0^{t_0}\chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(s)(1+s)^{(1/\widetilde{\kappa})}ds \leq \int_0^t \chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(s)(1+s)^{(1/\widetilde{\kappa})}ds, \quad (86)$$

and then

$$\frac{\int_0^t \chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(s)(1+s)^{(1/\widetilde{\kappa})}ds}{\nu} \geq 1, \quad \forall t \geq t_0. \quad (87)$$

Thus, (85) yields, $\forall t \geq t_0$,

$$c_2\chi^\eta(t)\gamma^{\widetilde{\kappa+1}}(t) \leq \psi'(t) + \frac{1}{\widetilde{\kappa}}c_2^{\widetilde{\kappa}}\nu^{\widetilde{\kappa}}c_2^{\widetilde{\kappa+1}}\left[(1+t)^{(-1/\widetilde{\kappa})}\right]^{\widetilde{\kappa+1}}\left[\int_0^t \chi^\eta(s)\gamma^{\widetilde{\kappa+1}}(s)(1+s)^{(1/\widetilde{\kappa})}ds\right]^{\widetilde{\kappa+1}}, \quad (88)$$

we can choose c_2 large enough so that $(1/\widetilde{\kappa})c_2^{\widetilde{\kappa}}\nu^{\widetilde{\kappa}} \leq c_1$, and then, we get

$$c_2\chi^{\eta}(t)\gamma^{\widetilde{\kappa}+1}(t) \leq \psi'(t) + c_1\psi^{\widetilde{\kappa}+1}, \quad \forall t \geq t_0. \tag{89}$$

Now, using (89) and the definition of H, we get, $\forall t \geq t_0$,

$$\begin{aligned} H'(t) = \phi'(t) - \psi'(t) &\leq -c_1\phi^{\widetilde{\kappa}+1}(t) + c_2\chi^{\eta}(t)\gamma^{\widetilde{\kappa}+1}(t) - \psi'(t) \\ &\leq -c_1\left[(H+\psi)^{\widetilde{\kappa}+1}(t)\right] + c_2\chi^{\eta}(t)\gamma^{\widetilde{\kappa}+1}(t) - \psi'(t). \end{aligned} \tag{90}$$

Since $H(0) > 0$, then there exists $t_1 > 0$ such that $\gamma(t) > 0, \forall t \in [0, t_1)$. Hence,

$$\begin{aligned} H'(t) &\leq -c_1\left[H^{\widetilde{\kappa}+1}(t) + \psi^{\widetilde{\kappa}+1}(t)\right] + c_2\chi^{\eta}(t)H^{\widetilde{\kappa}+1}(t) - \psi'(t), \forall\, t \in [t_0, t_1). \\ &\leq -c_1\left[H^{\widetilde{\kappa}+1}(t) + \psi^{\widetilde{\kappa}+1}(t) - \frac{c_2}{c_1}\chi^{\eta}(t)\gamma^{\widetilde{\kappa}+1}(t) + \frac{1}{c_1}\psi'(t)\right]. \end{aligned} \tag{91}$$

Thus,

$$H'(t) \leq -c_1H^{\widetilde{\kappa}+1}(t), \forall\, t \in [t_0, t_1). \tag{92}$$

This gives

$$-\frac{dH}{H^{\widetilde{\kappa}+1}} \geq c_1 ds. \tag{93}$$

Integrate over (t_0, t), we have

$$\frac{\widetilde{\kappa}}{H^{\widetilde{\kappa}}(t)} \geq c_1 t + c_2\|_{t_0}^{t}. \tag{94}$$

Therefore, we get

$$H^{\widetilde{\kappa}}(t) \leq \frac{\widetilde{\kappa}}{c_1 t + c_2}\|_{t_0}^{t}. \tag{95}$$

Hence, it follows that

$$H(t) \leq \frac{c}{(t-t_0)^{(1/\widetilde{\kappa})}}, \quad \forall\, t \in [t_0, t_1). \tag{96}$$

If $t_1 = +\infty$, then, using the definitions of H and ψ, we observe that, for sufficiently large t,

$$\phi(t) \leq C(1+t)^{(-1/\widetilde{\kappa})}\left[1 + \int_0^t \chi^{\eta}(s)\gamma^{\widetilde{\kappa}+1}(s)(1+s)^{(1/\widetilde{\kappa})}ds\right]. \tag{97}$$

By multiplying (97) with $\chi^{-\eta}$ and recalling the definition of ϕ, we obtain the following for $\eta = (\widetilde{\kappa} + 1/\widetilde{\kappa})$:

$$\mathscr{F}(t) \leq C(1+t)^{\frac{-1}{\widetilde{\kappa}}}\chi^{-(\widetilde{\kappa}+1/\widetilde{\kappa})}\left[1 + \int_0^t \chi^{(\widetilde{\kappa}+1/\widetilde{\kappa})}(s)\gamma^{\widetilde{\kappa}+1}(s)(1+s)^{(1/\widetilde{\kappa})}ds\right]. \tag{98}$$

Using the fact $\mathscr{F} \sim E$, we have two cases:

If $\widetilde{\kappa} = (2 - m_1/2m_1 - 2) > 0$, then $\widetilde{\kappa} + 1 = (m_1/2m_1 - 2)$ and $(\widetilde{\kappa} + 1/\widetilde{\kappa}) = (m_1/2 - m_1)$, we get

$$E(t) \leq C(1+t)^{-(2m_1-2/2-m_1)}\chi^{-(m_1/2-m_1)}\left(1 + \int_0^t (1+s)^{(2m_1-2/2-m_1)}\chi^{(m_1/2-m_1)}(s)\gamma^{(1/m_1-1)}ds\right). \tag{99}$$

If $\tilde{\kappa} = (m_2/2) - 1 > 0$, we have

$$E(t) \le C(1+t)^{-(2/m_2-2)}\chi^{-(m_2/m_2-2)}\left(1 + \int_0^t (1+s)^{(2/m_2-2)}\chi^{(m_2/m_2-2)}(s)\gamma^{(m_2/2)}ds\right). \tag{100}$$

This establishes (73). □

Theorem 11. *If conditions $(A_1 - A_5)$ hold, $m_1 \ge 2$, $m_2 > 2$, and $\xi_2(t) = c$. Then, the energy functional (21) satisfies for a positive constant C,*

$$E(t) \le C(1+t)^{-(2/m_2-2)}(\xi_1\xi_2\delta_3)^{-(m_2/m_2-2)}\left(1 + \int_0^t (1+s)^{(2/m_2-2)}(\xi_1\xi_2\delta_3)^{(m_2/m_2-2)}(s)\gamma^{(m_2/2)}ds\right), \tag{101}$$

where $\gamma(t) = c(\xi_1\xi_2\delta_3)(t)\int_t^{+\infty}\mu_1(s)ds$.

Proof. To prove the decay (101), we first multiplying (48) by δ_3, recalling the estimate of $c\delta_3\int_0^L w_t^2 dx$ in $(26)_2$, to get

$$\begin{aligned}\delta_3 L'(t) \le& -c\delta_3 E(t) + c\varepsilon\delta_3 E(t) - C_\varepsilon(E'(t))E^{-\hat{\kappa}} + c\delta_3\int_0^t \mu_1(s)|\eta_x|_2^2 ds + c\delta_3\int_t^\infty g(s)|\Psi_x|^2 dx\\ &+ c\delta_3\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 dx - c\delta_3 E'(t).\end{aligned} \tag{102}$$

Multiplying (102) by $E^{\hat{\kappa}}$ where $\hat{\kappa} = (m_2/2) - 1 > 0$, we get

$$\begin{aligned}L_1'(t) \le& -c\delta_3 E^{\hat{\kappa}+1}(t) + c\varepsilon E^{\hat{\kappa}+1} - C_\varepsilon(-E'(t))\\ &+ c\delta_3 E^{\hat{\kappa}}\int_0^t \mu_1(s)|\eta_x|_2^2 ds + c\delta_3 E^{\hat{\kappa}}\int_t^\infty \mu_1(s)|\Psi_x|^2 dx + c\delta_3\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 dx,\end{aligned} \tag{103}$$

where $L_1 = \delta_3 L + c\delta_3 E \sim E$. Choosing ε sufficiently small, we get

$$\begin{aligned}L_1(t) \le& -c\delta_3 E^{\hat{\kappa}+1}(t) + c\delta_3 E^{\hat{\kappa}}\int_0^t \mu_1(s)|\eta_x|_2^2 ds + c\delta_3\int_t^\infty \mu_1(s)|\eta_x|^2 dx\\ &+ c\delta_3\int_0^\infty \mu_2(s)|\tilde{\eta}_x|^2 dx.\end{aligned} \tag{104}$$

Using (21), (21) and the fact that ξ_1 and μ_1 are nonincreasing, we find that

$$c\xi_1(t)\int_0^t \mu_1(s)\|\eta_x\|^2 ds \leq -c\int_0^t \mu_1'(s)\|\eta_x\|^2 ds \leq -cE'(t), \quad \forall t \in \mathbb{R}^+. \tag{105}$$

Since $\xi_2 \equiv c$, we have

$$c\xi_2\int_0^\infty \mu_2(s)\left\|\widetilde{\eta}_x\right\|^2 ds \leq -c\int_0^\infty \xi_2\mu_2'(s)\left\|\widetilde{\eta}_x\right\|^2 ds \leq -cE'(t), \quad \forall t \in \mathbb{R}^+. \tag{106}$$

Now, multiplying (104) by $(\delta_3)^{\hat{\kappa}}(\xi_1\xi_2)^{\hat{\kappa}+1}$ and using (105) and (106), we get

$$\mathscr{L}'(t) \leq -c(\xi_1\xi_2\delta_3)^{\hat{\kappa}+1}E^{\hat{\kappa}+1}(t) + c\zeta(\xi_1\xi_2\delta_3)^{\hat{\kappa}}\int_t^\infty \mu_1(s)|\eta_x|_2^2 ds, \tag{107}$$

where $\mathscr{L} = (\delta_3)^{\hat{\kappa}}(\xi_1\xi_2)^{\hat{\kappa}+1}L_1 + c\delta_3 E \sim E$.

Recalling (21) and (25) and putting $\gamma(t) = c(\xi_1\xi_2\delta_3)(t)\int_t^{+\infty}\mu_1(s)ds$. Then, (107) becomes

$$\mathscr{L}'(t) \leq -c(\xi_1\xi_2\delta_3)^{\hat{\kappa}+1}(t)E^{\hat{\kappa}+1}(t) + c(\xi_1\xi_2\delta_3)^{\hat{\kappa}}(t)\gamma(t). \tag{108}$$

Setting $\chi := \xi_1\xi_2\delta_3$ which is a positive nonincreasing, then we get

$$\mathscr{L}'(t) \leq -c\chi^{\hat{\kappa}+1}(t)\mathscr{L}^{\hat{\kappa}+1}(t) + c(\chi)^{\hat{\kappa}}\gamma(t). \tag{109}$$

Using Young's inequality, with $q = \hat{\kappa}+1$ and $q^* = (\hat{\kappa}+1/\hat{\kappa})$, we get some positive constant c_1 and c_2

$$\mathscr{L}'(t) \leq -c_1\chi^{\hat{\kappa}+1}(t)\mathscr{L}^{\hat{\kappa}+1}(t) + c_2\gamma^{\hat{\kappa}+1}(t). \tag{110}$$

Repeating the procedures outlined in the last part of the proof of Theorem (58), substituting $\widetilde{\kappa}$ with $\hat{\kappa} = (m_2 - 2/2)$, concludes the verification of the decay (101). □

5. Examples and Remarks

In this section, we provide examples and remarks to illustrate and compare our stability results from Theorems 9–11 with some earlier findings in the literature.

Example 1. For, $i = 1, 2$, let $g_i(s) = ae^{-\sqrt{s}}$. Therefore, $\xi_i(s) = (1/2\sqrt{s})$. Then, in case if $\widetilde{\kappa} = (2 - m_1/2m_1 - 2) > 0$, the energy decay (73) becomes for $\delta_3(t) = (1+t)^{-\lambda}$, $0 \leq \lambda \leq 1$,

$$E(t) \leq C(1+t)^{-(2m_1-2/2-m_1)}(1+t)^{(m_1/4-2m_1)-\lambda}\left(1 + \int_0^t (1+s)^{(5m_1-4/2(2-m_1))-\lambda}(\xi\beta)^{(m_1/2-m_1)}(s)h^{(m_1/2-m_1)}ds\right). \tag{111}$$

Thus, we achieve polynomial stability in the form of

$$E(t) \leq C(1+t)^{(4-3m_1/2(2-m_1))-\lambda}. \tag{112}$$

For $\lambda = 0$ and $m_1 > (4/3)$, it is evident that $\lim_{t\longrightarrow\infty} E(t) = 0$. In the case of $\lambda = 1$, for any $1 < m_1 < 2$, we observe $\lim_{t\longrightarrow\infty} E(t) = 0$.

In the second scenario, when $\widetilde{\kappa} = (m_2/2) - 1 > 0$, and considering $g_i(s) = ae^{-\sqrt{s}}$, $\xi_i(s) = (1/2\sqrt{s})$, and $\delta_3(t) = (1+t)^{-\lambda}$, we obtain polynomial stability in the form of

$$E(t) \leq C(1+t)^{(m_2-4/2(m_2-2))-\lambda}. \tag{113}$$

Notably, for $\lambda = 0$ and $m_1 < m_2 < 4$, we have $\lim_{t\longrightarrow\infty} E(t) = 0$. Similarly, for $\lambda = 1$ and any $1 < m_1 \leq m_2$, we find $\lim_{t\longrightarrow\infty} E(t) = 0$.

Remark 12

(1) If $\xi_1 = c$, then the decay (57) becomes

$$E(t) \leq \delta_1\left(1 + \int_0^t (\mu_2(s))^{1-\delta_0}ds\right)e^{-\delta_0\int_0^t(\delta_3\xi_2)(s)ds} + \frac{c\widetilde{\zeta}}{\delta_0}\int_t^{+\infty}\mu_2(s)ds. \tag{114}$$

(2) If ξ_1 and ξ_2 are functions of t, then the decay (57) becomes

$$E(t) \le \delta_1\left(1 + \int_0^t (\mu(s))^{1-\delta_0} ds\right) e^{-\delta_0 \int_0^t (\delta_3 \xi)(s) ds} + \frac{c\overline{\zeta}}{\delta_0}\int_t^{+\infty} \mu(s) ds, \tag{115}$$

where $\xi = \max\{\xi_1, \xi_2\}$, $\mu = \max\{\mu_1, \mu_2\}$, and $\overline{\zeta} = \max\{\zeta, \widetilde{\zeta}\}$.

(3) The proofs of (114) and (115) are similar to the proof of decay in (57).

Remark 13

(1) In the case where $\xi_1 = c$, the energy decay estimate mirrors the decay (73), with the only difference being $\gamma(t) = c(\xi_1\xi_2\delta_3)(t)\int_t^{+\infty} \mu_2(s) ds$. The proof remains the same.

(2) If ξ_1 and ξ_2 are functions of t, then the energy decay will be the same as the *decay* (73) except $\gamma(t) = c(\xi\delta_3)(t)\int_t^{+\infty} \mu(s) ds$ where $\xi = \max\{\xi_1, \xi_2\}$ and $\mu = \max\{\mu_1, \mu_2\}$ and the proof will be the same.

Remark 14

(1) If $\xi_1 = c$, the energy decay aligns with the decay (101), with the only difference being $\gamma(t) = c(\xi_1\xi_2\delta_3)(t)\int_t^{+\infty} \mu_2(s) ds$. The proof remains the same.

(2) If ξ_1 and ξ_2 are functions of t, then the energy decay will be the same as the decay (101) except $\gamma(t) = c(\xi\delta_3)(t)\int_t^{+\infty} \mu(s) ds$ where $\xi = \max\{\xi_1, \xi_2\}$ and $\mu = \max\{\mu_1, \mu_2\}$. The proof will be the same.

6. Use of AI Tools Declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data Availability

No data were used to support this study.

Ethical Approval

Ethics approval was not required for this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors read and approved the final version of the manuscript.

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