Hindawi
Journal of Mathematics
Volume 2024, Article ID 1997549, 16 pages
https://doi.org/10.1155/2024/1997549

Research Article

New Developments of Hermite–Hadamard Type Inequalities via *s*-Convexity and Fractional Integrals

Khuram Ali Khan[1], **Saeeda Fatima**[1], **Ammara Nosheen**[1] **and Rostin Matendo Mabela**[2]

[1] *Department of Mathematics, University of Sargodha, Sargodha, Pakistan*
[2] *Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Kinshasa, Democratic Republic of the Congo*

Correspondence should be addressed to Rostin Matendo Mabela; rostin.mabela@unikin.ac.cd

Received 3 August 2022; Revised 9 April 2023; Accepted 18 December 2023; Published 16 January 2024

Academic Editor: Kenan Yildirim

In this paper, we present an identity for differentiable functions that has played an important role in proving Hermite–Hadamard type inequalities for functions whose absolute values of first derivatives are s-convex functions. Meanwhile, some Hermite–Hadamard type inequalities for the functions whose absolute values of second derivatives are s-convex are also established with the help of an existing identity in literature. Many limiting results are deduced from the main results which are stated in remarks. Some applications of proved results are also discussed in the present study.

1. Introduction

Inequalities have been proved to be the most efficient tools for the construction of several branches in mathematics. In the field of classical differential and integral equations, the inequalities have played an important role [1, 2]. Charles Hermite and Jacques Hadamard derived Hermite–Hadamard inequality which is stated as follows.

1.1. Hermite–Hadamard Inequality. Let function $\phi: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $\omega, \nu \in I\,([0,\infty) = I)$ with $\omega < \nu$, then the following inequalities hold:

$$\phi\left(\frac{\omega+\nu}{2}\right) \leq \frac{1}{\nu-\omega}\int_{\omega}^{\nu} \phi(r)\mathrm{d}r \leq \frac{\phi(\omega)+\phi(\nu)}{2}. \tag{1}$$

If ϕ is concave, then equation (1) holds in back direction. Barsam et al. [3] introduced the integral identities associated with Hermite–Hadamard inequality for s-convex functions. Barsam and Sattarzadeh [4] found Hermite–Hadamard type inequalities involving fractional integrals for uniformly convex functions. Because of many applications of Hermite–Hadamard type inequalities [5–9] and fractional calculus [10–14], it is intended to study the Hermite–Hadamard type inequalities involving fractional integrals. Mohammed [15] obtained the inequalities via factional integrals of convex functions with respect to increasing functions. Set [16] defined new Ostrowski type inequalities via Riemann–Liouville fractional integrals for s-convex functions. Abdeljawad et al. [17] introduced new Simpson-type inequalities for (s,m)-convex functions. Işcan [18] introduced some inequalities for s-convex functions involving fractional integrals. Usta et al. [19] introduced trapezoid type inequalities for s-convex functions with generalized fractional operators. Butt et al. [20] obtained integral identity; by using that identity, new inequalities were obtained via a general form of fractional integral operators. Agarwal et al. [21] gave Hermite–Hadamard type inequalities for generalized k-fractional integrals. Sahoo et al. [22] obtained integral inequalities by using k-Riemann–Liouville fractional operator for h-convex functions. Sarikaya et al. [23] introduced the following Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals.

Theorem 1 (see [23]). *Let ϕ: $[\omega, \nu] \longrightarrow \mathbb{R}$ be a positive function with $0 \leq \omega \leq \nu$ and $\phi \in L[\omega, \nu]$. If ϕ is a convex function on $[\omega, \nu]$, then the following inequality for fractional integrals holds:*

$$\phi\left(\frac{\omega+\nu}{2}\right) \leq \frac{\Gamma(\theta+1)}{2(\nu-\omega)}\left[J_{\omega^+}^{\theta}\phi(\nu) + J_{\nu^-}^{\theta}\phi(\omega)\right] \leq \frac{\phi(\omega)+\phi(\nu)}{2}, \tag{2}$$

where $J_{\omega^+}^{\theta}\phi$ and $J_{\nu^-}^{\theta}\phi$ indicate the left-sided and right-sided Riemann–Liouville fractional integrals of the order $\theta \in \mathbb{R}_+ = [0, \infty)$ which are as follows:

$$\begin{aligned}\left(J_{\omega^+}^{\theta}\phi\right)(r) &= \frac{1}{\Gamma(\theta)}\int_{\omega}^{r}(r-\gamma)^{\theta-1}\phi(\gamma)\mathrm{d}\gamma; \quad 0 \leq \omega < r \leq \nu,\\ \left(J_{\nu^-}^{\theta}\phi\right)(r) &= \frac{1}{\Gamma(\theta)}\int_{r}^{\nu}(\gamma-r)^{\theta-1}\phi(\gamma)\mathrm{d}\gamma; \quad 0 \leq \omega < r \leq \nu,\end{aligned} \tag{3}$$

respectively, and $\Gamma(\cdot)$ is the classical Euler gamma function.s-convex functions are generalization of classical convex function. It is remarkable that Özdemir et al. [24] defined the s-convex function as follows:

s − convex function [24]. Let ϕ: $\mathbb{R} \longrightarrow \mathbb{R}$ be a function, then ϕ is called s-convex function, if

$$\phi(\gamma\omega + (1-\gamma)\nu) \leq \gamma^s\phi(\omega) + (1-\gamma)^s\phi(\nu), \tag{4}$$

for each $\omega, \nu \in \mathbb{R}$ and $\gamma \in (0,1)$, $s \in (0,1]$.

Hölder Inequality for Integrals [25]. Let $p > 1$ and $1/p + 1/p = 1$. If ϕ and ξ are real functions on $[\omega, \nu]$ and if $|\phi|^p$, $|\xi|^q$ are integrable functions on $[\omega, \nu]$, $q \geq 1$, then

$$\int_{\omega}^{\nu}|\phi(r)\xi(r)|\mathrm{d}r \leq \left(\int_{\omega}^{\nu}|\phi(r)|^p\mathrm{d}r\right)^{1/p}\left(\int_{\omega}^{\nu}|\xi(r)|^q\mathrm{d}r\right)^{1/q}. \tag{5}$$

The following power-mean integral inequality is an elementary result of Hölder inequality:

Power-Mean Integral Inequality [25]. Let $q \geq 1$. If ϕ and ξ are real functions defined on $[\omega, \nu]$ and if $|\phi|, |\phi\|\xi|^q$ are integrable functions on $[\omega, \nu]$, then

$$\int_{\omega}^{\nu}|\phi(r)\xi(r)|\mathrm{d}r \leq \left(\int_{\omega}^{\nu}|\phi(r)|\mathrm{d}r\right)^{1-1/q}\left(\int_{\omega}^{\nu}|\phi(r)\|\xi(r)|^q\mathrm{d}r\right)^{1/q}. \tag{6}$$

Some authors applied classical inequalities such as Hölder inequality and power mean inequality and also applied the special functions like classical Euler–gamma and beta functions to fractional integrals to get new integral inequalities for the different classes of convex functions. Qaisar et al. obtained some new Hermite–Hadamard inequalities involving fractional integrals for convex functions [14]. Some refinements for integral and sum forms of Hölder inequality were elaborated by Işcan [1]. Authors are motivated by the results given in [3, 26]. The purpose of this paper is to establish new Hermite–Hadamard type inequalities involving fractional integrals via s-convex functions.

2. Hermite–Hadamard Type Inequalities Involving Fractional Integrals for the Class of Differentiable Functions

To prove our main results associated with Hermite–Hadamard type inequalities involving fractional integrals, we need the following lemma.

Lemma 2. *Let ϕ: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $\omega, \nu \in I^o$ with $\omega < \nu$. If $\phi' \in L[\omega, \nu]$, then the following equality for fractional integral with $\theta > 0$ holds:*

$$\begin{aligned}\phi(r) &- \frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^{\theta}\phi(\omega)}{(r-\omega)^{\theta}} + \frac{J_{r^+}^{\theta}\phi(\nu)}{(\nu-r)^{\theta}}\right]\\ &= \frac{r-\omega}{2}\int_0^1\gamma^{\theta}\phi'(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma\\ &\quad + \frac{r-\nu}{2}\int_0^1\gamma^{\theta}\phi'(\gamma r + (1-\gamma)\nu)\mathrm{d}\gamma,\end{aligned} \tag{7}$$

for all $r \in (\omega, \nu)$.

Proof. Consider

$$\begin{aligned}\int_0^1\gamma^{\theta}\phi'(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma &= \frac{\gamma^{\theta}}{(r-\omega)}\phi(\gamma r + (1-\gamma)\omega)\Big|_0^1 - \int_0^1\frac{\theta\gamma^{\theta-1}}{(r-\omega)}\phi(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma\\ &= \frac{1}{(r-\omega)}\left[\phi(r) - \theta\int_0^1\gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma\right],\\ I_1 &= (r-\omega)\int_0^1\gamma^{\theta}\phi'(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma,\\ I_1 &= \phi(r) - \theta\int_0^1\gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma.\end{aligned} \tag{8}$$

Substituting $\ell = \gamma r + (1-\gamma)\omega$ in the above equation, we get

$$\begin{aligned} I_1 &= \phi(r) - \theta\int_{\omega}^{r}\left(\frac{\ell-\omega}{r-\omega}\right)^{\theta-1}\phi(\ell)\frac{d\ell}{r-\omega},\\ I_1 &= \phi(r) - \frac{\theta}{(r-\omega)^{\theta}}\int_{\omega}^{r}(\ell-\omega)^{\theta-1}\phi(\ell)d\ell,\\ I_1 &= \phi(r) - \frac{\Gamma(\theta+1)}{(r-\omega)^{\theta}}\left[\frac{1}{\Gamma(\theta)}\int_{\omega}^{r}(\ell-\omega)^{\theta-1}\phi(\ell)d\ell\right],\\ I_1 &= \phi(r) - \frac{\Gamma(\theta+1)}{(r-\omega)^{\theta}}J_{r^-}^{\theta}\phi(\omega). \end{aligned} \tag{9}$$

Now, consider

$$\begin{aligned} \int_0^1 \gamma^{\theta}\phi'(\gamma r + (1-\gamma)\nu)d\gamma &= \frac{\gamma^{\theta}}{(r-\nu)}\phi(\gamma r + (1-\gamma)\nu)\Big|_0^1 - \frac{\theta}{(r-\nu)}\int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\nu)d\gamma\\ &= \frac{1}{(r-\nu)}\left[\phi(r) - \theta\int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\nu)d\gamma\right],\\ I_2 &= (r-\nu)\int_0^1 \gamma^{\theta}\phi'(\gamma r + (1-\gamma)\nu)d\gamma,\\ I_2 &= \phi(r) - \theta\int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\nu)d\gamma. \end{aligned} \tag{10}$$

Substituting $\mathcal{J} = \gamma r + (1-\gamma)\nu$ in the above equation, we get

$$\begin{aligned} I_2 &= \phi(r) - \theta\int_{\nu}^{r}\left(\frac{\nu-\mathcal{J}}{\nu-r}\right)^{\theta-1}\phi(\mathcal{J})\frac{d\mathcal{J}}{r-\nu},\\ I_2 &= \phi(r) - \frac{\Gamma(\theta+1)}{(\nu-r)^{\theta}}\left[\frac{1}{\Gamma(\theta)}\int_{r}^{\nu}(\nu-\mathcal{J})^{\theta-1}\phi(\mathcal{J})d\mathcal{J}\right],\\ I_2 &= \phi(r) - \frac{\Gamma(\theta+1)}{(\nu-r)^{\theta}}J_{r^+}^{\theta}\phi(\nu). \end{aligned} \tag{11}$$

Adding equations (9) and (11), we get

$$\begin{aligned} &(r-\omega)\int_0^1 \gamma^{\theta}\phi'(\gamma r + (1-\gamma)\omega)d\gamma\\ &\quad + (r-\nu)\int_0^1 \gamma^{\theta}\phi'(\gamma r + (1-\gamma)\nu)d\gamma\\ &= \phi(r) - \frac{\Gamma(\theta+1)}{(r-\omega)^{\theta}}J_{r^-}^{\theta}\phi(\omega) + \phi(r) - \frac{\Gamma(\theta+1)}{\nu-r}J_{r^+}^{\theta}\phi(\nu). \end{aligned} \tag{12}$$

The proof is completed. □

Remark 3. By replacing r with ν in equation (9) and r with ω in equation (11) and adding the resulting equations, we obtain the following equation:

$$\begin{aligned} &\frac{\phi(\omega)+\phi(\nu)}{2} - \frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J_{\nu^-}^{\theta}\phi(\omega) + J_{\omega^+}^{\theta}\phi(\nu)\right]\\ &= \frac{\nu-\omega}{2}\int_0^1 \gamma^{\theta}\phi'(\gamma\nu + (1-\gamma)\omega)d\gamma\\ &\quad + \frac{\omega-\nu}{2}\int_0^1 \gamma^{\theta}\phi'(\gamma\omega + (1-\gamma)\nu)d\gamma. \end{aligned} \tag{13}$$

Substituting $\gamma = 1-\ell$ in the second term of R. H. S of the equation (13), then equation (13) becomes ([26], Lemma 1.2).

The following two examples show that the class of functions whose absolute values are differentiable s-convex functions is nonempty.

Example 1. Let $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\phi(r) = r^4$, in this case, the function $|\phi'(r)| = \phi'(r)$ is a s–convex function for $0 < s \leq 1$. Because if for each $\omega, \nu \in \mathbb{R}$, we put

$$\xi(\gamma) = 4(\gamma\omega + (1-\gamma)\nu))^3 - 4\gamma^s\omega^3 - 4(1-\gamma)^s\nu^3. \tag{14}$$

It is easy to see that $\xi''(\gamma) \geq 0$ for $0 \leq \gamma \leq 1$ and $\xi(1) = \xi(0) = 0$, so $\xi(\gamma) \leq 0$, therefore

$$4(\gamma\omega + (1-\gamma)\nu)^3 \leq 4\gamma^s\omega^3 + 4(1-\gamma)^s\nu^3, \tag{15}$$

and that means

$$\phi'(\gamma\omega + (1-\gamma)\nu) \le \gamma^s \phi'(\omega) + (1-\gamma)^s \phi'(\nu). \tag{16}$$

Example 2. Let ϕ: $(0,\infty] \longrightarrow \mathbb{R}$ be defined by $\phi(r) = r - r\ln r$, $0 < s \le 1$, and $0 < \omega < \nu \le 1$, then the function $|\phi'(r)| = \phi'(r) = -\ln r$ is a s-convex function on the interval $[\omega, \nu]$. Because if we put

$$\begin{aligned}\xi(\gamma) &= \ln(\gamma\omega + (1-\gamma)\nu) - \gamma^s \ln(\omega) - (1-\gamma)^s \ln(\nu),\\ \xi''(\gamma) &= -\left(\frac{\omega-\nu}{\gamma\omega+(1-\gamma)\nu}\right) - s(s-1)\gamma^{s-2}\ln(\omega)\\ &\quad - s(s-1)(1-\gamma)^{s-2}\ln(\nu) \le 0,\end{aligned} \tag{17}$$

so $\xi(t) \ge 0$, therefore

$$\begin{aligned}&\ln(\gamma\omega + (1-\gamma)\nu) \ge \gamma^s \ln(\omega) + (1-\gamma)^s \ln(\nu),\\ &\phi'(\gamma\omega + (1-\gamma)\nu) \le \gamma^s \phi'(\omega) + (1-\gamma)^s \phi'(\nu).\end{aligned} \tag{18}$$

Theorem 4. *Let $\theta \ge 1$ and ϕ: $[\omega, \nu] \longrightarrow \mathbb{R}$ be a positive function with $0 \le \omega < r < \nu$ and $\phi \in L[\omega, \nu]$. If ϕ is a s-convex function on $[\omega, \nu]$, then the following inequality for fractional integrals holds:*

$$\Gamma(\theta+1)\left[\frac{J_{r^-}^{\theta}\phi(\omega)}{(r-\omega)^{\theta}} + \frac{J_{r^+}^{\theta}\phi(\nu)}{(\nu-r)^{\theta}}\right] \le \theta[[\phi(\nu)+\phi(\omega)]\beta(\theta, s+1) + 2\phi(r)\beta(\theta+s, 1)]. \tag{19}$$

Proof. Applying the s-convexity of ϕ, we get

$$\begin{aligned}&\phi(\gamma r + (1-\gamma)\omega) + \phi(\gamma r + (1-\gamma)\nu)\\ &\quad \le \gamma^s \phi(r) + (1-\gamma)^s \phi(\omega) + \gamma^s \phi(r) + (1-\gamma)^s \phi(\nu).\end{aligned} \tag{20}$$

Multiply both sides of equation (20) by $\gamma^{\theta-1}$ and integrate w.r.t γ over [0, 1].

$$\begin{aligned}\int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma + \int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\nu)\mathrm{d}\gamma \le \phi(r)\int_0^1 \gamma^{\theta+s-1}\mathrm{d}\gamma\\ + \phi(\omega)\int_0^1 \gamma^{\theta-1}(1-\gamma)^s \mathrm{d}\gamma\\ + \phi(r)\int_0^1 \gamma^{\theta+s-1}\mathrm{d}\gamma\\ + \phi(\nu)\int_0^1 \gamma^{\theta-1}(1-\gamma)^s \mathrm{d}\gamma,\end{aligned} \tag{21}$$

$$\begin{aligned}\int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\omega)\mathrm{d}\gamma + \int_0^1 \gamma^{\theta-1}\phi(\gamma r + (1-\gamma)\nu)\mathrm{d}\gamma \le \phi(r)\frac{1}{\theta+s} + \phi(\omega)\beta(\theta, s+1)\\ + \phi(r)\frac{1}{\theta+s} + \phi(\nu)\beta(\theta, s+1).\end{aligned}$$

Now, substituting $\ell = \gamma r + (1-\gamma)\omega$ in the first term of L. H. S of equation (21) and $\mathcal{J} = \gamma r + (1-\gamma)\nu$ in the second term of L. H. S of equation (21), we get

$$\begin{aligned}&\int_\omega^r \left(\frac{\ell-\omega}{r-\omega}\right)^{\theta-1}\phi(\ell)\frac{\mathrm{d}\ell}{r-\omega} + \int_\nu^r \left(\frac{\nu-\mathcal{J}}{\nu-r}\right)^{\theta-1}\phi(\mathcal{J})\frac{d\mathcal{J}}{r-\nu}\\ &\le \frac{2}{\theta+s}\phi(r) + [\phi(\omega)+\phi(\nu)]\beta(\theta, s+1).\end{aligned} \tag{22}$$

Multiplying both sides of the above equation by θ, we get

$$\begin{aligned}&\frac{\theta}{(r-\omega)^{\theta}}\int_\omega^r (\ell-\omega)^{\theta-1}\phi(\ell)\mathrm{d}\ell + \frac{\theta}{(\nu-r)^{\theta}}\int_r^\nu (\nu-\mathcal{J})^{\theta-1}\phi(\mathcal{J})\mathrm{d}\mathcal{J}\\ &\le \theta\left[\frac{2}{\theta+s}\phi(r) + [\phi(\omega)+\phi(\nu)]\beta(\theta, s+1)\right].\end{aligned} \tag{23}$$

The proof is completed. □

Theorem 5. *Let ϕ: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $\omega, \nu \in I^o$ with $\omega < r < \nu$ such that $\phi' \in L[\omega, \nu]$.*

If $|\phi'|$ is s-convex on $[\omega, \nu]$, then the following inequality for fractional integral holds:

$$\left|\phi(r) - \frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^{\theta}\phi(\omega)}{(r-\omega)^{\theta}} + \frac{J_{r^+}^{\theta}\phi(\nu)}{(\nu-r)^{\theta}}\right]\right| \leq \frac{r-\omega}{2}\left[\left|\phi'(r)\right|\beta(\theta+s+1,1) + \left|\phi'(\omega)\right|\beta(\theta+1,s+1)\right] + \frac{r-\nu}{2}\left[\left|\phi'(r)\right|\beta(\theta+s+1,1) + \left|\phi'(\nu)\right|\beta(\theta+1,s+1)\right], \tag{24}$$

for some fixed $s \in (0, 1]$.

Proof. Using Lemma 2,

$$\left|\phi(r) - \frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^{\theta}\phi(\omega)}{(r-\omega)^{\theta}} + \frac{J_{r^+}^{\theta}\phi(\nu)}{(\nu-r)^{\theta}}\right]\right| \leq \frac{r-\omega}{2}\int_0^1 \gamma^{\theta}\left|\phi'(\gamma r + (1-\gamma)\omega\right|\mathrm{d}\gamma + \frac{r-\nu}{2}\int_0^1 \gamma^{\theta}\left|\phi'(\gamma r + (1-\gamma)\nu\right|\mathrm{d}\gamma, \tag{25}$$

(since $|\phi'|$ is s-convex)

$$\begin{aligned}
&\leq \frac{r-\omega}{2}\left[\left|\phi'(r)\right|\int_0^1 \gamma^{\theta+s}\mathrm{d}\gamma + \left|\phi'(\omega)\right|\int_0^1 \gamma^{\theta}(1-\gamma)^s\mathrm{d}\gamma\right] \\
&\quad + \frac{r-\nu}{2}\left[\left|\phi'(r)\right|\int_0^1 \gamma^{\theta+s}\mathrm{d}\gamma + \left|\phi'(\nu)\right|\int_0^1 \gamma^{\theta}(1-\gamma)^s\mathrm{d}\gamma\right] \\
&\leq \frac{r-\omega}{2}\left[\left|\phi'(r)\right|\beta(\theta+s+1,1) + \left|\phi'(\omega)\right|\beta(\theta+1,s+1)\right] \\
&\quad + \frac{r-\nu}{2}\left[\left|\phi'(r)\right|\beta(\theta+s+1,1) + \left|\phi'(\nu)\right|\beta(\theta+1,s+1)\right].
\end{aligned} \tag{26}$$

The proof is completed. □

Theorem 6. *Let $\phi: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $\omega, \nu \in I^o$ with $\omega < r < \nu$ such that $\phi' \in L[\omega, \nu]$. If $|\phi'|^q\,(q > 1)$ is s-convex on $[\omega, \nu]$, then the following inequality for fractional integral holds:*

$$\begin{aligned}
&\left|\phi(r) - \frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^{\theta}\phi(\omega)}{(r-\omega)^{\theta}} + \frac{J_{r^+}^{\theta}\phi(\nu)}{(\nu-r)^{\theta}}\right]\right| \\
&\leq \left(\frac{1}{\theta+1}\right)^{1-1/q}\left[\frac{r-\omega}{2}\left[\left|\phi'(r)\right|^q\beta(\theta+s+1,1)\right.\right. \\
&\quad \left. + \left|\phi'(\omega)\right|^q\beta(\theta+1,s+1)\right]^{1/q} \\
&\quad + \frac{r-\nu}{2}\left[\left|\phi'(r)\right|^q\beta(\theta+s+1,1)\right. \\
&\quad \left. + \left|\phi'(\nu)\right|^q\beta(\theta+1,s+1)\right]^{1/q},
\end{aligned} \tag{27}$$

for some fixed $s \in (0, 1]$.

Proof. According to Lemma 2,

$$\begin{aligned}
\mathfrak{I} &= \left|\phi(r) - \frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^{\theta}\phi(\omega)}{(r-\omega)^{\theta}} + \frac{J_{r^+}^{\theta}\phi(\nu)}{(\nu-r)^{\theta}}\right]\right| \leq \frac{r-\omega}{2}\int_0^1 \gamma^{\theta}\left|\phi'(\gamma r + (1-\gamma)\omega\right|\mathrm{d}\gamma \\
&\quad + \frac{r-\nu}{2}\int_0^1 \gamma^{\theta}\left|\phi'(\gamma r + (1-\gamma)\nu\right|\mathrm{d}\gamma, \\
\mathfrak{I} &\leq \frac{r-\omega}{2}\left[\int_0^1 \left(\gamma^{\theta}\right)^{1-1/q}\gamma^{\theta/q}\left|\phi'(\gamma r + (1-\gamma)\omega)\right|\mathrm{d}\gamma\right] \\
&\quad + \frac{r-\nu}{2}\left[\int_0^1 \left(\gamma^{\theta}\right)^{1-1/q}\gamma^{\theta/q}\left|\phi'(\gamma r + (1-\gamma)\nu)\right|\mathrm{d}\gamma\right].
\end{aligned} \tag{28}$$

Applying the Hölder inequality (5) in equation (28), we get

$$\begin{aligned}\mathfrak{I} \leq & \frac{r-\omega}{2}\left(\int_0^1 \gamma^\theta \mathrm{d}\gamma\right)^{1-1/q}\left[\int_0^1 \gamma^\theta\left|\phi'(\gamma r+(1-\gamma)\omega)\right|^q \mathrm{d}\gamma\right]^{1/q} \\ & +\frac{r-\nu}{2}\left(\int_0^1 \gamma^\theta \mathrm{d}\gamma\right)^{1-1/q}\left[\int_0^1 \gamma^\theta\left|\phi'(\gamma r+(1-\gamma)\nu)\right|^q \mathrm{d}\gamma\right]^{1/q},\end{aligned} \tag{29}$$

(since $|\phi'|^q$ is s-convex)

$$\begin{aligned}\mathfrak{I} \leq & \frac{r-\omega}{2}\left(\frac{1}{\theta+1}\right)^{1-1/q}\left[\left|\phi'(r)\right|^q \int_0^1 \gamma^{\theta+s} \mathrm{d}\gamma+\left|\phi'(\omega)\right|^q \int_0^1 \gamma^\theta(1-\gamma)^s \mathrm{d}\gamma\right]^{1/q} \\ & +\frac{r-\nu}{2}\left(\frac{1}{\theta+1}\right)^{1-1/q}\left[\left|\phi'(r)\right|^q \int_0^1 \gamma^{\theta+s} \mathrm{d}\gamma+\left|\phi'(\nu)\right|^q \int_0^1 \gamma^\theta(1-\gamma)^s \mathrm{d}\gamma\right]^{1/q}, \\ \mathfrak{I} \leq & \left(\frac{1}{\theta+1}\right)^{1-1/q}\left[\frac{r-\omega}{2}\left(\left|\phi'(r)\right|^q \frac{1}{\theta+s+1}+\left|\phi'(\omega)\right|^q \beta(\theta+1, s+1)\right)^{1/q}\right. \\ & \left.+\frac{r-\nu}{2}\left(\left|\phi'(r)\right|^q \frac{1}{\theta+s+1}+\left|\phi'(\nu)\right|^q \beta(\theta+1, s+1)\right)^{1/q}\right].\end{aligned} \tag{30}$$

The proof is completed. □

Remark 7. By applying the power-mean integral inequality (6) in equation (28), then we also get the inequality (27).

Theorem 8. *Let $\phi: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $\omega, \nu \in I^o$ with $\omega < r < \nu$ such that $\phi' \in L[\omega, \nu]$. If $|\phi'|^q$ is s-convex on $[\omega, \nu]$, then the following inequality for fractional integral holds:*

$$\begin{aligned}\left|\phi(r)-\frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^\theta \phi(\omega)}{(r-\omega)^\theta}+\frac{J_{r^+}^\theta \phi(\nu)}{(\nu-r)^\theta}\right]\right| \leq & \left(\frac{1}{\theta p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\left[\frac{r-\omega}{2}\left[\left|\phi'(r)\right|^q+\left|\phi'(\omega)\right|^q\right]^{1/q}\right. \\ & \left.+\frac{r-\nu}{2}\left[\left|\phi'(r)\right|^q+\left|\phi'(\nu)\right|^q\right]^{1/q}\right],\end{aligned} \tag{31}$$

for some fixed $s \in (0,1]$.

Proof. According to Lemma 2,

$$\begin{aligned}\mathfrak{I}= & \left|\phi(r)-\frac{\Gamma(\theta+1)}{2}\left[\frac{J_{r^-}^\theta \phi(\omega)}{(r-\omega)^\theta}+\frac{J_{r^+}^\theta \phi(\nu)}{(\nu-r)^\theta}\right]\right| \leq \frac{r-\omega}{2} \int_0^1 \gamma^\theta\left|\phi'(\gamma r+(1-\gamma)\omega\right| \mathrm{d}\gamma \\ & +\frac{r-\nu}{2} \int_0^1 \gamma^\theta\left|\phi'(\gamma r+(1-\gamma)\nu\right| \mathrm{d}\gamma.\end{aligned} \tag{32}$$

According to Hölder inequality (5), we get

$$\begin{aligned}\mathfrak{J}\leq&\frac{r-\omega}{2}\left(\int_0^1\gamma^{\theta p}\mathrm{d}\gamma\right)^{1/p}\left[\int_0^1\left|\phi'(\gamma r+(1-\gamma)\omega)\right|^q\mathrm{d}\gamma\right]^{1/q}\\&+\frac{r-\nu}{2}\left(\int_0^1\gamma^{\theta p}\mathrm{d}\gamma\right)^{1/p}\left[\int_0^1\left|\phi'(\gamma r+(1-\gamma)\nu)\right|^q\mathrm{d}\gamma\right]^{1/q},\end{aligned}\tag{33}$$

(since $|\phi'|^q$ is s-convex)

$$\begin{aligned}\mathfrak{J}\leq&\frac{r-\omega}{2}\left(\frac{1}{\theta p+1}\right)^{1/p}\left[\left|\phi'(r)\right|^q\int_0^1\gamma^s\mathrm{d}\gamma+\left|\phi'(\omega)\right|^q\int_0^1(1-\gamma)^s\mathrm{d}\gamma\right]^{1/q}\\&+\frac{r-\nu}{2}\left(\frac{1}{\theta p+1}\right)^{1/p}\left[\left|\phi'(r)\right|^q\int_0^1\gamma^s\mathrm{d}\gamma+\left|\phi'(\nu)\right|^q\int_0^1(1-\gamma)^s\mathrm{d}\gamma\right]^{1/q},\\\mathfrak{J}\leq&\frac{r-\omega}{2}\left(\frac{1}{\theta p+1}\right)^{1/p}\left[\left|\phi'(r)\right|^q\frac{1}{s+1}+\left|\phi'(\omega)\right|^q\frac{1}{s+1}\right]^{1/q}\\&+\frac{r-\nu}{2}\left(\frac{1}{\theta p+1}\right)^{1/p}\left[\left|\phi'(r)\right|^q\frac{1}{s+1}+\left|\phi'(\nu)\right|^q\frac{1}{s+1}\right]^{1/q}.\end{aligned}\tag{34}$$

The proof is completed. □

3. Hermite–Hadamard Type Inequalities Involving Fractional Integrals for the Class of Twice Differentiable Functions

Dragomir et al. [27] defined the following identity involving Riemann–Liouville fractional integrals.

Lemma 9 (see [27]). *Let ϕ: $I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a twice differentiable function on I^o. Assume that $\omega,\nu\in I^o$ with $\omega<\nu$ and $\phi''\in L[\omega,\nu]$, then the following identity for fractional integral with $\theta>0$ holds:*

$$\begin{aligned}&\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{\Gamma(\theta+1)}{2(\nu-\omega)^\theta}\left[J_{\omega^+}^\theta\phi(\nu)+J_{\nu^-}^\theta\phi(\omega)\right]\\&=\frac{(\nu-\omega)^2}{2(\theta+1)}\int_0^1\gamma\left(1-\gamma^\theta\right)\left[\phi''(\gamma\omega+(1-\gamma)\nu)+\phi''((1-\gamma)\omega+\gamma\nu)\right]\mathrm{d}\gamma.\end{aligned}\tag{35}$$

To prove our results associated with Hermite–Hadamard inequalities involving fractional integrals for twice differentiable functions, we need Lemma 9.

Theorem 10. *Let ϕ: $I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a twice differentiable function on I^o such that $|\phi''|$ is s-convex function on I. Suppose that $\omega,\nu\in I^o$ with $\omega<\nu$, $\phi''\in L[\omega,\nu]$, and $\theta\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{\Gamma(\theta+1)}{2(\nu-\omega)^\theta}\left[J_{\omega^+}^\theta\phi(\nu)+J_{\nu^-}^\theta\phi(\omega)\right]\right|\\&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left[\left(\left|\phi''(\omega)\right|+\left|\phi''(\nu)\right|\right)\left[\beta(s+2,\theta+1)+\beta(2,\theta+s+1)\right]\right].\end{aligned}\tag{36}$$

Proof. According to Lemma 9, we get

$$\begin{aligned}\mathfrak{I} &= \left|\frac{\phi(\omega)+\phi(\nu)}{2} - \frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J_{\omega^+}^{\theta}\phi(\nu) + J_{\nu^-}^{\theta}\phi(\omega)\right]\right| \\ &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\int_0^1 \gamma\left(1-\gamma^{\theta}\right)\left[\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right| + \left|\phi''((1-\gamma)\omega+\gamma\nu)\right|\right]\mathrm{d}\gamma, \\ \mathfrak{I} &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\int_0^1 \gamma\left(1-\gamma^{\theta}\right)\left[\phi''(\gamma\omega+(1-\gamma)\nu)\right]\mathrm{d}\gamma + \int_0^1 \gamma\left(1-\gamma^{\theta}\right)\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|\right]\mathrm{d}\gamma\right].\end{aligned} \tag{37}$$

Applying the s-convexity of $|\phi''|$, we get

$$\begin{aligned}\mathfrak{I} &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\int_0^1 \gamma\left(1-\gamma^{\theta}\right)\left[\gamma^s\left|\phi''(\omega)\right| + (1-\gamma)^s\left|\phi''(\nu)\right|\right]\mathrm{d}\gamma\right. \\ &\quad \left. + \int_0^1 \gamma\left(1-\gamma^{\theta}\right)\left[(1-\gamma)^s\left|\phi''(\omega)\right| + \gamma^s\left|\phi''(\nu)\right|\right]\mathrm{d}\gamma\right], \\ \mathfrak{I} &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\left|\phi''(\omega)\right|\int_0^1 \gamma^{s+1}\left(1-\gamma^{\theta}\right)\mathrm{d}\gamma + \left|\phi''(\nu)\right|\int_0^1 \gamma\left(1-\gamma^{\theta}\right)(1-\gamma)^s\mathrm{d}\gamma\right. \\ &\quad \left. + \left|\phi''(\omega)\right|\int_0^1 \gamma\left(1-\gamma^{\theta}\right)(1-\gamma)^s\mathrm{d}\gamma + \left|\phi''(\nu)\right|\int_0^1 \gamma^{s+1}\left(1-\gamma^{\theta}\right)\mathrm{d}\gamma\right].\end{aligned} \tag{38}$$

Since $\gamma^{\theta} \geq \gamma$, $\theta \in (0,1]$, and $\gamma \in [0,1]$, we have $-\gamma^{\theta} \leq \gamma \Rightarrow 1-\gamma^{\theta} \leq 1-\gamma \leq (1-\gamma)^{\theta}$.

$$\begin{aligned}\mathfrak{I} &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\left|\phi''(\omega)\right|\int_0^1 \gamma^{s+1}(1-\gamma)^{\theta}\mathrm{d}\gamma + \left|\phi''(\nu)\right|\int_0^1 \gamma(1-\gamma)^{\theta+s}\mathrm{d}\gamma\right. \\ &\quad \left. + \left|\phi''(\omega)\right|\int_0^1 \gamma(1-\gamma)^{\theta+s}\mathrm{d}\gamma + \left|\phi''(\nu)\right|\int_0^1 \gamma^{s+1}(1-\gamma)^{\theta}\mathrm{d}\gamma\right], \\ \mathfrak{I} &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\left|\phi''(\omega)\right|\beta(s+2,\theta+1) + \left|\phi''(\nu)\right|\beta(2,\theta+s+1)\right. \\ &\quad \left. + \left|\phi''(\omega)\right|\beta(2,\theta+s+1) + \left|\phi''(\nu)\right|\beta(s+2,\theta+1)\right].\end{aligned} \tag{39}$$

The proof is completed. □

Theorem 11. *Let $\phi\colon I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable function on I^o. Assume that $\theta \in (0,1]$ and $q > 1$ such that $|\phi''|^q$ is s-convex function on I. Suppose that $\omega, \nu \in I^o$ with $\omega < \nu$ and $\phi'' \in L[\omega,\nu]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\phi(\omega)+\phi(\nu)}{2} - \frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J_{\omega^+}^{\theta}\phi(\nu) + J_{\nu^-}^{\theta}\phi(\omega)\right]\right| \\ &\quad \leq \frac{(\nu-\omega)^2}{(\theta+1)}\left[\left(\frac{\left|\phi''(\omega)\right|^q + \left|\phi''(\nu)\right|^q}{s+1}\right)^{1/q}\beta^{1/p}(p+1,\theta p+1)\right].\end{aligned} \tag{40}$$

Proof. According to Lemma 9, we get

$$\begin{aligned}\mathfrak{I} &= \left|\frac{\phi(\omega)+\phi(\nu)}{2} - \frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J_{\omega^+}^{\theta}\phi(\nu) + J_{\nu^-}^{\theta}\phi(\omega)\right]\right| \\ &\le \frac{(\nu-\omega)^2}{2(\theta+1)}\int_0^1 \gamma\left(1-\gamma^{\theta}\right)\left[\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right| + \left|\phi''((1-\gamma)\omega+\gamma\nu)\right|\right]\mathrm{d}\gamma.\end{aligned} \tag{41}$$

According to Hölder inequality (5),

$$\begin{aligned}\mathfrak{I} \le \frac{(\nu-\omega)^2}{2(\theta+1)}&\left[\left(\int_0^1 \gamma^p\left(1-\gamma^{\theta}\right)^p\mathrm{d}\gamma\right)^{1/p}\left(\int_0^1\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|^q\mathrm{d}\gamma\right)^{1/q}\right. \\ &\left.+\left(\int_0^1 \gamma^p\left(1-\gamma^{\theta}\right)^p\mathrm{d}\gamma\right)^{1/p}\left(\int_0^1 \phi''|(1-\gamma)\omega+\gamma\nu|^q\mathrm{d}\gamma\right)^{1/q}\right].\end{aligned} \tag{42}$$

Since $\gamma^{\theta}\ge\gamma$, $\theta\in(0,1]$ and $\gamma\in[0,1]$, we have $-\gamma^{\theta}\le\gamma\Rightarrow 1-\gamma^{\theta}\le 1-\gamma\le(1-\gamma)^{\theta}$.

$$\begin{aligned}\mathfrak{I} \le \frac{(\nu-\omega)^2}{2(\theta+1)}\left(\int_0^1 \gamma^p(1-\gamma)^{\theta p}\mathrm{d}\gamma\right)^{1/p}&\left[\left(\int_0^1\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|^q\mathrm{d}\gamma\right)^{1/q}\right. \\ &\left.+\left(\int_0^1\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|^q\mathrm{d}\gamma\right)^{1/q}\right],\end{aligned} \tag{43}$$

(since $|\phi''|^q$ is s-convex)

$$\begin{aligned}\int_0^1\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|^q\mathrm{d}\gamma &\le \left|\phi''(\omega)\right|^q\int_0^1\gamma^s\mathrm{d}\gamma + \left|\phi''(\nu)\right|^q\int_0^1(1-\gamma)^s\mathrm{d}\gamma \\ &= \frac{\left|\phi''(\omega)^q + \left|\phi''(\nu)\right|^q\right.}{s+1},\end{aligned} \tag{44}$$

$$\begin{aligned}\int_0^1\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|^q\mathrm{d}\gamma &\le \left|\phi''(\omega)\right|^q\int_0^1(1-\gamma)^s\mathrm{d}\gamma + \left|\phi''(\nu)\right|^q\int_0^1\gamma^s\mathrm{d}\gamma \\ &= \frac{\left|\phi''(\omega)^q + \left|\phi''(\nu)\right|^q\right.}{s+1},\end{aligned} \tag{45}$$

$$\beta(p+1,\theta p+1) = \int_0^1\gamma^p(1-\gamma)^{\theta p}\mathrm{d}\gamma. \tag{46}$$

Substituting equations (44)–(46) in equation (43), we get

$$\mathfrak{I} \leq \frac{(\nu-\omega)^2}{2(\theta+1)}[\beta(p+1,\theta p+1)]^{1/p}\left[\left(\frac{|\phi''(\omega)|^q+|\phi''(\nu)|^q}{s+1}\right)^{1/q}+\left(\frac{|\phi''(\omega)|^q+|\phi''(\nu)|^q}{s+1}\right)^{1/q}\right]. \tag{47}$$

The proof is completed. □

Remark 12. For $\theta = 1$, inequality (40) becomes as follows:

$$\begin{aligned}&\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{1}{\nu-\omega}\int_\omega^\nu \phi(\gamma)\mathrm{d}\gamma\right|\\ &\leq \frac{(\nu-\omega)^2}{(2)}\left[\left(\frac{|\phi''(\omega)|^q+|\phi''(\nu)|^q}{s+1}\right)^{1/q}\beta^{1/p}(p+1,p+1)\right].\end{aligned} \tag{48}$$

For $\alpha = 1$, Corollary 3.6 in [5] reduces to inequality (48) and, for $\alpha = m = 1$, Corollary 3.5 in [5] reduces to inequality (48).

3.1. Comparison. Here, we have compared Theorem 11 with Corollary 5.8 in [19].

For $\theta = 1$, Theorem 11 becomes as follows:

$$\begin{aligned}&\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{1}{\nu-\omega}\int_\omega^\nu \phi(\gamma)\mathrm{d}\gamma\right|\\ &\leq \frac{(\nu-\omega)^2}{(2)}\left[\left(\frac{|\phi''(\omega)|^q+|\phi''(\nu)|^q}{s+1}\right)^{1/q}\beta^{1/p}(p+1,p+1)\right].\end{aligned} \tag{49}$$

Let $\phi(\gamma) = 1/\gamma$, $p = q = 2, s = 0.3$, and $\omega = 1, \nu = 3$.

$$\begin{aligned}&\left|\frac{1/\omega+1/\nu}{2}-\frac{1}{\nu-\omega}\int_\omega^\nu \frac{1}{\gamma}\mathrm{d}\gamma\right|\\ &\leq \frac{(\nu-\omega)^2}{(2)}\left[\left(\frac{|2/\omega^3|^2+|1/\nu^3|^2}{s+1}\right)^{1/2}\beta^{1/2}(3,3)\right],\\ &s = 0.3,\\ &\left|\frac{2}{3}-\frac{1.0986}{2}\right| \leq 2\left(\frac{4.005487}{1.3}\right)^{1/2}\left(\frac{1}{30}\right)^{1/2},\\ &0.117367 \leq 1.281875.\end{aligned} \tag{50}$$

Similarly, Corollary 5.8 in [19] for $\theta = 1$ becomes as follows:

$$\begin{aligned}&\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{1}{\nu-\omega}\int_\omega^\nu \phi(\gamma)\mathrm{d}\gamma\right|\\ &\leq \frac{(\nu-\omega)^2}{(4)}\left(\frac{|\phi''(\omega)|^q+|\phi''(\nu)|^q}{s+1}\right)^{1/q}\left(1-\frac{2}{2p+1}\right)^{1/p}.\end{aligned} \tag{51}$$

Let $\phi(\gamma) = 1/\gamma$, $p = q = 2, s = 0.3$, and $\omega = 1, \nu = 3$.

$$\begin{aligned}&\left|\frac{1/\omega+1/\nu}{2}-\frac{1}{\nu-\omega}\int_\omega^\nu \frac{1}{\gamma}\mathrm{d}\gamma\right|\\ &\leq \frac{(\nu-\omega)^2}{(4)}\left(\frac{|2/\omega^3|^2+|2/\nu^3|^2}{s+1}\right)^{1/2}\left(1-\frac{2}{5}\right)^{1/2},\\ &\left|\frac{2}{3}-\frac{1.09861}{2}\right| \leq \left(\frac{3}{5}\right)^{1/2}\left(\frac{4.27216}{1.3}\right)^{1/2},\\ &0.117361 \leq 1.404197.\end{aligned} \tag{52}$$

The difference of bounds of Theorem 11 is 1.164508 and difference of bounds of Corollary 5.8 in [19] is 1.28683. Hence, our result is more efficient.

Theorem 13. *Let ϕ: $I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I^o. Assume that $\theta \in (0,1]$ and $q \geq 1$ such that $|\phi''|^q$ is s-convex function on I. Suppose that $\omega, \nu \in I^o$, with $\omega < \nu$ and $\phi'' \in L[\omega, \nu]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J_{\omega^+}^{\theta}\phi(\nu)+J_{\nu^-}^{\theta}\phi(\omega)\right]\right|\\
&\leq\frac{\theta(\nu-\omega)^2}{4(\theta+1)(\theta+2)}\left(\frac{2(\theta+2)}{\theta}\right)^{1/q}\left(\left|\phi''(\omega)\right|^q\beta(s+2,\theta+1)\right.\\
&\quad\left.+\left|\phi''(\nu)\right|^q\beta(2,\theta+s+1)\right]^{1/q}+\left[\left|\phi''(\omega)\right|^q\beta(2,\theta+s+1)+\left|\phi''(\nu)\right|^q\beta(s+2,\theta+1)\right]^{1/q}\Big).
\end{aligned} \tag{53}$$

Proof. According to Lemma 9, we get

$$\begin{aligned}
\mathfrak{I}&=\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J_{\omega^+}^{\theta}\phi(\nu)+J_{\nu^-}^{\theta}\phi(\omega)\right]\right|\\
&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left[\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|d\gamma+\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|d\gamma\right].
\end{aligned} \tag{54}$$

Applying the power-mean integral inequality (6), we get

$$\begin{aligned}
\mathfrak{I}&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left[\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)d\gamma\right)^{1-1/q}\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|^q d\gamma\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)d\gamma\right)^{1-1/q}\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|^q d\gamma\right)^{1/q}\right],\\
\mathfrak{I}&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)d\gamma\right)^{1-1/q}\left[\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|^q d\gamma\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|^q d\gamma\right)^{1/q}\right].
\end{aligned} \tag{55}$$

Simplifying

$$\int_0^1\gamma\left(1-\gamma^{\theta}\right)d\gamma=\int_0^1\left(\gamma-\gamma^{\theta+1}\right)d\gamma=\frac{\theta}{2(\theta+2)}, \tag{56}$$

(since $|\phi''|^q$ is s-convex)

$$\begin{aligned}
\mathfrak{I}&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left(\frac{\theta}{2(\theta+2)}\right)^{1-1/q}\left[\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left(\gamma^s\left|\phi''(\omega)\right|^q+(1-\gamma)^s\left|\phi''(\nu)\right|^q\right)d\gamma\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\gamma\left(1-\gamma^{\theta}\right)\left((1-\gamma)^s\left|\phi''(\omega)\right|^q+\gamma^s\left|\phi''(\nu)\right|^q\right)d\gamma\right)^{1/q}\right].
\end{aligned} \tag{57}$$

Since $\gamma^{\theta} \geq \gamma$, $\theta \in (0,1]$, and $\gamma \in [0,1]$, we have $-\gamma^{\theta} \leq \gamma \Rightarrow 1-\gamma^{\theta} \leq 1-\gamma \leq (1-\gamma)^{\theta}$.

$$\begin{aligned}
\mathfrak{I} &\leq \frac{\theta(\nu-\omega)^2}{4(\theta+1)(\theta+2)}\left(\frac{2(\theta+2)}{\theta}\right)^{1/q}\left[\left(|\phi''(\omega)|^q\int_0^1\gamma^{s+1}(1-\gamma)^{\theta}d\gamma+|\phi''(\nu)|^q\int_0^1\gamma(1-\gamma)^{\theta+s}\mathrm{d}\gamma\right)^{1/q}\right.\\
&\quad\left.+\left(|\phi''(\omega)|^q\int_0^1\gamma(1-\gamma)^{\theta+s}d\gamma+|\phi''(\nu)|^q\int_0^1\gamma^{s+1}(1-\gamma)^{\theta}\mathrm{d}\gamma\right)^{1/q}\right],\\
\mathfrak{I} &\leq \frac{\theta(\nu-\omega)^2}{4(\theta+1)(\theta+2)}\left(\frac{2(\theta+2)}{\theta}\right)^{1/q}\left[\left(|\phi''(\omega)|^q\beta(s+2,\theta+1)+|\phi''(\nu)|^q\beta(2,\theta+s+1)\right)^{1/q}\right.\\
&\quad\left.+\left(|\phi''(\omega)|^q\beta(2,\theta+s+1)+|\phi''(\nu)|^q\beta(s+2,\theta+1)\right)^{1/q}\right].
\end{aligned}\tag{58}$$

The proof is completed. □

Theorem 14. *Suppose that $\phi\colon I\subset\mathbb{R}\longrightarrow\mathbb{R}$ is a differentiable on I^o such that $\phi''\in L[\omega,\nu]$, where $\omega,\nu\in I^o$ with $\omega<\nu$. Assume that $|\phi''|^q$ is s-convex on $[\omega,\nu]$ with $q>1$ and $\theta\in(0,1]$, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J^{\theta}_{\omega^+}\phi(\nu)+J^{\theta}_{\nu^-}\phi(\omega)\right]\right|\\
&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left(\frac{1}{p+1}\right)^{1/p}\left[\left(|\phi''(\omega)|^q\beta(s+1,q\theta+1)+|\phi''(\nu)|^q\beta(1,q\theta+s+1)\right)^{1/q}\right.\\
&\quad\left.+\left(|\phi''(\omega)|^q\beta(1,q\theta+s+1)+|\phi''(\nu)|^q\beta(s+1,q\theta+1)\right)^{1/q}\right],
\end{aligned}\tag{59}$$

with $1/p+1/p=1$.

Proof. According to Lemma 9, we get

$$\begin{aligned}
\mathfrak{I}&=\left|\frac{\phi(\omega)+\phi(\nu)}{2}-\frac{\Gamma(\theta+1)}{2(\nu-\omega)^{\theta}}\left[J^{\theta}_{\omega^+}\phi(\nu)+J^{\theta}_{\nu^-}\phi(\omega)\right]\right|\\
&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left[\int_0^1\gamma\left(1-\gamma^{\theta}\right)|\phi''(\gamma\omega+(1-\gamma)\nu)|\mathrm{d}\gamma+\int_0^1\gamma\left(1-\gamma^{\theta}\right)|\phi''((1-\gamma)\omega+\gamma\nu)|\mathrm{d}\gamma\right].
\end{aligned}\tag{60}$$

Applying the Hölder inequality (5), we get

$$\begin{aligned}
\mathfrak{I}&\leq\frac{(\nu-\omega)^2}{2(\theta+1)}\left[\left(\int_0^1\gamma^p\mathrm{d}\gamma\right)^{1/p}\left(\int_0^1\left(1-\gamma^{\theta}\right)^q|\phi''(\gamma\omega+(1-\gamma)\nu)|^q\mathrm{d}\gamma\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\gamma^p\mathrm{d}\gamma\right)^{1/p}\left(\int_0^1\left(1-\gamma^{\theta}\right)^q|\phi''((1-\gamma)\omega+\gamma\nu)|^q\mathrm{d}\gamma\right)^{1/q}\right],
\end{aligned}\tag{61}$$

(since $|\phi''|^q$ is s-convex)

$$\mathfrak{I} \leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left(\int_0^1 \gamma^p \mathrm{d}t\right)^{1/p}\left[\left(|\phi''(\omega)|^q \int_0^1 \gamma^s\left(1-\gamma^\theta\right)^q d\gamma + |\phi''(\nu)|^q \int_0^1 \left(1-\gamma^\theta\right)^q (1-\gamma)^s \mathrm{d}\gamma\right)^{1/q} + \left(|\phi''(\omega)|^q \int_0^1 \left(1-\gamma^\theta\right)^q (1-\gamma)^s d\gamma + |\phi''(\nu)|^q \int_0^1 \left(1-\gamma^\theta\right)^q \gamma^s \mathrm{d}\gamma\right)^{1/q}\right]. \tag{62}$$

Since $\gamma^\theta \geq \gamma$, $\theta \in (0,1]$, and $\gamma \in [0,1]$, we have $-\gamma^\theta \leq \gamma \Rightarrow 1-\gamma^\theta \leq 1-\gamma \leq (1-\gamma)^\theta$.

$$\mathfrak{I} \leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left(\frac{1}{p+1}\right)^{1/p}\left(|\phi''(\omega)|^q \int_0^1 \gamma^s (1-\gamma)^{\theta q} \mathrm{d}\gamma + |\phi''(\nu)|^q \int_0^1 (1-\gamma)^{\theta q+s} \mathrm{d}\gamma\right)^{1/q} + \left(|\phi''(\omega)|^q \int_0^1 (1-\gamma)^{\theta q+s} \mathrm{d}\gamma + |\phi''(\nu)|^q \int_0^1 (1-\gamma)^{\theta q}\gamma^s \mathrm{d}\gamma\right)^{1/q}\Bigg]. \tag{63}$$

The proof is completed. □

Theorem 15. *Assume that $\phi\colon I \subseteq [0,\infty) \longrightarrow \mathbb{R}$ is a differentiable on I^o such that $\phi'' \in L[\omega,\nu]$, where $\omega,\nu \in I^o$ with $\omega < \nu$. Assume that $|\phi''|^q$ is s-convex on $[\omega,\nu]$ with $q \geq 1$ and $\theta \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\phi(\omega)+\phi(\nu)}{2} - \frac{\Gamma(\theta+1)}{2(\nu-\omega)^\theta}\left[J_{\omega^+}^\theta \phi(\nu) + J_{\nu^-}^\theta \phi(\omega)\right]\right| \\ &\leq \frac{(\nu-\omega)^2}{4(\theta+1)}\left[\left(2\left(|\phi''(\omega)|^q \beta(s+2,\theta q+1) + |\phi''(\nu)|^q \beta(2,\theta q+s+1)\right)\right)^{1/q}\right. \\ &\quad \left.+\left(2\left(|\phi''(\omega)|^q \beta(2,\theta q+s+1) + |\phi''(\nu)|^q \beta(s+2,\theta q+1)\right)\right)^{1/q}\right],\end{aligned} \tag{64}$$

with $1/p + 1/q = 1$.

Proof. According to Lemma 9, we get

$$\begin{aligned}\mathfrak{I} &= \left|\frac{\phi(\omega)+\phi(\nu)}{2} - \frac{\Gamma(\theta+1)}{2(\nu-\omega)^\theta}\left[J_{\omega^+}^\theta \phi(\nu) + J_{\nu^-}^\theta \phi(\omega)\right]\right| \\ &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\int_0^1 \gamma\left(1-\gamma^\theta\right)|\phi''(\gamma\omega+(1-\gamma)\nu)|\mathrm{d}\gamma + \int_0^1 \gamma\left(1-\gamma^\theta\right)|\phi''((1-\gamma)\omega+\gamma\nu)|\mathrm{d}\gamma\right], \\ \mathfrak{I} &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left[\int_0^1 \gamma^{1-1/q}\gamma^{1/q}\left(1-\gamma^\theta\right)\left[|\phi''(\gamma\omega+(1-\gamma)\nu)|\right]\mathrm{d}\gamma\right. \\ &\quad \left.+\int_0^1 \gamma^{1-1/q}\gamma^{1/q}\left(1-\gamma^\theta\right)|\phi''((1-\gamma)\omega+\gamma\nu)|\right]\mathrm{d}\gamma\Bigg].\end{aligned} \tag{65}$$

Applying the Hölder inequality (5), we get

$$\begin{aligned}\Im \leq \frac{(\nu-\omega)^2}{2(\theta+1)}\Bigg[&\left(\int_0^1 \gamma \mathrm{d}\gamma\right)^{1-1/q}\left(\int_0^1 \gamma\left(1-\gamma^\theta\right)^q\left|\phi''(\gamma\omega+(1-\gamma)\nu)\right|^q\mathrm{d}\gamma\right)^{1/q}\\&+\left(\int_0^1 \gamma \mathrm{d}\gamma\right)^{1-1/q}\left(\int_0^1 \gamma\left(1-\gamma^\theta\right)^q\left|\phi''((1-\gamma)\omega+\gamma\nu)\right|^q\mathrm{d}\gamma\right)^{1/q}\Bigg],\end{aligned}\tag{66}$$

(since $|\phi''|^q$ is the s-convex)

$$\begin{aligned}\Im \leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left(\int_0^1 \gamma \mathrm{d}\gamma\right)^{1-1/q}\Bigg[&\left(\int_0^1 \gamma\left(1-\gamma^\theta\right)^q\left(\gamma^s\left|\phi''(\omega)\right|^q+(1-\gamma)^s\left|\phi''(\nu)\right|^q\right)\mathrm{d}\gamma\right)^{1/q}\\&+\left(\int_0^1 \gamma\left(1-\gamma^\theta\right)^q\left((1-\gamma)^s\left|\phi''(\omega)\right|^q+\gamma^s\left|\phi''(\nu)\right|^q\right)\mathrm{d}\gamma\right)^{1/q}\Bigg].\end{aligned}\tag{67}$$

Since $\gamma^\theta \geq \gamma$, $\theta \in (0,1]$, and $\gamma \in [0,1]$, we have $-\gamma^\theta \leq \gamma \Rightarrow 1-\gamma^\theta \leq 1-\gamma \leq (1-\gamma)^\theta$.

$$\begin{aligned}\Im &\leq \frac{(\nu-\omega)^2}{2(\theta+1)}\left(\frac{1}{2}\right)^{1-1/q}\Bigg[\left(\left|\phi''(\omega)\right|^q\int_0^1 \gamma^{s+1}(1-\gamma)^{\theta q}+\left|\phi''(\nu)\right|^q\int_0^1 \gamma(1-\gamma)^{\theta q+s}\mathrm{d}\gamma\right)^{1/q}\\&\quad+\left(\left|\phi''(\omega)\right|^q\int_0^1 \gamma(1-\gamma)^{\theta q+s}\mathrm{d}\gamma+\left|\phi''(\nu)\right|^q\int_0^1 \gamma^{s+1}(1-\gamma)^{\theta q}\mathrm{d}\gamma\right)^{1/q}\Bigg],\\ \Im &\leq \frac{(\nu-\omega)^2}{4(\theta+1)}\Big[\left(2\left(\left|\phi''(\omega)\right|^q\beta(s+2,\theta q+1)+\left|\phi''(\nu)\right|^q\beta(2,\theta q+s+1)\right)\right)^{1/q}\\&\quad+\left(2\left(\left|\phi''(\omega)\right|^q\beta(2,\theta q+s+1)+\left|\phi''(\nu)\right|^q\beta(s+2,\theta q+1)\right)\right)^{1/q}\Big].\end{aligned}\tag{68}$$

The proof is competed. □

Remark 16. For $s=1$, Theorem 10 reduces to Theorem 2 in [27] and Theorem 11 reduces to Theorem 3 in [27]. When both $\theta=s=1$, then Theorem 10 reduces to Theorem 2 [28].

4. Applications to Some Special Means

Consider the following special means for arbitrary real numbers f,g and $f \neq g$ as follows:

$$\begin{aligned}A(f,g)&=\frac{f+g}{2},\quad f,g\in\mathbb{R},\\ H(f,g)&=\frac{2}{1/f+1/g},\quad f,g\in\frac{\mathbb{R}}{0},\\ L(f,g)&=\frac{g-f}{\ln|g|-\ln|f|},\quad |f|\neq|g|.\end{aligned}\tag{69}$$

Proposition 17. *Let $\omega, \nu \in \mathbb{R}$, $\omega < \nu$, $\omega, \nu > 0$, and $s \in (0,1]$, then*

$$\left|A\left(e^{\omega}, e^{\nu}\right) - L\left(e^{\omega}, e^{\nu}\right)\right| \leq \frac{(\nu-\omega)}{4}\left[\left|e^{\omega}\right| + \left|e^{\nu}\right|\right]\left[\beta(s+2,2) + \beta(2,s+2)\right]. \tag{70}$$

Proof. This statement follows from Theorem 10, by using $\phi(\gamma) = e^{\gamma}$ and $\theta = 1$. □

Proposition 18. *Let $\omega, \nu \in \mathbb{R}$, $\omega < \nu$, $0 \in [\omega, \nu]$, and $s \in (0,1]$, then*

$$\left|H^{-1}(\omega,\nu) - L^{-1}(\omega,\nu)\right| \leq \frac{(\nu-\omega)^2}{2}\left[\frac{1}{s+1}\left(\left|\frac{2}{\omega^3}\right|^q + \left|\frac{2}{\nu^3}\right|^q\right)\right]^{1/q}\beta^{1/p}(p+1,p+1). \tag{71}$$

Proof. This statement follows from Theorem 11, by using $\phi(\gamma) = 1/\gamma$, $\gamma \neq 0$, and $\theta = 1$. □

Remark 19. For $\alpha = m = 1$ and $r = 0$, Proposition 4.4 in [5] reduces to inequality (71).

Proposition 20. *Let $\omega, \nu \in \mathbb{R}$, $\omega < \nu$, $0 \in [\omega, \nu]$, and $s \in (0,1]$, then*

$$\begin{aligned}\left|H^{-1}(\omega,\nu) - L^{-1}(\omega,\nu)\right| \leq \frac{(\nu-\omega)^2}{4}\left(\frac{1}{6}\right)^{1-1/q}&\left[\left(\left|\frac{2}{\omega^3}\right|^q\beta(s+2,2) + \left|\frac{2}{\nu^3}\right|^q\beta(2,s+2)\right)^{1/q}\right.\\ &\left.+\left(\left|\frac{2}{\omega^3}\right|^q\beta(2,s+2)\right) + \left|\frac{2}{\nu^3}\right|^q\beta(s+2,2)\right)^{1/q}\right].\end{aligned} \tag{72}$$

Proof. This statement follows from Theorem 13, by using $\phi(\gamma) = 1/\gamma$, $\gamma \neq 0$, and $\theta = 1$. □

5. Conclusion

In this paper, authors have established Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals via s-convex functions by applying two different techniques. In first part, an identity is proved in which a differentiable function is presented in the form of Riemann–Liouville fractional integrals of first derivatives of function. Furthermore, this identity is used to establish Hermite–Hadamard type inequalities in which the absolute values of first derivatives are s-convex functions. In the second part, an identity in which a function in the form of integral of double derivative of function is used to establish Hermite–Hadamard inequalities in which the absolute values of second derivatives are s-convex functions. The limiting cases included some existing results in the literature. Some applications of the obtained results are also described in the form of means. This method can also be applicable for other classes of convex functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare they have no conflicts of interest.

References

[1] I. Işcan, "New refinements for integral and sum forms of Hölder inequality," *Journal of Inequalities and Applications*, vol. 2019, 2019.

[2] M. Kadakal, I. Işcan, H. Kadakal, and K. Bekar, "On improvements of some integral inequalities," *Honam Mathematical Journal*, vol. 43, no. 3, pp. 441–452, 2019.

[3] H. Barsam, S. M. Ramezani, and Y. Sayyari, "On the new Hermite–Hadamard type inequalities for s-convex functions," *Afrika Matematika*, vol. 32, no. 7-8, pp. 1355–1367, 2021.

[4] H. Barsam and A. R. Sattarzadeh, "Hermite-Hadamard inequalities for uniformly convex functions and its applications in means," *Miskolc Mathematical Notes*, vol. 2, pp. 1787–2413, 2020.

[5] M. Emin Özdemir, S. I Butt, B. Bayraktar, and J. Nasir, "Several integral inequalities for (α, s, m)-convex functions," *American Institute of Mathematical Sciences Mathematics*, vol. 5, no. 4, pp. 3906–3921, 2020.

[6] H. Barsam and A. Sattarzadeh, "Some results on Hermite-Hadamard inequalities," *Journal of Mahani Mathematical Research*, vol. 9, no. 2, pp. 79–86, 2020.

[7] B. Y. Xi and F. Qi, "Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means,"

Journal of Nonlinear and Convex Analysis, vol. 16, pp. 873–890, 2015.

[8] Y. B. Xi, D. D. Gao, and F. Qi, "Integral inequalities of Hermite-Hadamard type for α, s-convex and α, s, m-convex function," *Italian Journal of Pure and Applied Mathematics*, vol. 44, pp. 499–510, 2020.

[9] M. A. Khan, Y. Chu, T. U. Khan, and J. Khan, "Some new inequalities of Hermite-Hadamard type for s-convex functions with applications," *Open Mathematics*, vol. 15, no. 1, pp. 1414–1430, 2017.

[10] C. Zhu, M. Feckan, and J. Wang, "Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula," *Journal of Applied Mathematics, Statistics and Informatics*, vol. 8, no. 2, pp. 21–28, 2012.

[11] P. O. Mohammed and M. Z. Sarikaya, "Hermite–Hadamard type inequalities for F-convex function involving fractional integrals," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 359, 2018.

[12] T. Lian, W. Tang, and R. Zhou, "Fractional Hermite–Hadamard inequalities for (s, m)-convex or s-concave functions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 240, 2018.

[13] Y. Khurshid, M. Adil Khan, and Y. M. Chu, "Conformable fractional integral inequalities for GG-and GA-convex function," *American Institute of Mathematical Sciences Mathematics*, vol. 5, no. 5, pp. 5012–5030, 2020.

[14] S. Qaisar, J. Nasir, S. I. Butt, and S. Hussain, "On some fractional integral inequalities of Hermite-Hadamard's type through convexity," *Symmetry*, vol. 11, no. 2, p. 137, 2019.

[15] P. O. Mohammed, "Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 3, pp. 2314–2324, 2021.

[16] E. Set, "New inequalities of Ostrowski type for the mappings whosederivatives are s-convex in the second sensevia fractional integrals," *Computers and Mathematics with Applications*, vol. 63, no. 7, pp. 1147–1154, 2012.

[17] T. Abdeljawad, S. Rashid, Z. Hammouch, and Y. Chu, "Some new local fractional inequalities associated with generalized s, m-convex functions and applications," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–27, 2020.

[18] I. Işcan, "Generalization of different type integral inequalities for *s*-convex functions via fractional integrals," *Applicable Analysis*, vol. 93, no. 9, pp. 1846–1862, 2014.

[19] F. Usta, H. Budak, M. Z. Sarikaya, and E. Set, "On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators," *Filomat*, vol. 32, no. 6, pp. 2153–2171, 2018.

[20] S. I. Butt, S. Yousaf, A. O. Akdemir, and M. A. Dokuyucu, "New Hadamard-type integral inequalities via a general form of fractional integral operators," *Chaos, Solitons and Fractals*, vol. 148, Article ID 111025, 2021.

[21] P. Agarwal, M. Jleli, and M. Tomar, "Certain Hermite-Hadamard type inequalities via generalized K-fractional integrals," *Journal of Inequalities and Applications*, vol. 2017, no. 1, pp. 1–10, 2017.

[22] S. K. Sahoo, M. Tariq, H. Ahmad, A. A. Aly, B. F. Felemban, and P. Thounthong, "Some Hermite-Hadamard type fractional integral inequalities involving twice differentiable mappings," *Symmetry*, vol. 13, no. 11, p. 2209, 2021.

[23] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[24] M. E. Özdemir, C. Yildiz, A. O. Akdemir, and E. Set, "On some inequalities for s-convex functions and applications," *Journal of Inequalities and Applications*, vol. 2013, no. 1, p. 333, 2013.

[25] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Klumer Akademic Publishers, Dordrecht, Boston, London, 1993.

[26] J. R. Wang, X. Li, and Y. Zhou, "Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals via s-convex functions and applications to special means," *Filomat*, vol. 30, no. 5, pp. 1143–1150, 2016.

[27] S. S. Dargomir, M. I. Bhatti, and M. Muddassar, "Some new Hermite-Hadamard's type fractional integral inequalities," *Journal of Computational Analysis and Applications*, vol. 18, 2015.

[28] M. E. Özdemir, M. Avci, and H. Kavurmaci, "Hermite-Hadamard-type inequalities via α-m-convexity," *Computers and Mathematics with Applications*, vol. 6, pp. 2614–2620, 2011.