Hindawi
Journal of Mathematics
Volume 2024, Article ID 2214422, 10 pages
https://doi.org/10.1155/2024/2214422

Research Article

Some Topological Approaches of Rough Sets through Minimal Neighborhoods and Decision Making

Ismail T. Shbair , Amgad S. Salama , Osama A. Embaby , and Abdelfattah A. El-Atik

Department of Mathematics, Faculty of Science, Tanta University, Tanta 31511, Egypt

Correspondence should be addressed to Ismail T. Shbair; shbairismail@gmail.com

Received 19 June 2023; Revised 11 September 2023; Accepted 1 February 2024; Published 19 March 2024

Academic Editor: Chiranjibe Jana

Rough set has an important role to deal with uncertainty objects. The aim of this article is to introduce some kinds of generalization for rough sets through minimal neighborhoods using special kinds of binary relations. Moreover, four different types of dual approximation operators will be constructed in terms of minimal neighborhoods. The comparison between these types of approximation operators is discussed. Some new kinds of topological structures induced by minimal neighborhoods are established and some of their properties are studied. Finally, we give a comparison between these topologies that help for determining the major components of COVID-19 infections. In this application, the components of infections help the expert in decision making in medicine.

1. Introduction and Preliminaries

Topology is an area of mathematics that is very important, whose definitions and concepts exist inside branches of mathematics and many applications. Topological spaces and their generalizations [1] are considered as basic definitions in system analysis. Recently, topological structures have been used to study graphs [2]. Also, many researchers suggested topological models in analysis [3], chemistry [4], medicine [5], and physics [6] and for determining COVID-19 [7]. Furthermore, Lashin et al. [8] have generated other topologies using a general binary relation. Some researchers have used a topology to represent structures such as fractals [2] in terms of binary relations.

Rough set theory (RST, for short) is initiated in 1982 by Pawlak [9]. RST is a mathematical method to study the uncertainty data, but not qualitatively. Also, Pawlak studied the relation between topology and its generalization and RST. Minimal structure in topology and rough sets are studied in [10], and many applications are discussed in [11]. The fundamental idea of RST is the common lower and upper approximations, which have been constructed using many types of neighborhoods, such as right and left neighborhoods [12], minimal right [13], and minimal left [14] neighborhoods. New kinds of neighborhoods called E_j-neighborhoods are defined by Al-Shami et al. [15]. Al-Shami has defined new kinds of neighborhoods called C_j-neighborhoods and studied them in medical applications [16]. Also, Al-Shami has defined maximal neighborhoods and studied their characteristics and applications in the medical field [17]. The range of rough sets applications today is far broader than it was in the past, and it has been used in many science and engineering domains, including computer network [18], solution of missing attribute values [19], and medical application [16].

In this paper, some ideas of minimal neighborhoods in topology are integrated. Other types of minimal structures will be investigated from a view point of minimal neighborhoods. We think that topological space will serve as a crucial foundation for modifying information extraction and processing. Several foundational ideas regarding topology and RST are introduced. Finally, the present approximations will be applied to have the best tools for determining the major components of COVID-19 infections. This medical application may be useful for experts in decision making.

Definition 1 (see [1]). If τ is a family from a nonempty universal set $\mathfrak{U}$. τ is a topology on $\mathfrak{U}$ if τ satisfies the following: (i) ϕ and $\mathfrak{U}$ are in τ; (ii) if $\mathfrak{B}_i \in \tau$ for $i \in \mathcal{J}$, then $\cup_{i\in\mathcal{J}}\mathfrak{B}_i \in \tau$; and (iii) if $\mathfrak{B}_1, \mathfrak{B}_2 \in \tau$, then $\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \tau$.

RST emerged out of the necessity to depict subsets of $\mathfrak{U}$ in units of equivalence classes as a partition, which defines a topological structure known as approximation space symbolized by $K = (\mathfrak{U}, \psi)$, where $\psi \subseteq \mathfrak{U} \times \mathfrak{U}$ is an equivalence relation [20, 21]. To represent the equivalence class containing x, we will use $[x] \subseteq \mathfrak{U}$. RST has come from the need to represent subsets from the universe set in terms of the equivalence class called approximation space $K = (\mathfrak{U}, \psi)$, where ψ is a knowledge about an element of $\mathfrak{U}$. The equivalence class of ψ is also known as the granules, elementary sets, or blocks.

Definition 2 (see [21]). In $K = (\mathfrak{U}, \psi)$ with $\mathfrak{B} \subseteq \mathfrak{U}$, the ψ–lower and ψ–upper approximations of $\mathfrak{B}$ are defined by $\underline{\psi}(\mathfrak{B}) = \{x \in \mathfrak{U}: [x] \subseteq \mathfrak{B}\}$ and $\overline{\psi}(\mathfrak{B}) = \{x \in \mathfrak{U}: [x] \cap \mathfrak{B} \neq \phi\}$, respectively.

$\mathfrak{U}$ is divided to three disjoint regions in $K = (\mathfrak{U}, \psi)$ by using Definition 2; the boundary region (briefly $B_\psi(\mathfrak{B})$), the positive region (briefly, $P_\psi(\mathfrak{B})$), and the negative region (briefly $N_\psi(\mathfrak{B})$) are defined by $B_\psi(\mathfrak{B}) = \overline{\psi}(\mathfrak{B}) - \underline{\psi}(\mathfrak{B})$, $P_\psi(\mathfrak{B}) = \underline{\psi}(\mathfrak{B})$, and $N_\psi(\mathfrak{B}) = \mathfrak{B} - \overline{\psi}(\mathfrak{B})$, respectively.

Definition 3 (see [9]). If $K = (\mathfrak{U}, \psi)$ is an approximation space with $\mathfrak{B} \subseteq \mathfrak{U}$, then the accuracy of $\mathfrak{B}$ is defined by $\xi(\mathfrak{B}) = |\underline{\psi}(\mathfrak{B})|/|\overline{\psi}(\mathfrak{B})|$, where $|\overline{\psi}(\mathfrak{B})| \neq 0$ and $|.|$ denotes the cardinality.

The definition of approximations uses the concept of knowledge granules. $\underline{\psi}(\mathfrak{B})$ is the set of all granules that exist in $\mathfrak{B}$, $\overline{\psi}(\mathfrak{B})$ is the set of all granules that have a nonempty intersection with $\mathfrak{B}$, and $B_\psi(\mathfrak{B})$ is the set of all granules that exist in the upper approximation and do not exist in the lower approximation. These definitions of approximation operators were given by Pawlak [9].

In $K = (\mathfrak{U}, \psi)$, if $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{U}$, then each of the following is true [21], where $\mathfrak{A}^c$ is the complement.

(L1)	$\underline{\psi}(\mathfrak{U}) = \mathfrak{U}$	(H1)	$\overline{\psi}(\mathfrak{U}) = \mathfrak{U}$
(L2)	$\underline{\psi}(\phi) = \phi$	(H2)	$\overline{\psi}(\phi) = \phi$
(L3)	$\underline{\psi}(\mathfrak{A}) \subseteq \mathfrak{A}$	(H3)	$\mathfrak{A} \subseteq \overline{\psi}(\mathfrak{A})$
(L4)	$\underline{\psi}(\mathfrak{A}) \cap \underline{\psi}(\mathfrak{B}) = \underline{\psi}(\mathfrak{A} \cap \mathfrak{B})$	(H4)	$\overline{\psi}(\mathfrak{A} \cup \mathfrak{B}) = \overline{\psi}(\mathfrak{A}) \cup \overline{\psi}(\mathfrak{B})$
(L5)	$\underline{\psi}(\mathfrak{A}^c) = [\overline{\psi}(\mathfrak{A})]^c$		
(L6)	$\underline{\psi}(\underline{\psi}(\mathfrak{A})) = \underline{\psi}(\mathfrak{A})$	(H6)	$\overline{\psi}(\overline{\psi}(\mathfrak{A})) = \overline{\psi}(\mathfrak{A})$
(L7)	If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\underline{\psi}(\mathfrak{A}) \subseteq \underline{\psi}(\mathfrak{B})$	(H7)	If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\overline{\psi}(\mathfrak{A}) \subseteq \overline{\psi}(\mathfrak{B})$
(L8)	$\underline{\psi}(\mathfrak{A}) \cup \underline{\psi}(\mathfrak{B}) \subseteq \underline{\psi}(\mathfrak{A} \cup \mathfrak{B})$	(H8)	$\overline{\psi}(\mathfrak{A} \cap \mathfrak{B}) \subseteq \overline{\psi}(\mathfrak{A}) \cap \overline{\psi}(\mathfrak{B})$

Definition 4 (see [22]). The binary relation ψ is called

(i) serial: $\forall x \in \mathfrak{U}$, $\exists y \in \mathfrak{U}$ s.t $x\psi y$, $\mathrm{RN}(x) \neq \phi$

(ii) inverse serial: $\forall x \in \mathfrak{U}$, $\exists y \in \mathfrak{U}$ s.t $y\psi x$, $\cup \mathrm{RN}(x) = \mathfrak{U}$

(iii) reflexive: $\forall x \in \mathfrak{U}$, $x\psi x$

(iv) symmetric: $\forall x \in \mathfrak{U}$, if $x\psi y$, then $y\psi x$

Definition 5 (see [23]). Let ψ be a binary relation and $x, y \in \mathfrak{U}$. Then, $\mathrm{RN}(x) = \{y \in \mathfrak{U}: x\psi y\}$ and $\mathrm{LN}(x) = \{y \in \mathfrak{U}: y\psi x\}$ are called right neighborhood and left neighborhood of x, respectively.

Definition 6 (see [24]). If ψ is a binary relation and $x \in \mathfrak{U}$, then minimal right neighborhood is $\mathrm{MN}_r(x) = \cap_y\{\mathrm{RN}(y): y\psi x\}$.

2. Generalization for Rough Sets Based on Minimal Neighborhood Systems

This section aims to investigate RST based on the minimal neighborhood system. Four types of approximation operators are generated. In addition, a comparison between our study and another approach is investigated.

Definition 7. If ψ is a binary relation, then four kinds of minimal neighborhood systems are defined as follows:

(i) Minimal right neighborhood $\mathrm{MN}_r(x) = \cap_y\{\mathrm{RN}(y): y\psi x\}$

(ii) Minimal left neighborhood $\mathrm{MN}_l(x) = \cap_y\{\mathrm{LN}(y): x\psi y\}$

(iii) Minimal of union neighborhood $\mathrm{MN}_u(x) = \mathrm{MN}_r(x) \cup \mathrm{MN}_l(x)$

(iv) Minimal of intersection neighborhood $\mathrm{MN}_i(x) = \mathrm{MN}_r(x) \cap \mathrm{MN}_l(x)$

Remark 8. It is clear that $\forall x \in \mathfrak{U}$,

$$\mathrm{MN}_r(x) = \begin{cases} \bigcap_{x\in \mathrm{RN}(y)} \mathrm{RN}(y), & \text{if } \exists y \text{ s.t } y\psi x, \\ \phi, & \text{otherwise}, \end{cases}$$
$$\mathrm{MN}_l(x) = \begin{cases} \bigcap_{x\in \mathrm{LN}(y)} \mathrm{LN}(y) & \text{if } \exists y \text{ s.t } x\psi y, \\ \phi, & \text{otherwise}, \end{cases} \quad (1)$$

Definition 9. Let $\mathrm{MN}_j(x)$ be minimal neighborhood systems with binary relation ψ, where $x \in \mathfrak{U}$ and $j \in \mathfrak{J} = \{r, l, u, i\}$. Then, $(\mathfrak{U}, \psi, MN_j)$ is an approximation space (briefly, MN_j–approximation space).

Remark 10. In Definition 9, $\mathrm{MN}_j(x)$ is the same for all $j \in \mathfrak{J}$, when ψ is an equivalence relation.

Lemma 11. *If $y \in MN_j(x)$, then $MN_j(y) \subseteq MN_j(x)$, where $x, y \in \mathfrak{U}$ and $j \in \{r, l, i\}$.*

Proof. Consider $j = r$. Let $y \in \mathrm{MN}_r(x)$ and $z \in \mathrm{MN}_r(y)$. Then, $z \in \mathrm{MN}_r(x)$. Therefore, $\mathrm{MN}_r(y) \subseteq \mathrm{MN}_r(x)$. By the same manner, the proof is verified for $j = l$. In the case $j = i$, if $y \in \mathrm{MN}_i(x)$, then $y \in \mathrm{MN}_r(x)$ and $y \in \mathrm{MN}_l(x)$. Thus, $\mathrm{MN}_r(y) \subseteq \mathrm{MN}_r(x)$ and $\mathrm{MN}_l(y) \subseteq \mathrm{MN}_l(x)$. Hence, $\mathrm{MN}_r(y) \cap \mathrm{MN}_l(y) \subseteq \mathrm{MN}_r(x) \cap \mathrm{MN}_l(x)$. Therefore, $\mathrm{MN}_i(y) \subseteq \mathrm{MN}_i(x)$.

Lemma 11 does not hold for $j = u$, in general. □

Example 1. If $\mathfrak{U} = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}\}$ with $\psi = \{(\mathfrak{e}, \mathfrak{e}), (\mathfrak{g}, \mathfrak{g}), (\mathfrak{e}, \mathfrak{f}), (\mathfrak{f}, \mathfrak{f}), (\mathfrak{f}, \mathfrak{g}), (\mathfrak{g}, \mathfrak{f})\}$, then $\mathrm{RN}(\mathfrak{U}, \psi) = \{\{\mathfrak{e}, \mathfrak{f}\}, \{\mathfrak{f}, \mathfrak{g}\}, \phi\}$ and $\mathrm{LN}(\mathfrak{U}, \psi) = \{\{\mathfrak{e}\}, \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}, \{\mathfrak{f}, \mathfrak{g}\}, \phi\}$. Hence, $\mathrm{MN}_r(\mathfrak{e}) = \{\mathfrak{e}, \mathfrak{f}\}$, $\mathrm{MN}_r(\mathfrak{f}) = \{\mathfrak{f}\}$, $\mathrm{MN}_r(\mathfrak{g}) = \{\mathfrak{f}, \mathfrak{g}\}$, $\mathrm{MN}_r(\mathfrak{h}) = \phi$, $\mathrm{MN}_l(\mathfrak{e}) = \{\mathfrak{e}\}$, $\mathrm{MN}_l(\mathfrak{f}) = \mathrm{MN}_l(\mathfrak{g}) = \{\mathfrak{f}, \mathfrak{g}\}$, $\mathrm{MN}_l(\mathfrak{h}) = \phi$, $\mathrm{MN}_u(\mathfrak{e}) = \{\mathfrak{e}, \mathfrak{f}\}$, $\mathrm{MN}_u(\mathfrak{f}) = \mathrm{MN}_u(\mathfrak{g}) = \{\mathfrak{f}, \mathfrak{g}\}$, $\mathrm{MN}_u(\mathfrak{h}) = \phi$, $\mathrm{MN}_i(\mathfrak{e}) = \{\mathfrak{e}\}$, $\mathrm{MN}_i(\mathfrak{f}) = \{\mathfrak{f}\}$, $\mathrm{MN}_i(\mathfrak{g}) = \{\mathfrak{f}, \mathfrak{g}\}$, and $\mathrm{MN}_i(\mathfrak{h}) = \phi$. Clearly, $\mathfrak{f} \in \mathrm{MN}_u(\mathfrak{e})$, but $\mathrm{MN}_u(\mathfrak{f}) \nsubseteq \mathrm{MN}_u(\mathfrak{e})$.

The proof of Lemma 12 is clear, so it is omitted.

Lemma 12. *Let ψ be a symmetric relation and $y \in MN_u(x)$. Then, $MN_u(y) \subseteq MN_u(x)$ for each $x, y \in \mathfrak{U}$.*

Lemma 13. *If ψ is an inverse serial and a serial relation, then $MN_r(x) \neq \phi$ and $MN_l(x) \neq \phi$, respectively.*

Proof. Let ψ be a serial relation. Then, $\forall x \in \mathfrak{U}$, $\exists y \in \mathfrak{U}$ s.t $x\psi y$. Hence, $x \in \mathrm{LN}(y)$ for some $y \in \mathfrak{U}$. Therefore, $\mathrm{MN}_l(x) \neq \phi$. □

Let ψ be an inverse serial and serial relation. Then, $\mathrm{MN}_l(x) \neq \phi$ and $\mathrm{MN}_r(x) \neq \phi$ are not true, in general.

Example 2. If $\mathfrak{U} = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$ and $\psi_1 = \{(\mathfrak{e}, \mathfrak{f}), (\mathfrak{f}, \mathfrak{f}), (\mathfrak{g}, \mathfrak{f})\}$, then $\mathrm{RN}(\mathfrak{U}, \psi_1) = \{\{\mathfrak{f}\}\}$ and $\mathrm{LN}(\mathfrak{U}, \psi_1) = \{\phi, \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}\}$. Hence, $\mathrm{MN}_{r_1}(\mathfrak{e}) = \mathrm{MN}_{r_1}(\mathfrak{g}) = \phi, \mathrm{MN}_{r_1}(\mathfrak{f}) = \{\mathfrak{f}\}$ and $\mathrm{MN}_{l_1}(\mathfrak{e}) = \mathrm{MN}_{l_1}(\mathfrak{f}) = \mathrm{MN}_{l_1}(\mathfrak{g}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$. Clearly, $\forall\, x \in \mathfrak{U}$, ψ_1 is serial and $\mathrm{MN}_{l_1}(x) \neq \phi$, but $\mathrm{MN}_{r_1}(\mathfrak{e}) = \phi$. Now, let $\psi_2 = \{(\mathfrak{g}, \mathfrak{e}), (\mathfrak{g}, \mathfrak{f}), (\mathfrak{g}, \mathfrak{g})\}$. Then, $(\mathfrak{U}, \psi_2) = \{\phi, \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}\}$ and $\mathrm{LN}(\mathfrak{U}, \psi_2) = \{\{\mathfrak{g}\}\}$. Hence, $\mathrm{MN}_{r_2}(\mathfrak{e}) = \mathrm{MN}_{r_2}(\mathfrak{f}) = \mathrm{MN}_{r_2}(\mathfrak{g}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$, $\mathrm{MN}_{l_2}(\mathfrak{e}) = \mathrm{MN}_{l_2}(\mathfrak{f}) = \phi$, and $\mathrm{MN}_{l_2}(\mathfrak{g}) = \{\mathfrak{g}\}$. Clearly, $\forall x \in \mathfrak{U}$, ψ_2 is inverse serial and $\mathrm{MN}_{r_2}(x) \neq \phi$, but $\mathrm{MN}_{l_2}(\mathfrak{e}) = \phi$.

Lemma 14. *If ψ is a reflexive relation, then $y \in MN_j(y)$ and $MN_j(y) \neq \phi$, $\forall y \in \mathfrak{U}$ and $j \in \mathfrak{J}$.*

Proof. Assume that ψ is reflexive and $y \in \mathfrak{U}$. Then, $y \in \mathrm{RN}(y)$ and $y \in \mathrm{LN}(y)$. Thus, $y \in MN_j(y)$. Therefore, $MN_j(y) \neq \phi$, $\forall j \in \mathfrak{J}$. □

Proposition 15. *If ψ is a reflexive relation and $y \in \mathfrak{U}$, then*

(i) $MN_r(y) \subseteq RN(y)$
(ii) $MN_l(y) \subseteq LN(y)$
(iii) $MN_i(y) \subseteq RN(y)$
(iv) $MN_i(y) \subseteq LN(y)$
(v) $MN_u(y) \subseteq RN(y) \cup LN(y)$

Proof. (i) Let ψ be a reflexive relation. Then, $y \in \mathrm{RN}(y)$ for all $y \in \mathfrak{U}$. Consider $x \in MN_r(y)$. Then, $x \in \mathrm{RN}(y)$. Therefore, $MN_r(y) \subseteq \mathrm{RN}(y)$. By the same manner, proofs of (ii), (iii), (iv), and (v) are verified. □

The converse of Proposition 15 and equality are not true, in general.

Example 3. In Example 1, all properties of Proposition 15 are satisfied. However, ψ is not reflexive relation, since $(\mathfrak{h}, \mathfrak{h}) \notin \psi$. Also, $MN_r(\mathfrak{f}) \neq \mathrm{RN}(\mathfrak{f})$, $MN_l(\mathfrak{f}) \neq \mathrm{LN}(\mathfrak{f})$, $MN_i(\mathfrak{f}) \neq \mathrm{RN}(\mathfrak{f}) \neq \mathrm{LN}(\mathfrak{f})$, and $MN_u(\mathfrak{f}) \neq \mathrm{RN}(\mathfrak{f}) \cup \mathrm{LN}(\mathfrak{f})$.

Proposition 16. *Let ψ be a reflexive and symmetric relation. Then, $MN_j(y) \subseteq RN(y)$ and $MN_j(y) \subseteq LN(y)$, $\forall y \in \mathfrak{U}$ and $j \in \mathfrak{J}$.*

Proof. Suppose that ψ is reflexive and symmetric relation. Then, $y \in \mathrm{RN}(y) = \mathrm{LN}(y)$, $\forall y \in \mathfrak{U}$. Hence, the proof is clear by Proposition 15. □

Definition 17. In $(\mathfrak{U}, \psi, MN_j)$ with $\mathfrak{B} \subseteq \mathfrak{U}$, the MN_j–lower and MN_j–upper approximations of $\mathfrak{B}$ are defined by

$\underline{\psi}_j(\mathfrak{B}) = \{x \in \mathfrak{U}: MN_j(x) \subseteq \mathfrak{B}\}$ and $\overline{\psi}_j(\mathfrak{B}) = \{x \in \mathfrak{U}: MN_j(x) \cap \mathfrak{B} \neq \phi\}$, respectively.

Definition 18. For all $j \in \mathfrak{J}$, if $\underline{\psi}_j(\mathfrak{B}) = \overline{\psi}_j(\mathfrak{B})$, then the set $\mathfrak{B}$ is called MN_j-exact. Otherwise, the set $\mathfrak{B}$ is called MN_j-rough.

Definition 19. For each $j \in \mathfrak{J}$, the MN_j-boundary set, the MN_j-positive set, and the MN_j-negative set are defined by $B_j(\mathfrak{B}) = \overline{\psi}_j(\mathfrak{B}) - \underline{\psi}_j(\mathfrak{B})$, $P_j(\mathfrak{B}) = \underline{\psi}_j(\mathfrak{B})$, and $N_j(\mathfrak{B}) = \mathfrak{B} - \overline{\psi}_j(\mathfrak{B})$, respectively.

Theorem 20. *Let ψ be a binary relation and $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{U}$. Then, MN_j-lower approximation and MN_j-upper approximation have the following characteristics, where $\mathfrak{A}^c$ is the complement.*

(L1)	$\underline{\psi}_j(\mathfrak{U}) = \mathfrak{U}$	(H2)	$\overline{\psi}_j(\phi) = \phi$
(L4)	$\underline{\psi}_j(\mathfrak{A}) \cap \underline{\psi}_j(\mathfrak{B}) = \underline{\psi}_j(\mathfrak{A} \cap \mathfrak{B})$	(H4)	$\overline{\psi}_j(\mathfrak{A} \cup \mathfrak{B}) = \overline{\psi}_j(\mathfrak{A}) \cup \overline{\psi}_j(\mathfrak{B})$
(L5)	$\underline{\psi}_j(\mathfrak{A}^c) = [\overline{\psi}_j(\mathfrak{A})]^c$		
(L6)	$\underline{\psi}_j(\underline{\psi}_j(\mathfrak{A})) = \underline{\psi}_j(\mathfrak{A}), j \neq u$	(H6)	$\overline{\psi}_j(\overline{\psi}_j(\mathfrak{A})) = \overline{\psi}_j(\mathfrak{A}), j \neq u$
(L7)	If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\underline{\psi}_j(\mathfrak{A}) \subseteq \underline{\psi}_j(\mathfrak{B})$	(H7)	If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\overline{\psi}_j(\mathfrak{A}) \subseteq \overline{\psi}_j(\mathfrak{B})$
(L8)	$\underline{\psi}_j(\mathfrak{A}) \cup \underline{\psi}_j(\mathfrak{B}) \subseteq \underline{\psi}_j(\mathfrak{A} \cup \mathfrak{B})$	(H8)	$\overline{\psi}_j(\mathfrak{A} \cap \mathfrak{B}) \subseteq \overline{\psi}_j(\mathfrak{A}) \cap \overline{\psi}_j(\mathfrak{B})$

Proof. The properties (L1), (L4), (H2), and (H4) are clear. Therefore, the remaining properties can be proved as follows:

(L5) $\underline{\psi}_j(\mathfrak{A}^c) = \{x \in \mathfrak{U}: MN_j(x) \subseteq \mathfrak{A}^c\} = \{x \in \mathfrak{U}: MN_j(x) \cap \mathfrak{A} = \phi\} = \{x \in \mathfrak{U}: MN_j(x) \cap \mathfrak{A} \neq \phi\}^c = [\overline{\psi}_j(\mathfrak{A})]^c$.

(L6) Let $y \in \underline{\psi}_j(\underline{\psi}_j(\mathfrak{A}))$, where $j \neq u$. Then, $MN_j(y) \subseteq \underline{\psi}_j(\mathfrak{A}) = \{x \in \mathfrak{U}: MN_j(x) \subseteq \mathfrak{A}\}$. Hence, $MN_j(y) \subseteq \mathfrak{A}$. So, $y \in \underline{\psi}_j(\mathfrak{A})$. Therefore, $\underline{\psi}_j(\underline{\psi}_j(\mathfrak{A})) \subseteq \underline{\psi}_j(\mathfrak{A})$. Conversely, let $x \in \underline{\psi}_j(\mathfrak{A})$, where $j \neq u$. Then, $MN_j(x) \subseteq \mathfrak{A}$. We want to show that $MN_j(x) \subseteq \underline{\psi}_j(\mathfrak{A})$. Let $y \in MN_j(x)$. Then, by Lemma 11, we have $MN_j(y) \subseteq MN_j(x)$. Thus, $MN_j(y) \subseteq \mathfrak{A}$. Hence, $y \in \underline{\psi}_j(\mathfrak{A})$ and then $MN_j(x) \subseteq \underline{\psi}_j(\mathfrak{A})$. So, $x \in \underline{\psi}_j(\underline{\psi}_j(\mathfrak{A}))$ and $\underline{\psi}_j(\mathfrak{A}) \subseteq \underline{\psi}_j(\underline{\psi}_j(\mathfrak{A}))$. Therefore, $\underline{\psi}_j(\underline{\psi}_j(\mathfrak{A})) = \underline{\psi}_j(\mathfrak{A})$.

(H6) Similar to the proof of (L6).

(L7) Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then,

$\underline{\psi}_j(\mathfrak{A}) = \{x \in \mathfrak{U}: MN_j(x) \subseteq \mathfrak{A}\} \subseteq \{x \in \mathfrak{U}: MN_j(x) \subseteq \mathfrak{B}\} = \underline{\psi}_j(\mathfrak{B})$.

(H7) Similar to the proof of (L7).

(L8) and (H8) can be proved directly by using (L7) and (H7).

Properties (L2), (L3), (H1), and (H3) are not true, in general. □

(L2)	$\underline{\psi}_j(\phi) = \phi$	(H1)	$\overline{\psi}_j(\mathfrak{U}) = \mathfrak{U}$
(L3)	$\underline{\psi}_j(\mathfrak{A}) \subseteq \mathfrak{A}$	(H3)	$\mathfrak{A} \subseteq \overline{\psi}_j(\mathfrak{A})$

Example 4. In Example 1, if $\mathfrak{X} = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$ and $\mathfrak{Y} = \{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\}$, then $\underline{\psi}_r(\mathfrak{X}) = \underline{\psi}_l(\mathfrak{X}) = \underline{\psi}_u(\mathfrak{X}) = \underline{\psi}_i(\mathfrak{X}) = \mathfrak{U} \nsubseteq \mathfrak{X}$, $\mathfrak{Y} \nsubseteq \overline{\psi}_r(\mathfrak{Y}) = \overline{\psi}_i(\mathfrak{Y}) = \{\mathfrak{e}, \mathfrak{g}\}$, $\mathfrak{Y} \nsubseteq \overline{\psi}_l(\mathfrak{Y}) = \overline{\psi}_u(\mathfrak{Y}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$, $\underline{\psi}_r(\phi) = \underline{\psi}_l(\phi) = \underline{\psi}_u(\phi) = \underline{\psi}_i(\phi) = \{\mathfrak{h}\} \neq \phi$, and $\overline{\psi}_r(\mathfrak{U}) = \overline{\psi}_l(\mathfrak{U}) = \overline{\psi}_u(\mathfrak{U}) = \overline{\psi}_i(\mathfrak{U}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\} \neq \mathfrak{U}$.

Proposition 21. *Let ψ be a reflexive relation. Then,* *Proof.* It is sufficient to prove (L3) and proofs of (L2), (H1), and (H3) are similar. Assume that ψ is reflexive and $y \in \underline{\psi}_j(\mathfrak{A})$ for all $y \in \mathfrak{U}$. Thus, $MN_j(y) \subseteq \mathfrak{A}$. However, $y \in MN_j(y)$. Hence, $y \in \mathfrak{A}$. Therefore, $\underline{\psi}_j(\mathfrak{A}) \subseteq \mathfrak{A}$.

Properties (L6) and (H6) are not true, for $j = u$ with reflexive relation, in general. □

Example 5. Let $\mathfrak{U} = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}\}$ and $= \{(\mathfrak{e}, \mathfrak{e}), (\mathfrak{f}, \mathfrak{f}), (\mathfrak{g}, \mathfrak{g}), (\mathfrak{h}, \mathfrak{h}), (\mathfrak{e}, \mathfrak{g}), (\mathfrak{e}, \mathfrak{h}), (\mathfrak{f}, \mathfrak{h}), (\mathfrak{g}, \mathfrak{f})\}$. Then, $\mathrm{RN}(\mathfrak{U}, \psi) = \{\{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\}, \{\mathfrak{f}, \mathfrak{h}\}, \{\mathfrak{f}, \mathfrak{g}\}, \{\mathfrak{h}\}\}$ and $\mathrm{LN}(\mathfrak{U}, \psi) = \{\{\mathfrak{e}\}, \{\mathfrak{f}, \mathfrak{g}\}, \{\mathfrak{e}, \mathfrak{g}\}, \{\mathfrak{e}, \mathfrak{f}, \mathfrak{h}\}\}$. Hence, $MN_r(\mathfrak{e}) = \{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\}$, $MN_r(\mathfrak{f}) = \{\mathfrak{f}\}$, $MN_r(\mathfrak{g}) = \{\mathfrak{g}\}$, $MN_r(\mathfrak{h}) = \{\mathfrak{h}\}$, $MN_l(\mathfrak{e}) = \{\mathfrak{e}\}$, $MN_l(\mathfrak{f}) = \{\mathfrak{f}\}$, $MN_l(\mathfrak{g}) = \{\mathfrak{g}\}$, $MN_l(\mathfrak{h}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{h}\}$, $MN_u(\mathfrak{e}) = \{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\}$, $MN_u(\mathfrak{f}) = \{\mathfrak{f}\}$, $MN_u(\mathfrak{g}) = \{\mathfrak{g}\}$, $MN_u(\mathfrak{h}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{h}\}$, $MN_i(\mathfrak{e}) = \{\mathfrak{e}\}$, $MN_i(\mathfrak{f}) = \{\mathfrak{f}\}$, $MN_i(\mathfrak{g}) = \{\mathfrak{g}\}$, and $MN_i(\mathfrak{h}) = \{\mathfrak{h}\}$. However, $\underline{\psi}_u(\underline{\psi}_u(\{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\})) = \{\mathfrak{g}\} \neq \underline{\psi}_u(\{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\}) = \{\mathfrak{e}, \mathfrak{g}\}$ and $\overline{\psi}_u(\overline{\psi}_u(\{\mathfrak{g}\})) = \{\mathfrak{e}, \mathfrak{g}, \mathfrak{h}\} \neq \overline{\psi}_u(\{\mathfrak{g}\}) = \{\mathfrak{e}, \mathfrak{g}\}$.

Proposition 22. *If ψ is a symmetric relation and $\mathfrak{A} \subseteq \mathfrak{U}$, then*

(L6)	$\underline{\psi}_u(\underline{\psi}_u(\mathfrak{A})) = \underline{\psi}_u(\mathfrak{A})$	(H6)	$\overline{\psi}_u(\overline{\psi}_u(\mathfrak{A})) = \overline{\psi}_u(\mathfrak{A})$

Proof. The proof is obvious using Lemma 12 and Theorem 20.

Equality in the properties (L6) and (H6) are not true, in general. □

Example 6. In Example 1, consider $\mathfrak{X} = \{\mathfrak{e}, \mathfrak{f}\}$, $\mathfrak{Y} = \{\mathfrak{g}\}$, and $\mathfrak{Z} = \{\mathfrak{e}, \mathfrak{g}\}$. Then, $\underline{\psi}_r(\mathfrak{X}) \cup \underline{\psi}_r(\mathfrak{Y}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{h}\} \neq \underline{\psi}_r(\mathfrak{X} \cup \mathfrak{Y}) = \mathfrak{U}$, $\underline{\psi}_l(\mathfrak{X}) \cup \underline{\psi}_l(\mathfrak{Y}) = \{\mathfrak{e}, \mathfrak{h}\} \neq \underline{\psi}_l(\mathfrak{X} \cup \mathfrak{Y}) = \mathfrak{U}$, $\underline{\psi}_u(\mathfrak{X}) \cup \underline{\psi}_u(\mathfrak{Y}) = \{\mathfrak{e}, \mathfrak{h}\} \neq \underline{\psi}_u(\mathfrak{X} \cup \mathfrak{Y}) = \mathfrak{U}$, $\underline{\psi}_i(\mathfrak{X}) \cup \underline{\psi}_i(\mathfrak{Y}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{h}\} \neq \underline{\psi}_i(\mathfrak{X} \cup \mathfrak{Y}) = \mathfrak{U}$, $\overline{\psi}_r(\mathfrak{X} \cap \mathfrak{Z}) = \{\mathfrak{e}\} \neq \overline{\psi}_r(\mathfrak{X}) \cap \overline{\psi}_r(\mathfrak{Z}) = \{\mathfrak{e}, \mathfrak{g}\}$, $\overline{\psi}_l(\mathfrak{X} \cap \mathfrak{Z}) = \{\mathfrak{e}\} \neq \overline{\psi}_l(\mathfrak{X}) \cap \overline{\psi}_l(\mathfrak{Z}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$, $\overline{\psi}_u(\mathfrak{X} \cap \mathfrak{Z}) = \{\mathfrak{e}\} \neq \overline{\psi}_l(\mathfrak{X}) \cap \overline{\psi}_u(\mathfrak{Z}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$, and $\overline{\psi}_i(\mathfrak{X} \cap \mathfrak{Z}) = \{\mathfrak{e}\} \neq \overline{\psi}_i(\mathfrak{X}) \cap \overline{\psi}_i(\mathfrak{Z}) = \{\mathfrak{e}, \mathfrak{g}\}$.

To satisfy the majority of Pawlak's approximations qualities from Theorem 20 and Proposition 21, ψ must be a reflexive relation. Therefore, minimal approximations are a generalization for rough approximations. In Table 1, we compare our study with other techniques of rough approximations. The symbol √ indicates that Pawlak's property is verified.

Definition 23. Let ψ be a reflexive relation with $\mathfrak{B} \subseteq \mathfrak{U}$ and $j \in \mathfrak{J}$. Then, MN_j–accuracy of approximation of the subset $\mathfrak{B}$ is $\xi_j(\mathfrak{B}) = |\underline{\psi}_j(\mathfrak{B})| / |\overline{\psi}_j(\mathfrak{B})|$, where $|\overline{\psi}_j(\mathfrak{B})| \neq 0$ and $|.|$ denotes the cardinality.

Corollary 24. *From Definition 23 and Proposition 21, we deduce that with a reflexive relation ψ, (i) $0 \leq \xi_j(\mathfrak{B}) \leq 1$; (ii) if $\xi_j(\mathfrak{B}) = 1$, then the subset $\mathfrak{B}$ is MN_j–exact. Otherwise, $\mathfrak{B}$ is MN_j–rough.*

3. Relationship between MNj – Approximation Operators

In this section, several kinds of MNj–approximation operators are compared. Also, boundary and accuracy of MNj–approximations are investigated.

Remark 25. From Tables 2 and 3 and Example 5, different kinds of MN_j–approximations operators, MN_j–boundary and MN_j–accuracy, are compared. $\xi_i(\mathfrak{A})$ is the best accuracy.

Theorem 26. *If ψ is a binary relation and $\mathfrak{A} \subseteq \mathfrak{U}$, then*

(i) $\underline{\psi}_u(\mathfrak{A}) \subseteq \underline{\psi}_r(\mathfrak{A}) \subseteq \underline{\psi}_i(\mathfrak{A})$
(ii) $\underline{\psi}_u(\mathfrak{A}) \subseteq \underline{\psi}_l(\mathfrak{A}) \subseteq \underline{\psi}_i(\mathfrak{A})$
(iii) $\overline{\psi}_i(\mathfrak{A}) \subseteq \overline{\psi}_r(\mathfrak{A}) \subseteq \overline{\psi}_u(\mathfrak{A})$
(iv) $\overline{\psi}_i(\mathfrak{A}) \subseteq \overline{\psi}_l(\mathfrak{A}) \subseteq \overline{\psi}_u(\mathfrak{A})$

Proof. (i) If $x \in \underline{\psi}_u(\mathfrak{A})$, then $MN_u(x) = [MN_r(x) \cup MN_l(x)] \subseteq \mathfrak{A}$. Thus, $MN_r(x) \subseteq \mathfrak{A}$. Hence, $x \in \underline{\psi}_r(\mathfrak{A})$. Therefore, $\underline{\psi}_u(\mathfrak{A}) \subseteq \underline{\psi}_r(\mathfrak{A})$. Now, let $x \in \underline{\psi}_r(\mathfrak{A})$. Then, $MN_r(x) \subseteq \mathfrak{A}$. However, $MN_i(x) = [MN_r(x) \cap MN_l(x)] \subseteq \mathfrak{A}$. Hence, $x \in \underline{\psi}_i(\mathfrak{A})$. Therefore, $\underline{\psi}_r(\mathfrak{A}) \subseteq \underline{\psi}_i(\mathfrak{A})$. (ii) If $x \in \overline{\psi}_i(\mathfrak{A})$, then $MN_i(x) \cap \mathfrak{A} \neq \phi$. However, $MN_i(x) = MN_r(x) \cap MN_l(x)$ and thus $MN_r(x) \cap \mathfrak{A} \neq \phi$. Hence, $x \in \overline{\psi}_r(\mathfrak{A})$. Therefore, $\overline{\psi}_i(\mathfrak{A}) \subseteq \overline{\psi}_r(\mathfrak{A})$. Now, let $x \in \overline{\psi}_r(\mathfrak{A})$. Then, $MN_r(x) \cap \mathfrak{A} \neq \phi$. However, $MN_u(x) = MN_r(x) \cup MN_l(x)$. Thus, $MN_u(x) \cap \mathfrak{A} \neq \phi$. Hence, $x \in \overline{\psi}_u(\mathfrak{A})$. Therefore, $\overline{\psi}_r(\mathfrak{A}) \subseteq \overline{\psi}_u(\mathfrak{A})$. By the same manner, the proof is verified for (ii) and (iv).

Equality in Theorem 26 does not hold, in general. □

Example 7. In Example 1, $\underline{\psi}_u(\{\mathfrak{f}\}) = \{\mathfrak{h}\} \neq \underline{\psi}_r(\{\mathfrak{f}\}) = \{\mathfrak{f}, \mathfrak{h}\}$, $\underline{\psi}_r(\{\mathfrak{e}\}) = \{\mathfrak{h}\} \neq \underline{\psi}_i(\{\mathfrak{e}\}) = \{\mathfrak{e}, \mathfrak{h}\}$, $\underline{\psi}_u(\{\mathfrak{e}\}) = \{\mathfrak{h}\} \neq \underline{\psi}_l(\{\mathfrak{e}\}) = \{\mathfrak{e}, \mathfrak{h}\}$, $\underline{\psi}_l(\{\mathfrak{f}\}) = \{\mathfrak{h}\} \neq \underline{\psi}_i(\{\mathfrak{f}\}) = \{\mathfrak{f}, \mathfrak{h}\}$, $\overline{\psi}_i(\{\mathfrak{f}\}) = \{\mathfrak{f}, \mathfrak{g}\} \neq \overline{\psi}_r(\{\mathfrak{f}\}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$, $\overline{\psi}_r(\{\mathfrak{g}\}) = \{\mathfrak{g}\} \neq \overline{\psi}_u(\{\mathfrak{g}\}) = \{\mathfrak{f}, \mathfrak{g}\}$, $\overline{\psi}_i(\{\mathfrak{g}\}) = \{\mathfrak{g}\} \neq \overline{\psi}_l(\{\mathfrak{g}\}) = \{\mathfrak{f}, \mathfrak{g}\}$, and $\overline{\psi}_l(\{\mathfrak{f}\}) = \{\mathfrak{f}, \mathfrak{g}\} \neq \overline{\psi}_u(\{\mathfrak{f}\}) = \{\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\}$.

Theorem 27. *Let ψ be a reflexive relation and $\mathfrak{A} \subseteq \mathfrak{U}$. Then,*

(i) $\underline{\psi}_u(\mathfrak{A}) \subseteq \underline{\psi}_r(\mathfrak{A}) \subseteq \underline{\psi}_i(\mathfrak{A}) \subseteq \mathfrak{A} \subseteq \overline{\psi}_i(\mathfrak{A}) \subseteq \overline{\psi}_r(\mathfrak{A}) \subseteq \overline{\psi}_u(\mathfrak{A})$

Table 1: A comparison between various techniques of rough set properties.

Pawlak's properties	Yao and Lin [23]	Yu et al. [25]	Mareay [26]	Our technique
(L1)	√	√	√	
(H1)		√	√	√
(L2)		√	√	
(H2)		√	√	
(L3)		√	√	
(H3)		√	√	√
(L4)	√		√	√
(H4)	√	√	√	√
(L5)	√		√	√
(L6)		√	√	
(H6)			√	
(L7)	√	√	√	√
(H7)	√	√	√	√
(L8)	√	√	√	√
(H8)	√	√	√	√

Table 2: A comparison between several kinds of MN_j−approximations.

$\mathfrak{A}$	$\underline{\psi}_r(\mathfrak{A})$	$\overline{\psi}_r(\mathfrak{A})$	$B_r(\mathfrak{A})$	$\xi_r(\mathfrak{A})$	$\underline{\psi}_l(\mathfrak{A})$	$\overline{\psi}_l(\mathfrak{A})$	$B_l(\mathfrak{A})$	$\xi_l(\mathfrak{A})$
$\{\mathfrak{e}\}$	ϕ	$\{\mathfrak{e}\}$	$\{\mathfrak{e}\}$	0	$\{\mathfrak{e}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	1/2
$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	ϕ	1	$\{\mathfrak{f}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	1/2
$\{\mathfrak{g}\}$	$\{\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e}\}$	1/2	$\{\mathfrak{g}\}$	$\{\mathfrak{g}\}$	ϕ	1
$\{\mathfrak{h}\}$	$\{\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{e}\}$	1/2	ϕ	$\{\mathfrak{h}\}$	$\{\mathfrak{h}\}$	0
$\{\mathfrak{e},\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{e},\mathfrak{f}\}$	$\{\mathfrak{e}\}$	1/2	$\{\mathfrak{e},\mathfrak{f}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	2/3
$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e}\}$	1/2	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	2/3
$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{e}\}$	1/2	$\{\mathfrak{e}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	1/2
$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{e}\}$	2/3	$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	2/3
$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e}\}$	2/3	$\{\mathfrak{f}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	1/2
$\{\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e}\}$	2/3	$\{\mathfrak{g}\}$	$\{\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	1/2
$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{e}\}$	2/3	$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	$\mathfrak{U}$	$\{\mathfrak{h}\}$	3/4
$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e}\}$	2/3	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	ϕ	1	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	2/3
$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	$\mathfrak{U}$	$\{\mathfrak{e}\}$	3/4	$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	2/3
$\mathfrak{U}$	$\mathfrak{U}$	$\mathfrak{U}$	ϕ	1	$\mathfrak{U}$	$\mathfrak{U}$	ϕ	1

Table 3: A comparison between several kinds of MN_j−approximations.

$\mathfrak{A}$	$\underline{\psi}_u(\mathfrak{A})$	$\overline{\psi}_u(\mathfrak{A})$	$B_u(\mathfrak{A})$	$\xi_u(\mathfrak{A})$	$\underline{\psi}_i(\mathfrak{A})$	$\overline{\psi}_i(\mathfrak{A})$	$B_i(\mathfrak{A})$	$\xi_i(\mathfrak{A})$
$\{\mathfrak{e}\}$	ϕ	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	0	$\{\mathfrak{e}\}$	$\{\mathfrak{e}\}$	ϕ	1
$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	1/2	$\{\mathfrak{f}\}$	$\{\mathfrak{f}\}$	ϕ	1
$\{\mathfrak{g}\}$	$\{\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e}\}$	1/2	$\{\mathfrak{g}\}$	$\{\mathfrak{g}\}$	ϕ	1
$\{\mathfrak{h}\}$	ϕ	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	0	$\{\mathfrak{h}\}$	$\{\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{f}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/3	$\{\mathfrak{e},\mathfrak{f}\}$	$\{\mathfrak{e},\mathfrak{f}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/3	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{h}\}$	ϕ	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	0	$\{\mathfrak{e},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g}\}$	$\mathfrak{U}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/2	$\{\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g}\}$	ϕ	1
$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{f}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/3	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/3	$\{\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{g},\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{f},\mathfrak{g}\}$	$\mathfrak{U}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/2	$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e}\}$	2/3	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{g}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{h}\}$	2/3	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{e},\mathfrak{g},\mathfrak{h}\}$	ϕ	1
$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{g}\}$	$\mathfrak{U}$	$\{\mathfrak{e},\mathfrak{h}\}$	1/2	$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	$\{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}$	ϕ	1
$\mathfrak{U}$	$\mathfrak{U}$	$\mathfrak{U}$	ϕ	1	$\mathfrak{U}$	$\mathfrak{U}$	ϕ	1

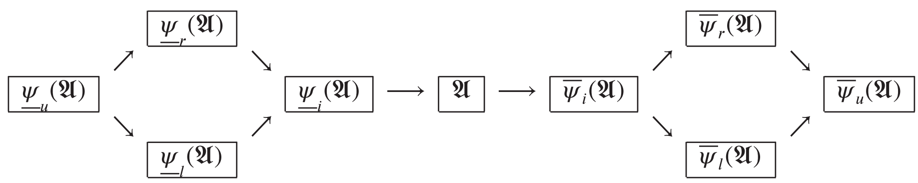
(ii) $\underline{\psi}_u(\mathfrak{A}) \subseteq \underline{\psi}_l(\mathfrak{A}) \subseteq \underline{\psi}_i(\mathfrak{A}) \subseteq \mathfrak{A} \subseteq \overline{\psi}_i(\mathfrak{A}) \subseteq \overline{\psi}_l(\mathfrak{A}) \subseteq \overline{\psi}_u(\mathfrak{A})$

Proof. By using Proposition 21 and Theorem 26, the proof is obvious.

The equality in Theorem 27 is not true, in general. □

Example 8. In Example 5 and Tables 2 and 3, $\underline{\psi}_u(\{\mathfrak{e},\mathfrak{h}\}) \neq \underline{\psi}_r(\{\mathfrak{e},\mathfrak{h}\}) \neq \underline{\psi}_i(\{\mathfrak{e},\mathfrak{h}\})$, $\overline{\psi}_i(\{\mathfrak{f},\mathfrak{g}\}) \neq \overline{\psi}_r(\{\mathfrak{f},\mathfrak{g}\}) \neq \overline{\psi}_u(\{\mathfrak{f},\mathfrak{g}\})$, $\underline{\psi}_u(\{\mathfrak{e},\mathfrak{h}\}) \neq \underline{\psi}_l(\{\mathfrak{e},\mathfrak{h}\}) \neq \underline{\psi}_i(\{\mathfrak{e},\mathfrak{h}\})$, and $\overline{\psi}_i(\{\mathfrak{f},\mathfrak{g}\}) \neq \overline{\psi}_l(\{\mathfrak{f},\mathfrak{g}\}) \neq \overline{\psi}_u(\{\mathfrak{f},\mathfrak{g}\})$.

In the following implications, several kinds of MN_j−approximations operators with a reflexive relation ψ and $\mathfrak{A} \subseteq \mathfrak{U}$ are compared.



Theorem 28. *If ψ is a reflexive relation and $\mathfrak{A} \subseteq \mathfrak{U}$, then*

(i) $B_i(\mathfrak{A}) \subseteq B_r(\mathfrak{A}) \subseteq B_u(\mathfrak{A})$

(ii) $B_i(\mathfrak{A}) \subseteq B_l(\mathfrak{A}) \subseteq B_u(\mathfrak{A})$

Proof. (i) Consider $y \in B_i(\mathfrak{A})$. Thus, $y \in \overline{\psi}_i(\mathfrak{A})$ and $y \notin \underline{\psi}_i(\mathfrak{A})$. By using Theorem 30, $y \in \overline{\psi}_r(\mathfrak{A})$ and $y \notin \underline{\psi}_r(\mathfrak{A})$. Hence, $y \in B_r(\mathfrak{A})$. Therefore, $B_i(\mathfrak{A}) \subseteq B_r(\mathfrak{A})$. Now, let $y \in B_r(\mathfrak{A})$. Then, $y \in \overline{\psi}_r(\mathfrak{A})$ and $y \notin \underline{\psi}_r(\mathfrak{A})$. By using Theorem 27, $y \in \overline{\psi}_u(\mathfrak{A})$ and $y \notin \underline{\psi}_u(\mathfrak{A})$. Hence, $y \in B_u(\mathfrak{A})$. Therefore, $B_r(\mathfrak{A}) \subseteq B_u(\mathfrak{A})$. By the same manner, (ii) is verified. □

Corollary 29. *Let ψ be a reflexive relation and $\mathfrak{A} \subseteq \mathfrak{U}$. Then,*

(i) $\xi_u(\mathfrak{A}) \leqslant \xi_r(\mathfrak{A}) \leqslant \xi_i(\mathfrak{A})$

(ii) $\xi_u(\mathfrak{A}) \leqslant \xi_l(\mathfrak{A}) \leqslant \xi_i(\mathfrak{A})$

The proof of Theorem 30 is clear. So, the proof is omitted.

Theorem 30. *Let ψ be a reflexive relation and $\mathfrak{A} \subseteq \mathfrak{U}$. Then,*

(i) *$\mathfrak{A}$ is MN_u−exact $\Rightarrow \mathfrak{A}$ is MN_r−exact $\Rightarrow \mathfrak{A}$ is MN_i−exact*

(ii) *$\mathfrak{A}$ is MN_u−exact $\Rightarrow \mathfrak{A}$ is MN_l−exact $\Rightarrow \mathfrak{A}$ is MN_i−exact*

The equality in Theorem 28 and Corollary 29 does not hold, in general.

Example 9. In Example 5 and Tables 2 and 3, $B_i(\{\mathfrak{e}\}) \neq B_r(\{\mathfrak{e}\}) \neq B_u(\{\mathfrak{e}\})$, $B_i(\{\mathfrak{e}\}) \neq B_l(\{\mathfrak{e}\}) \neq B_u(\{\mathfrak{e}\})$, $\xi_u(\{\mathfrak{h}\}) \neq \xi_r(\{\mathfrak{h}\}) \neq \xi_i(\{\mathfrak{h}\})$, and $\xi_u(\{\mathfrak{e}\}) \neq \xi_l(\{\mathfrak{e}\}) \neq \xi_i(\{\mathfrak{e}\})$.

The converse in Theorem 30 is not true, in general.

Example 10. In Example 5 and Tables 2 and 3, we have $\{\mathfrak{h}\}$ is MN_i−exact, but $\{\mathfrak{h}\}$ is neither MN_r−exact nor MN_u−exact. Also, $\{\mathfrak{f}\}$ is MN_r−exact, but $\{\mathfrak{f}\}$ is not MN_u−exact. Furthermore, $\{\mathfrak{h}\}$ is MN_i−exact, but $\{\mathfrak{h}\}$ is neither MN_l−exact nor MN_u−exact. Finally, $\{\mathfrak{g}\}$ is MN_l−exact, but $\{\mathfrak{g}\}$ is not MN_u−exact.

4. Topological Spaces Induced by Minimal Neighborhoods

In this section, various topologies are created by using the minimal of neighborhoods. The comparison between these new types of topologies is studied.

It is easy to prove the conditions of topology for the class τ_j in Theorem 31, so the proof must be omitted.

Theorem 31. *If $(\mathfrak{U}, \psi, MN_j)$ is MN_j−approximation space and ψ is a binary relation, then the families $\tau_j = \{\mathfrak{B} \subseteq \mathfrak{U}\colon MN_j(x) \subseteq \mathfrak{B}, x \in \mathfrak{B}\}$ are topologies on $\mathfrak{U}$, for all $j \in \mathfrak{J}$.*

Example 11. In Example 1, we have

$$
\begin{aligned}
\tau_r &= \{\mathfrak{U}, \phi, \{\mathfrak{f}\}, \{\mathfrak{h}\}, \{\mathfrak{e},\mathfrak{f}\}, \{\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{f},\mathfrak{h}\}, \{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}, \{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}\},\\
\tau_l &= \{\mathfrak{U}, \phi, \{\mathfrak{e}\}, \{\mathfrak{h}\}, \{\mathfrak{e},\mathfrak{h}\}, \{\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}\},\\
\tau_u &= \{\mathfrak{U}, \phi, , \{\mathfrak{h}\}\{\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}\},\\
\tau_r &= \{\mathfrak{U}, \phi, \{\mathfrak{e}\}, \{\mathfrak{f}\}, \{\mathfrak{h}\}, \{\mathfrak{e},\mathfrak{f}\}, \{\mathfrak{e},\mathfrak{h}\}, \{\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{f},\mathfrak{h}\}, \{\mathfrak{e},\mathfrak{f},\mathfrak{g}\}, \{\mathfrak{e},\mathfrak{f},\mathfrak{h}\}, \{\mathfrak{f},\mathfrak{g},\mathfrak{h}\}\}.
\end{aligned}
\tag{2}
$$

Theorem 32. *If τ_j are topologies and ψ is a binary relation, then*

(i) $\tau_u \subseteq \tau_r \subseteq \tau_i$

(ii) $\tau_u \subseteq \tau_l \subseteq \tau_i$

Proof. If $\mathfrak{B} \in \tau_u$, then $MN_u(x) \subseteq \mathfrak{B}$ for all $x \in \mathfrak{B}$. However, $MN_u(x) = MN_r(x) \cup MN_l(x)$ and then $MN_r(x) \subseteq \mathfrak{B}$ for all $x \in \mathfrak{B}$. Hence, $\mathfrak{B} \in \tau_r$. Therefore, $\tau_u \subseteq \tau_r$. Now, let $\mathfrak{B} \in \tau_r$, then $MN_r(x) \subseteq \mathfrak{B}$ for all $x \in \mathfrak{B}$. However, $MN_i(x) = MN_r(x) \cap MN_l(x)$ and then $MN_i(x) \subseteq \mathfrak{B}$ for all $x \in \mathfrak{B}$. Hence, $\mathfrak{B} \in \tau_i$. Therefore, $\tau_r \subseteq \tau_i$. By the same manner, the proof is true for (ii).

The equality in Theorem 32 is not true, in general. □

Example 12. In Example 11, we have $\tau_u \neq \tau_r \neq \tau_l \neq \tau_i$.

From Theorem 32, it is easy to prove Theorem 33. So, we omit the proof.

Theorem 33. *If τ_j are topologies and ψ is a symmetric relation, then $\tau_u = \tau_r = \tau_l = \tau_i$.*

5. Applications: COVID-19 Infections and Heart Attacks

In this section, some suggested methodologies to practical issues are presented and particularly in the area of patient diagnostics where more precise judgments are required. Therefore, medical applications are examined for the proposed approximations characteristics in terms of minimal neighborhoods. These examples show how the generalization of rough sets using minimal neighborhoods can effectively handle and represent a variety of real-world issues. It is illustrated that the utilization of minimal neighborhoods in the context of RST aids in the elimination of data uncertainty and ambiguity.

Example 13. This example aims to illustrate the significance of present approximations in order to obtain the best tools for determining the major components of COVID-19 infections in humans. The World Health Organization and medical organizations with expertise in COVID-19 gathered the data in Table 4 [27]. Due to the similar properties in the rows (objects), data from 500 patients were reduced to 10 patients. Therefore, the set of objects is $\mathfrak{U} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9, \mathfrak{q}_{10}\}$.

The attributes (most common symptoms) of COVID-19 are given as follows: {Difficulty breathing = $\mathfrak{s}_1$, Chest pain = $\mathfrak{s}_2$, High Headache = $\mathfrak{s}_3$, Dry cough = $\mathfrak{s}_4$, Temperature = $\mathfrak{s}_5$, Loss of smell or taste = $\mathfrak{s}_6$} and Decision COVID-19, as shown in Table 4.

From Table 4, the symptoms are given as follows: $F(\mathfrak{q}_1) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6\}$, $F(\mathfrak{q}_2) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6\}$, $F(\mathfrak{q}_3) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_4, \mathfrak{s}_6\}$, $F(\mathfrak{q}_4) = \{\mathfrak{s}_1, \mathfrak{s}_2\}$, $F(\mathfrak{q}_5) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_4\}$, $F(\mathfrak{q}_6) = \{\mathfrak{s}_1, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$, $F(\mathfrak{q}_7) = F(\mathfrak{q}_8) = \{\mathfrak{s}_4, \mathfrak{s}_5\}$, $F(\mathfrak{q}_9) = \{\mathfrak{s}_6\}$, and $F(\mathfrak{q}_{10}) = \{\mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$.

The relation is given as follows: $\mathfrak{q}_n \psi \mathfrak{q}_m \Leftrightarrow F(\mathfrak{q}_n) \subseteq F(\mathfrak{q}_m)$. Consequently, $\psi = \{(\mathfrak{q}_1, \mathfrak{q}_1), (\mathfrak{q}_1, \mathfrak{q}_2), (\mathfrak{q}_2, \mathfrak{q}_2), (\mathfrak{q}_3, \mathfrak{q}_1), (\mathfrak{q}_3, \mathfrak{q}_2), (\mathfrak{q}_3, \mathfrak{q}_3), (\mathfrak{q}_4, \mathfrak{q}_1), (\mathfrak{q}_4, \mathfrak{q}_2), (\mathfrak{q}_4, \mathfrak{q}_3), (\mathfrak{q}_4, \mathfrak{q}_4), (\mathfrak{q}_4, \mathfrak{q}_5), (\mathfrak{q}_5, \mathfrak{q}_1), (\mathfrak{q}_5, \mathfrak{q}_2), (\mathfrak{q}_5, \mathfrak{q}_3)\}, (\mathfrak{q}_5, \mathfrak{q}_5), (\mathfrak{q}_6, \mathfrak{q}_2), (\mathfrak{q}_6, \mathfrak{q}_6), (\mathfrak{q}_7, \mathfrak{q}_1), (\mathfrak{q}_7, \mathfrak{q}_2), (\mathfrak{q}_7, \mathfrak{q}_6), (\mathfrak{q}_7, \mathfrak{q}_7), (\mathfrak{q}_7, \mathfrak{q}_8), (\mathfrak{q}_7, \mathfrak{q}_{10}), (\mathfrak{q}_8, \mathfrak{q}_1), (\mathfrak{q}_8, \mathfrak{q}_2), (\mathfrak{q}_8, \mathfrak{q}_6), (\mathfrak{q}_8, \mathfrak{q}_7), (\mathfrak{q}_8, \mathfrak{q}_8), (\mathfrak{q}_8, \mathfrak{q}_{10}), (\mathfrak{q}_9, \mathfrak{q}_1), (\mathfrak{q}_9, \mathfrak{q}_2), (\mathfrak{q}_9, \mathfrak{q}_3), (\mathfrak{q}_9, \mathfrak{q}_9), (\mathfrak{q}_{10}, \mathfrak{q}_2), (\mathfrak{q}_{10}, \mathfrak{q}_6)$, and $(\mathfrak{q}_{10}, \mathfrak{q}_{10})$.

Then, $\mathrm{RN}(\mathfrak{U}, \psi) = \{\mathfrak{q}_1, \mathfrak{q}_2\}, \{\mathfrak{q}_2\}, \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}, \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5\}, \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_5\}, \{\mathfrak{q}_2, \mathfrak{q}_6\}, \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}, \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_9\}, \{\mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_{10}\}$, and $\mathrm{LN}(\mathfrak{U}, \psi) = \{\{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9\}, \mathfrak{U}, \{\mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_9\}, \{\mathfrak{q}_4\}, \{\mathfrak{q}_4, \mathfrak{q}_5\}, \{\mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}, \{\mathfrak{q}_7, \mathfrak{q}_8\}, \{\mathfrak{q}_9\}, \{\mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}\}$.

Hence, $MN_r(\mathfrak{q}_1) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$, $MN_r(\mathfrak{q}_2) = \{\mathfrak{q}_2\}$, $MN_r(\mathfrak{q}_3) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$, $MN_r(\mathfrak{q}_4) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5\}$, $MN_r(\mathfrak{q}_5) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_5\}$, $MN_r(\mathfrak{q}_6) = \{\mathfrak{q}_2, \mathfrak{q}_6\}$, $MN_r(\mathfrak{q}_7) = MN_r(\mathfrak{q}_8) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, $MN_r(\mathfrak{q}_9) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_9\}$, $MN_r(\mathfrak{q}_{10}) = \{\mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_{10}\}$, $MN_l(\mathfrak{q}_1) = \{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9\}$, $MN_l(\mathfrak{q}_2) = \mathfrak{U}$, $MN_l(\mathfrak{q}_3) = \{\mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_9\}$, $MN_l(\mathfrak{q}_4) = \{\mathfrak{q}_4\}$, $MN_l(\mathfrak{q}_5) = \{\mathfrak{q}_4, \mathfrak{q}_5\}$, $MN_l(\mathfrak{q}_6) = \{\mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, $MN_l(\mathfrak{q}_7) = MN_l(\mathfrak{q}_8) = \{\mathfrak{q}_7, \mathfrak{q}_8\}$, $MN_l(\mathfrak{q}_9) = \{\mathfrak{q}_9\}$, $MN_l(\mathfrak{q}_{10}) = \{\mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, $MN_u(\mathfrak{q}_1) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9\}$, $MN_u(\mathfrak{q}_2) = \mathfrak{U}$, $MN_u(\mathfrak{q}_3) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_9\}$, $MN_u(\mathfrak{q}_4) = MN_u(\mathfrak{q}_5) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5\}$, $MN_u(\mathfrak{q}_6) = \{\mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, $MN_u(\mathfrak{q}_7) = MN_u(\mathfrak{q}_8) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, $MN_u(\mathfrak{q}_9) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_9\}$, $MN_u(\mathfrak{q}_{10}) = \{\mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, $MN_i(\mathfrak{q}_1) = \{\mathfrak{q}_1\}$, $MN_i(\mathfrak{q}_2) = \{\mathfrak{q}_2\}$, $MN_i(\mathfrak{q}_3) = \{\mathfrak{q}_3\}$, $MN_i(\mathfrak{q}_4) = \{\mathfrak{q}_4\}$, $MN_i(\mathfrak{q}_5) = \{\mathfrak{q}_5\}$, $MN_i(\mathfrak{q}_6) = \{\mathfrak{q}_6\}$, $MN_i(\mathfrak{q}_7) = MN_i(\mathfrak{q}_8) = \{\mathfrak{q}_7, \mathfrak{q}_8\}$, $MN_i(\mathfrak{q}_9) = \{\mathfrak{q}_9\}$, and $MN_i(\mathfrak{q}_{10}) = \{\mathfrak{q}_{10}\}$.

Patients with confirmed COVID-19 infections are $\mathfrak{A} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_9, \mathfrak{q}_{10}\}$ and then (i) $\underline{\psi}_r(\mathfrak{A}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_{10}\}$, $\overline{\psi}_r(\mathfrak{A}) = \mathfrak{U}$, $B_r(\mathfrak{A}) = \{\mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9\}$, and $\xi_r(\mathfrak{A}) = 2/5$; (ii) $\underline{\psi}_l(\mathfrak{A}) = \{\mathfrak{q}_9\}$, $\overline{\psi}_l(\mathfrak{A}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9, \mathfrak{q}_{10}\}$, $B(\mathfrak{A})_l = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_{10}\}$, and $\xi_l(\mathfrak{A}) = 1/8$; (iii) $\underline{\psi}_u(\mathfrak{A}) = \phi$, $\overline{\psi}_u(\mathfrak{A}) = \mathfrak{U}$, $B(\mathfrak{A})_u = \mathfrak{U}$, and $\xi_u(\mathfrak{A}) = 0$; and (iv) $\underline{\psi}_i(\mathfrak{A}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_9, \mathfrak{q}_{10}\}$, $\overline{\psi}_i(\mathfrak{A}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9, \mathfrak{q}_{10}\}$, $B(\mathfrak{A})_i = \{\mathfrak{q}_7, \mathfrak{q}_8\}$, and $\xi_i(\mathfrak{A}) = 5/7$.

According to the proposed fourth type, patients $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_6, \mathfrak{q}_9$, and $\mathfrak{q}_{10}$ are unquestionably infected with COVID-19 using the current technique, as shown in Table 4.

Example 14. The presented methodologies are used to make decisions on heart attacks. Table 5 shows the set of objects (patients) as $\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_6, \mathfrak{q}_7, \mathfrak{q}_8, \mathfrak{q}_9, \mathfrak{q}_{10}\}$ collected from Al-Azhar University's cardiology department (Hospital of Sayed Glal University, Cairo, Egypt) [11]. It was shortened to $\mathfrak{U} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_8, \mathfrak{q}_9\}$ since the properties in rows (objects) are same. The study covered patients with a variety of symptoms, and the set of attributes = {Breathlessness = $\mathfrak{s}_1$, Orthopnea = $\mathfrak{s}_2$, Paroxysmal nocturnal dyspnea = $\mathfrak{s}_3$, Reduced exercise tolerance = $\mathfrak{s}_4$, Ankle swelling = $\mathfrak{s}_5$} and decision of heart attacks is ruled out or confirmed = D as illustrated in Table 5.

From Table 5, the symptoms are given as follows: $F(\mathfrak{q}_1) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4\}$, $F(\mathfrak{q}_2) = \{\mathfrak{s}_4, \mathfrak{s}_5\}$, $F(\mathfrak{q}_3) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$, $F(\mathfrak{q}_4) = \{\mathfrak{s}_4\}$, $F(\mathfrak{q}_5) = \{\mathfrak{s}_1, \mathfrak{s}_4, \mathfrak{s}_5\}$, $F(\mathfrak{q}_8) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_4, \mathfrak{s}_5\}$, and $F(\mathfrak{q}_9) = \{\mathfrak{s}_1, \mathfrak{s}_3, \mathfrak{s}_4\}$.

The relation is given as follows: $\mathfrak{q}_n \psi \mathfrak{q}_m \Leftrightarrow F(\mathfrak{q}_n) \subseteq F(\mathfrak{q}_m)$. Consequently, $\psi = \{(\mathfrak{q}_1, \mathfrak{q}_1), (\mathfrak{q}_1, \mathfrak{q}_3), (\mathfrak{q}_2, \mathfrak{q}_2), (\mathfrak{q}_2, \mathfrak{q}_3), (\mathfrak{q}_2,$

Table 4: Decision data set.

Patients	Serious symptoms		Most common symptoms				Decision COVID-19
	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$	$\mathfrak{s}_5$	$\mathfrak{s}_6$	
$\{\mathfrak{q}_1\}$	Yes	Yes	No	Yes	Yes	Yes	Yes
$\{\mathfrak{q}_2\}$	Yes	Yes	Yes	Yes	Yes	Yes	Yes
$\{\mathfrak{q}_3\}$	Yes	Yes	No	Yes	No	Yes	No
$\{\mathfrak{q}_4\}$	Yes	Yes	No	No	No	No	No
$\{\mathfrak{q}_5\}$	Yes	Yes	No	Yes	No	No	No
$\{\mathfrak{q}_6\}$	Yes	No	Yes	Yes	Yes	No	Yes
$\{\mathfrak{q}_7\}$	No	No	No	Yes	Yes	No	Yes
$\{\mathfrak{q}_8\}$	No	No	No	Yes	Yes	No	No
$\{\mathfrak{q}_9\}$	No	No	No	No	No	Yes	Yes
$\{\mathfrak{q}_{10}\}$	No	No	Yes	Yes	Yes	No	Yes

Table 5: Decision information data set.

Patients	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$	$\mathfrak{s}_5$	D
$\{\mathfrak{q}_1\}$	Yes	Yes	Yes	Yes	No	Yes
$\{\mathfrak{q}_2\}$	No	No	No	Yes	Yes	No
$\{\mathfrak{q}_3\}$	Yes	Yes	Yes	Yes	Yes	Yes
$\{\mathfrak{q}_4\}$	No	No	No	Yes	No	No
$\{\mathfrak{q}_5\}$	Yes	No	No	Yes	Yes	No
$\{\mathfrak{q}_8\}$	Yes	Yes	No	Yes	Yes	Yes
$\{\mathfrak{q}_9\}$	Yes	No	Yes	Yes	No	Yes

$\mathfrak{q}_5)$, $(\mathfrak{q}_2, \mathfrak{q}_8)$, $(\mathfrak{q}_3, \mathfrak{q}_3)$, $(\mathfrak{q}_4, \mathfrak{q}_1)$, $(\mathfrak{q}_4, \mathfrak{q}_2)$, $(\mathfrak{q}_4, \mathfrak{q}_3)$, $(\mathfrak{q}_4, \mathfrak{q}_4)$, $(\mathfrak{q}_4, \mathfrak{q}_5)$, $(\mathfrak{q}_4, \mathfrak{q}_8)\}$, $(\mathfrak{q}_4, \mathfrak{q}_9)$, $(\mathfrak{q}_5, \mathfrak{q}_3)$, $(\mathfrak{q}_5, \mathfrak{q}_5)$, $(\mathfrak{q}_5, \mathfrak{q}_8)$, $(\mathfrak{q}_8, \mathfrak{q}_3)$, $(\mathfrak{q}_8, \mathfrak{q}_8)$, $(\mathfrak{q}_9, \mathfrak{q}_1)$, $(\mathfrak{q}_9, \mathfrak{q}_3)$, and $(\mathfrak{q}_9, \mathfrak{q}_9)$.

Then, $\mathrm{RN}(\mathfrak{U}, \psi) = \{\{\mathfrak{q}_1, \mathfrak{q}_3\}, \{\mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_5, \mathfrak{q}_8\}, \mathfrak{U}, \{\mathfrak{q}_3, \mathfrak{q}_5, \mathfrak{q}_8\}, \{\mathfrak{q}_3\}, \{\mathfrak{q}_3, \mathfrak{q}_8\}, \{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_9\}\}$ and $\mathrm{LN}(\mathfrak{U}, \psi) = \{\{\mathfrak{q}_1, \mathfrak{q}_4, \mathfrak{q}_9\}, \{\mathfrak{q}_2, \mathfrak{q}_4\}, \mathfrak{U}, \{\mathfrak{q}_4\}, \{\mathfrak{q}_2, \mathfrak{q}_4, \mathfrak{q}_5\}, \{\mathfrak{q}_2, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_8\}, \{\mathfrak{q}_4, \mathfrak{q}_9\}\}$.

Hence, $MN_r(\mathfrak{q}_1) = \{\mathfrak{q}_1, \mathfrak{q}_3\}$, $MN_r(\mathfrak{q}_2) = \{\mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_5, \mathfrak{q}_8\}$, $MN_r(\mathfrak{q}_3) = \{\mathfrak{q}_3\}$, $MN_r(\mathfrak{q}_4) = \mathfrak{U}$, $MN_r(\mathfrak{q}_5) = \{\mathfrak{q}_3, \mathfrak{q}_5, \mathfrak{q}_8\}$, $MN_r(\mathfrak{q}_8) = \{\mathfrak{q}_3, \mathfrak{q}_8\}$, $MN_r(\mathfrak{q}_9) = \{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_9\}$, $MN_l(\mathfrak{q}_1) = \{\mathfrak{q}_1, \mathfrak{q}_4, \mathfrak{q}_9\}$, $MN_l(\mathfrak{q}_2) = \{\mathfrak{q}_2, \mathfrak{q}_4\}$, $MN_l(\mathfrak{q}_3) = \mathfrak{U}$, $MN_l(\mathfrak{q}_4) = \{\mathfrak{q}_4\}$, $MN_l(\mathfrak{q}_5) = \{\mathfrak{q}_2, \mathfrak{q}_4, \mathfrak{q}_5\}$, $MN_l(\mathfrak{q}_8) = \{\mathfrak{q}_2, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_8\}$, $MN_l(\mathfrak{q}_9) = \{\mathfrak{q}_4, \mathfrak{q}_9\}$, $MN_u(\mathfrak{q}_1) = MN_u(\mathfrak{q}_9) = \{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_9\}$, $MN_u(\mathfrak{q}_2) = MN_u(\mathfrak{q}_5) = MN_u(\mathfrak{q}_8) = \{\mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4, \mathfrak{q}_5, \mathfrak{q}_8\}$, $MN_u(\mathfrak{q}_3) = MN_u(\mathfrak{q}_4) = \mathfrak{U}$, $MN_i(\mathfrak{q}_1) = \{\mathfrak{q}_1\}$, $MN_i(\mathfrak{q}_2) = \{\mathfrak{q}_2\}$, $MN_i(\mathfrak{q}_3) = \{\mathfrak{q}_3\}$, $MN_i(\mathfrak{q}_4) = \{\mathfrak{q}_4\}$, $MN_i(\mathfrak{q}_5) = \{\mathfrak{q}_5\}$, $MN_i(\mathfrak{q}_8) = \{\mathfrak{q}_8\}$, and $MN_i(\mathfrak{q}_9) = \{\mathfrak{q}_9\}$.

In Table 5, patients with confirmed heart attacks are $\mathfrak{A} = \{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_8, \mathfrak{q}_9\}$. Then,

(i) $\underline{\psi}_r(\mathfrak{A}) = \mathfrak{A}$, $\overline{\psi}_r(\mathfrak{A}) = \mathfrak{U}$, $B_r(\mathfrak{A}) = \{\mathfrak{q}_2, \mathfrak{q}_4, \mathfrak{q}_5\}$, and $\xi_r(\mathfrak{A}) = 4/7$

(ii) $\underline{\psi}_l(\mathfrak{A}) = \phi$, $\overline{\psi}_l(\mathfrak{A}) = \mathfrak{A}$, $B_l(\mathfrak{A} = \mathfrak{A})$, and $\xi_l(\mathfrak{A}) = 0$

(iii) $\underline{\psi}_u(\mathfrak{A}) = \phi$, $\overline{\psi}_u(\mathfrak{A}) = \mathfrak{U}$, $B_u(\mathfrak{A}) = \mathfrak{U}$, and $\xi_u(\mathfrak{A}) = 0$

(iv) $\underline{\psi}_i(\mathfrak{A}) = \mathfrak{A}$, $\overline{\psi}_i(\mathfrak{A}) = \mathfrak{A}$, $B_i(\mathfrak{A}) = \phi$, and $\xi_i(\mathfrak{A}) = 1$

From the proposed first and fourth types, the patients $\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_8$, and $\mathfrak{q}_9$ have certainly undergone heart attacks, which is consistent with Table 5. In addition, the fourth type is a best accuracy and the topology which constructed by MN_i is the best choice for decision making.

6. Conclusion and Future Work

The current paper examines four different kinds of generalization for rough sets which contain four different types of lower and upper approximations that construct by minimal neighborhoods. The properties of these approximations are discussed. Many comparisons have been made between our generalization and other generalizations. The approximation operators pave the way for additional topological advances to RST and applications. There are four topologies established through our study. Medical applications are shown in two examples and used for decision making. In a future work, these results can be studied in bi-neighborhoods. Also, this research will be helpful and will open new doors in the study of topologies which approach rough sets through minimal neighborhoods, as well as applications for graphs [2] and in the study of minimal neighborhoods as applications of these new concepts.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors completed this study and wrote and approved the final version of the manuscript.

References

[1] J. Keley, *General Topology*, Van Nostrand Company, New York, NY, USA, 1955.

[2] A. A. El-Atik and A. A. Nasef, "Some topological structures of fractals and their related graphs," *Filomat*, vol. 34, no. 1, pp. 1–24, 2020.

[3] C. Largeron and S. Bonnevay, "A pretopological approach for structural analysis," *Information Sciences*, vol. 144, no. 1-4, pp. 169–185, 2002.

[4] B. M. R. Stadler and P. F. Stadler, "Generalized topological spaces in evolutionary theory and combinatorial chemistry," *Journal of Chemical Information and Computer Sciences*, vol. 42, no. 3, pp. 577–585, 2002.

[5] A. S. Nawar and A. E. F. A. El Atik, "A model of a human heart via graph nano topological spaces," *International Journal of Biomathematics*, vol. 12, no. 01, Article ID 1950006, 2019.

[6] A. Galton, "A generalized topological view of motion in discrete space," *Theoretical Computer Science*, vol. 305, no. 1-3, pp. 111–134, 2003.

[7] M. K. El-Bably and A. E. F. A. El Atik, "Soft β-rough sets and their application to determine COVID-19," *Turkish Journal of Mathematics*, vol. 45, no. 3, pp. 1133–1148, 2021.

[8] E. Lashin, A. Kozae, A. Abo Khadra, and T. Medhat, "Rough set theory for topological spaces," *International Journal of Approximate Reasoning*, vol. 40, no. 1-2, pp. 35–43, 2005.

[9] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[10] M. M. El-Sharkasy, "Minimal structure approximation space and some of its application," *Journal of Intelligent and Fuzzy Systems*, vol. 40, no. 1, pp. 973–982, 2021.

[11] A. A. Azzam, A. M. Khalil, and S. G. Li, "Medical applications via minimal topological structure," *Journal of Intelligent and Fuzzy Systems*, vol. 39, no. 3, pp. 4723–4730, 2020.

[12] Y. Yao, "Two views of the theory of rough sets in finite universes," *International Journal of Approximate Reasoning*, vol. 15, no. 4, pp. 291–317, 1996.

[13] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, *Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing*, Springer, Berlin, Germany, 2005.

[14] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, "New approach for closure spaces by relations," *Acta Mathematica Academiae Paedagogicae Nyregyhziensis*, vol. 22, no. 3, pp. 285–304, 2006.

[15] T. M. Al-Shami, W. Q. Fu, and E. A. Abo-Tabl, "New rough approximations based on E-neighborhoods," *Complexity*, vol. 2021, Article ID 6666853, 6 pages, 2021.

[16] T. M. Al-Shami, "An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application," *Information Sciences*, vol. 569, no. 2021, pp. 110–124, 2021.

[17] T. M. Al-Shami, "Maximal rough neighborhoods with a medical application," *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, pp. 16373–16384, 2022.

[18] A. M. Kozae, S. A. El-Sheikh, E. A. Aly, and M. Hosny, "Rough sets and its applications in a computer network," *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 3, pp. 605–624, 2013.

[19] A. S. Salama, "Topological solution of missing attribute values problem in incomplete information tables," *Information Sciences*, vol. 180, no. 5, pp. 631–639, 2010.

[20] T. Y. Lin, *Topological and Fuzzy Rough Sets, Intelligent Decision Support*, Springer, Berlin, Germany, 1992.

[21] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*, Springer Science & Business Media, Berlin, Germany, 1991.

[22] B. De Baets and E. Kerre, "A revision of Bandler-Kohout compositions of relations," *Journal of Mathematica Pannonica*, vol. 39, pp. 59–78, 1993.

[23] Y. Yao and T. Lin, "Generalization of rough sets using modal logics," *Intelligent Automation and Soft Computing*, vol. 2, no. 2, pp. 103–119, 1996.

[24] A. A. El-Atik, M. E. Abd El-Monsef, and E. I. Lashin, "On finite T_0 topological spaces," in *Proceeding of Ninth Prague Topological Symposium Contributed papers from the Symposium held in Prague*, Czech Republic, August, 2001.

[25] Z. Yu, X. Bai, and Z. Yun, "A study of rough sets based on 1-neighborhood systems," *Information Sciences*, vol. 248, pp. 103–113, 2013.

[26] R. Mareay, "Generalized rough sets based on neighborhood systems and topological spaces," *Journal of the Egyptian Mathematical Society*, vol. 24, no. 4, pp. 603–608, 2016.

[27] M. A. El-Safty and S. Al-Zahrani, "Topological modeling for symptom reduction of Corona virus," *Punjab University Journal of Mathematics*, vol. 53, no. 3, pp. 1–15, 2021.