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Research Article

Study of Nonlinear Second-Order Differential Inclusion Driven by a Φ – Laplacian Operator Using the Lower and Upper Solutions Method

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In this paper, we study a second-order differential inclusion under boundary conditions governed by maximal monotone multivalued operators. These boundary conditions incorporate the classical Dirichlet, Neumann, and Sturm–Liouville problems. Our method of study combines the method of lower and upper solutions, the analysis of multivalued functions, and the theory of monotone operators. We show the existence of solutions when the lower solution σ and the upper solution γ are well ordered. Next, we show how our arguments of proof can be easily exploited to establish the existence of extremal solutions in the functional interval $[\sigma, \gamma]$. We also show that our method can be applied to the periodic case.

1. Introduction

We consider the nonlinear second-order problem

$$\begin{cases} (\Phi(\Theta(x)z'(x)))' \in A(z(x)) + H(x, z(x), z'(x)) + g(z(x)) \ a.e \text{ on } \Pi = [0, \alpha], \\ \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)), \end{cases} \tag{1}$$

where $A\colon \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a subdifferential of a lower semicontinuous, proper, and convex function which are not identically equal to $+\infty$, $H\colon \Pi \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ is L^p-Caratheodory multifunction, $p \geq 2$, $G_i\colon \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, $i = 1, 2$, is a maximal monotone operator, $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a not necessarily continuous map, $\Phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a monotone homeomorphism, and $\Theta\colon \Pi \longrightarrow \mathbb{R}$ is a continuous and positive function.

Introduced by Picard in 1890 in [1, 2], the method of lower and upper solutions is still intensely used today to establish existence and multiplicity results for boundary problems of second or other order. For example, in [3–5], it was combined, respectively, with the topological degree theory, a fixed point theorem for ordered Banach spaces and fixed point index theory to establish existence and multiplicity results for nonlinear second-order differential equations while in [6], it is used to study the fractional evolution equation with order σ, $1<\sigma<2$. Also, recently in [7, 8], the method has been extended, respectively, to semilinear and generalized second-order random impulsive differential problem driven, respectively, by the scalar Laplacian operator $u\mapsto u''$ and $u\mapsto (p(.)u')'$ with linear boundary conditions. Much earlier, Frigon [9, 10] generalized this method to differential inclusions but her study was only focused on a semilinear problem with linear boundary conditions. It was followed by other authors such as Bader-Papageorgiou [11] and Staicu-Papageorgiou [12] who worked only with a nonlinear homogeneous differential operator, the p-Laplacian operator $u\mapsto (|u'|^{p-2}u')'$ with nonlinear and multivalued boundary conditions that encompass the Dirichlet, Neumann, and Sturm–Liouville problems. They also show that their method stay true for the periodic problems but like the references given above, their work does not include variational inequalities.

The aim of this paper is to extend the aforementioned works to a large class of problems incorporating the operators used and variational inequalities with nonlinear and multivalued boundary conditions. At this end, we deal with a nonhomogeneous and nonlinear differential operator, the $\Phi-$ Laplacian operator $u\mapsto (\Phi(\Theta(.)u'))'$, in a problem that incorporates variational inequalities. Our proof is based on a fixed point theorem for ordered Banach spaces due to Heikilla and Hu [13].

The $\Phi-$ Laplacian operator under consideration applies to several areas such as nonlinear elasticity, non-Newtonian fluid theory, theory of capillary surfaces, and diffusion of flows in porous media (see [14]). As for differential inclusions, they arise in the mathematical modelling of certain problems in the control theory, optimisation, mathematical economics, sweeping process, stochastic analysis, and many other fields (see [15–17]). Finally, variational inequalities models many applied problems, such as differential, Nash games electrical circuits with ideal diodes, dynamic traffic networks and hybrid engineering systems with variable structures, and Coulomb friction for contacting bodies (see [18]).

2. Notations and Preliminaries

Here, we will take stock of the notations and results we will be using in the rest of the article. Our main sources are the books of Hu-Papageorgiou [19] and Zeidler [20].

We denote by (Π,Σ,ζ) a finite measure space and U a separable Banach space; $\mathscr{P}(U)\backslash\{\varnothing\}$ is the set of nonempty parts of U; $B(U)$ is the Borel $\sigma-$ field of U; $\mathscr{P}_{nf}(U)$ is the set of nonempty and closed parts of U; $\mathscr{P}_{nfc}(U)$ is the set of nonempty, closed, and convex parts of U; $\mathscr{P}_{nwkc}(U)$ is the set of nonempty, weakly compact, and convex parts of U; H' is a multifunction defined on Π with values in $\mathscr{P}_{nf}(U)\backslash\{\varnothing\}$; δ is a positive real function defined by, for any element t of U, for any element z of Π, $\delta(t,z)=d(t,H'(z))=\inf\{\|t-u\|\colon u\in H'(z)\}$; H is a multifunction defined on Π with values in $\mathscr{P}(U)\backslash\{\varnothing\}$, Gr$H$ is the graph of the multifunction H, i.e., the set of pairs (u,w) belonging to $\Pi\times U$ such that w belongs to $H(u)$; S_H^P is the set of functions belonging to $L^p(\Pi,U)$ such that for almost all element w of Π, $f(w)$ belongs to $H(w)$, with $2\le p\le +\infty$; V and W are Hausdorff topological spaces; C is a closed subset of W; I is a multifunction defined on V with values in $\mathscr{P}(W)\backslash\{\varnothing\}$; and $I^-(C)$ is the set of elements v of V such that $I(v)\cap C\neq\varnothing$.

If $I^-(C)$ is closed, we say that I is upper semicontinuous (usc in the abbreviated form). If I has closed values and W is regular, we say that I has a closed graph. If I is locally compact, then I has a closed graph which implies that I has closed values.

If for all $t\in U$ and for all $z\in\Pi$, $\delta(t,z)$ is $\Sigma-$ measurable, we say that the multifunction H' is measurable. If a multifunction of type H' is measurable, then its graph is measurable. On the other hand, the opposite is true only if Σ is $\zeta-$ complete.

GrH belongs to $\Sigma\times B(U)$. The set S_H^P may be empty. If GrH is measurable, then the set S_H^P is nonempty if only if $z\mapsto\inf\{\|u\|\colon u\in H'(z)\}$ belongs to $L^p(\Pi)_+$.

U and U' are, respectively, a reflexive Banach space and its topological dual. $\langle\rangle$ is the duality bracket between U and U'. $D(A)$ is the domain of A. Let $D(A)$ be a subset of U, then the map $A\colon D(A)\longrightarrow\mathscr{P}(U')$ is said to be

(1) monotone if for all $(u,v)\in[D(A)]^2$, for all $(u',v')\in A(u)\times A(v)$, $\langle u'-v',u-v\rangle\ge 0$;
(2) strictly monotone if A is monotone and for all $(u,v)\in[D(A)]^2$, for all $(u',v')\in A(u)\times A(v)$, $\langle u'-v',u-v\rangle=0$ leads to $u=v$;
(3) maximal monotone if A is monotone and for all $(u,u')\in D(A)\times A(u)$, $\langle u'-v',u-v\rangle=0$ leads to $(v,v')\in D(A)\times A(v)$.

The maximal monotony of A implies that for any element u of $D(A)$, the set $A(u)$ is nonempty, closed, and convex. In addition, GrA is demiclosed. This means that if the sequence $(z_n,z_n')_{n\ge1}$ is in GrA, either the sequence $(z_n)_{n\ge1}$ converges strongly to z in U and the sequence $(z_n')_{n\ge1}$ converges weakly to z' in U' or the sequence $(z_n)_{n\ge1}$ converges weakly to z in U and the sequence $(z_n')_{n\ge1}$ converge strongly to z' in U', then the pair (z,z') belongs to GrA. A map A defined on U with values in U' is said to be demicontinuous, if for every sequence $(u_n)_{n\ge1}$ that converges to u in U, we have the sequence $(A(u_n))_{n\ge1}$ that converges weakly to $A(u)$ in U'. If a map A is monotone and demicontinuous, then it is maximal monotone. If A is a map defined on $D(A)\subseteq U$ with values in $\mathscr{P}(U')$, with $D(A)\subseteq U$ bounded or $D(A)$ unbounded such that for $u\in D(A)$, $(\inf\{\langle u',u\rangle\colon u'\in A(u)\}/\|u\|_U)$ converges to $+\infty$ as $\|u\|_U$ converges to $+\infty$, then A is coercive. A maximal monotone and coercive map is surjective.

Let U_1,U_2 be Banach spaces and Ψ a map defined on U_1 with values in U_2. Ψ is said to be completely continuous if the sequence $(v_n)_{n\ge1}$ converges weakly to v in U_1; then, the sequence $(\Psi(v_n))_{n\ge1}$ converges strongly to $\Psi(v)$ in U_2 and Ψ is said to be compact if it is continuous and maps bounded sets into relatively compact sets. Complete continuity is different from compactness but if U_1 is reflexive, then complete continuity leads to compactness. In addition, if U_1

is reflexive and Ψ is linear, then the complete continuity equals compactness.

Let U be a reflexive Banach space, $\varphi: U \longrightarrow (-\infty, +\infty]$ a proper, convex, and lower semicontinuous map. Let $u \in D(\varphi)$. The subdifferential of φ at u is the multifunction $\partial\varphi: U \longrightarrow \mathscr{P}(U')$ defined by

$$\partial\varphi(u) = \{u' \in U': \langle u', v - u\rangle \leq \varphi(v) - \varphi(u) \quad \forall v \in U\}, \tag{2}$$

where $\partial\varphi$ is a maximal monotone map.

$\|y\| = (\int_0^\alpha |y(x)|^p + |y'(x)|^p)^{1/p}$ denotes the norm on $W^{1,p}(\Pi)$.

$\langle\,\rangle_0$ denotes the duality brackets for the pair $(W^{-1,q}(\Pi), W_0^{1,p}(\Pi))$.

Theorem 1. *If U and V are Banach spaces, $I: U \longrightarrow \mathscr{P}_{nwkc}(V)$ is usc from V into V_w, $F: V \longrightarrow U$ and is completely continuous and if $F^\circ I$ maps bounded sets into relatively compact sets, then one of the following statements holds:*

(a) the set $X = \{u \in U: u \in \nu(F^\circ I) \text{for some } \nu \in (0, 1)\}$ is unbounded or

(b) $F^\circ I$ has a fixed point.

To establish the existence of a solution for problem (1), we will need the following fixed point theorem for multifunctions in ordered Banach spaces due to Heikkila-Hu [13].

Theorem 2. *Let U be a separable, reflexive, and ordered Banach space and $V \subseteq U$ a nonempty and weakly closed set. Let $S: V \longrightarrow \mathscr{P}(V)\backslash\{\varnothing\}$ be a multifunction with weakly closed values. We suppose that $S(V)$ is bounded and*

(i) $Y = \{u \in V: u \leq v, \text{for some } v \in S(u)\}$ is nonempty;

(ii) If $u_1 \leq v_1, v_1 \in S(u_1)$ and $u_1 \leq u_2$, then we can find $v_2 \in S(u_2)$ such that $v_1 \leq v_2$.

Then, S has a fixed point, that means there exists $u \in V$ such that $u \in S(u)$.

2.1. Auxiliary Results. Our respective definitions of solutions, lower solution and upper solution, of problem (1) are as follows.

Definition 3. A function $z \in C^1(\Pi)$ such that $\Phi(\Theta(.)z'(.)) \in W^{1,q}(0,\alpha)$, with $(1/p) + (1/q) = 1$ and $p \geq 2$, is said to be a solution of problem (1) if it verifies

$$\begin{gathered}(\Phi(\Theta(x)z'(x)))' = j(x) + h(x) + g(z(x)) \text{a.e on } \Pi = [0,\alpha], \\ j \in S^q_{A(z(.))}, h \in S^q_{H(.,z(.),z'(.))}, \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)).\end{gathered} \tag{3}$$

Definition 4

(a) A function $\sigma \in C^1(\Pi)$ such that $\Phi(\Theta(.)\sigma'(.)) \in W^{1,q}(0,\alpha)$ is said to be a lower solution of problem (1) if there exist $j_1 \in S^q_{A(\sigma(.))}, h_1 \in S^q_{H(.,\sigma(.),\sigma'(.))}$ such that

$$\begin{cases}(\Phi(\Theta(x)\sigma'(x)))' \geq j_1(x) + h_1(x) + g(\sigma(x)) \text{a.e on } \Pi = [0,\alpha], \\ \Theta(0)\sigma'(0) \in G_1(\sigma(0)) + \mathbb{R}_+, -\Theta(\alpha)\sigma'(\alpha) \in G_2(\sigma(\alpha)) + \mathbb{R}_+.\end{cases} \tag{4}$$

(b) A function $\gamma \in C^1(\Pi)$ such that $\Phi(\Theta(.)\gamma'(.)) \in W^{1,q}(0,\alpha)$ is said to be an upper solution of problem (1) if there exist $j_2 \in S^q_{A(\gamma(.))}, h_2 \in S^q_{H(.,\gamma(.),\gamma'(.))}$ such that

$$\begin{cases}(\Phi(\Theta(x)\gamma'(x)))' \leq j_2(x) + h_2(x) + g(\gamma(x)) \text{ a.e on } \Pi = [0,\alpha], \\ \Theta(0)\gamma'(0) \in G_1(\gamma(0)) - \mathbb{R}_+, -\Theta(\alpha)\gamma'(\alpha) \in G_2(\gamma(\alpha)) - \mathbb{R}_+.\end{cases} \tag{5}$$

Our hypotheses on the data of (1) are the following:

(H_0): problem (1) admits a pair of well-ordered lower and upper solutions $\sigma \in C^1(\Pi)$ and $\gamma \in C^1(\Pi)$.

(H_Θ): Θ: $\Pi \longrightarrow \mathbb{R}$ is a continuous positive function such that there exist $m, M > 0$ satisfying

$$0 < m \leq \Theta(x) \leq M \quad \text{for every } x \in \Pi. \tag{6}$$

(H_Φ) Φ: $\mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism map such that

(a) $\Phi(0) = 0$;

(b) there exist $d_1, d_2, d_3 > 0$ such that $d_1|y|^p \leq \Phi(\Theta(x)y)\Theta(x)y \leq d_2 + d_3|y|^p$, for all $(x, y) \in \Pi \times \mathbb{R}$.

Remark 5. Any increasing homeomorphism Φ of the form, for all $u \in \mathbb{R}$, $\Phi(\Theta(x)u) = b(u)\Theta^{p-1}(x)\Phi_p(u)$ with b: $\mathbb{R} \longrightarrow]0, +\infty[$ a continuous map and $\Phi_p(u) = |u|^{p-2}u$, for all $u \in \mathbb{R}$, satisfies hypotheses (H_Φ).

(H_A): A: $\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a maximal monotone multivalued map defined by

$$A(u) = \partial J(., u) \quad \text{for all } u \in \mathbb{R}, \tag{7}$$

where J: $\Pi \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that

(i) for all $u \in \mathbb{R}, x \mapsto J(x, u)$ is measurable;

(ii) for almost all $x \in \Pi, x \mapsto J(x, u)$ is a proper, convex, and lower semicontinuous function.

(iii) for every $r > 0$, there exists $g_r \in L^q(\Pi)$ such that for a.e $x \in \Pi$ and for all $u \in \mathbb{R}$ with $|u| \leq r$ and for all $v \in A(z(x))$, we have $|v| \leq g_r(x)$.

Remark 6. There exists a nondecreasing function a such that

$$A(u) = [a(u^-), a(u^+)] \quad \text{for all } u \in \mathbb{R}, \tag{8}$$

where

$$\begin{aligned} a(u^-) &= \lim_{\epsilon \longrightarrow 0^+} a(u - \epsilon), \\ a(u^+) &= \lim_{\epsilon \longrightarrow 0^+} a(u + \epsilon). \end{aligned} \tag{9}$$

(H_H) H: $\Pi \times \mathbb{R}^2 \longrightarrow \mathcal{P}_{nfc}(\mathbb{R})$ is a multifunction such that

(i) for all $u, v \in \mathbb{R}, x \mapsto H(x, u, v)$ is a graph measurable;

(ii) for almost all $x \in \Pi$, $(u, v) \mapsto H(x, u, v)$ has a closed graph;

(iii) for almost all $x \in \Pi$, for all $(u, v) \in [\sigma(x), \gamma(x)] \times \mathbb{R}$, we can find $w \in S^p_{H(x,u,v)}$ in such a manner that

$$|w| < \kappa(|\Phi(v)|)(\lambda(x) + a|v|), \tag{10}$$

where $\lambda \in L^1(\Pi)_+, a > 0$, and κ: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+ \backslash \{0\}$ a nondecreasing function that can be measured in the Borel sense in such a manner that

$$\min\left\{\int_{\Phi(M\rho)}^{+\infty} \frac{ds}{\kappa(s)}, \int_{-\Phi(-M\rho)}^{+\infty} \frac{ds}{\kappa(s)}\right\} > \|\lambda\|_1 + a\left(\max_\Pi\{\sigma, \gamma\} - \min_\Pi\{\sigma, \gamma\}\right) + \|\beta\|_1, \tag{11}$$

with $\rho = (\max\{|\sigma(\alpha - x) - \gamma(x)|, x \in \{0, \alpha\}\}/\alpha)$ and $\beta = (\alpha/\kappa(\rho))\sup\{|j| + |g(z)|: j \in S^q_{A(z)}, \quad |z| \leq \max\{\|\sigma\|_\infty, \|\gamma\|_\infty\}\}$

(iv) for all $r > 0$, we can find $\psi_r \in L^q(\Pi)$ such that for almost all $x \in \Pi$ and for all $u, v \in \mathbb{R}$ with $|u|, |v| \leq r$ and for every $l \in H(x, u, v)$, $|l| \leq \psi_r(x)$.

Remark 7. β is measurable. In addition, Remark 1.2 and hypothesis (H_g) *lead to* $\beta \in L^q(\Pi) \subset L^1(\Pi)$. The hypothesis $(H_F)(iii)$ shows that the derivatives of the solution functions of (1) are uniformly bounded. This is the Bernstein–Nagumo–Wintner growth condition.

(H_G): for $i = 1, 2$, the map G_i: $\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a maximal monotone and $0 \in G_i(0)$.

Remark 8. We can find an increasing positive function $k_i, i = 1, 2$ such that $G_i(z) = [k_i(z-); k_i(z+)]$, where $k_i(z-) = \lim_{\varepsilon \longrightarrow 0^+} k_i(z - \epsilon)$ and $k_i(z+) = \lim_{\varepsilon \longrightarrow 0^+} k_i(z + \epsilon)$, $i = 1, 2$.

(H_g): g: $\mathbb{R} \longrightarrow \mathbb{R}$ is a not necessarily continuous function such that we can find $M > 0$ and $1 \leq l < +\infty$ in such a manner that $x \mapsto g(x) + Mx^l$ is decreasing. Also, it maps bounded sets to bounded sets.

Lemma 9. *Suppose that $z \in C^1(\Pi)$ and hypotheses (H_Θ), (H_Φ) and $(H_H)(iii)$ are satisfied and*

$$(\Phi(\Theta(x)z'(x)))' = j(x) + h(x) + g(z(x)) \; a.e \text{ on } \Pi = [0, \alpha], \tag{12}$$

with $j \in S^q_{A(z(.))}$, $f \in S^q_{H(.,z(.),z'(.))}$. *If*

$$\sigma(x) \le z(x) \le \gamma(x) \quad \text{for all } x \in \Pi, \tag{13}$$

then there exists $M^* > 0$ *that depends to* $\sigma, \gamma, \xi, \lambda, g, a, j, \Phi$ *such that* $|z'(x)| \le M^*$, *for all* $x \in \Pi$.

Proof. We set

$$\beta = \frac{1}{\kappa(\rho)} \sup\{|j| + |g(z)|: \ j \in A(z) \text{ and } |z| \le \max\{\|\sigma\|_\infty, \|\gamma\|_\infty\}\}. \tag{14}$$

By hypothesis (iii) of (H_H), we can find $M^* > M/m\rho$ such that

$$\min\left\{\int_{-\Phi(-M\rho)}^{-\Phi(-mM^*)} \frac{ds}{\kappa(s)}, \int_{\Phi(M\rho)}^{\Phi(mM^*)} \frac{ds}{\kappa(s)}\right\} > \|\kappa\|_1 + a\left(\max_\Pi\{\sigma, \gamma\} - \min_\Pi\{\sigma, \gamma\}\right) + \beta\alpha. \tag{15}$$

As in the proof of Lemma 1 of [3] or Lemma 5 of [12], we show that $|z'(x)| \le M^*$ for all $x \in \Pi$. In fact, if we reason by the absurd, we arrive at the following contradiction:

$$\max\left\{\int_{-\Phi(-M\rho)}^{-\Phi(-mM^*)} \frac{ds}{\kappa(s)}, \int_{\Phi(M\rho)}^{\Phi(mM^*)} \frac{ds}{\kappa(s)}\right\} \le \|\kappa\|_1 + a\left(\max_\Pi\{\sigma, \gamma\} - \min_\Pi\{\sigma, \gamma\}\right) + \beta\alpha. \tag{16}$$

Let us introduce, respectively, the truncation map ϱ: $\Pi \times \mathbb{R}$: $\mathbb{R} \longrightarrow \mathbb{R}^2$, the penalty function Λ: $\Pi \times \mathbb{R} \longrightarrow \mathbb{R}$, and the map Q: $\Pi \times \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})\backslash\{\varnothing\}$ defined by

$$\varrho(x, u, v) = \begin{cases} (\sigma(x), \sigma'(x)), & \text{if } u < \sigma(x), \\ (\gamma(x), \gamma'(x)), & \text{if } u > \gamma(x), \\ (u, M_0), & \text{if } \sigma(x) \le u \le \gamma(x), v > M_0, \\ (u, -M_0), & \text{if } \sigma(x) \le u \le \gamma(x), v < -M_0, \\ (u, v), & \text{if } \sigma(x) \le u \le \gamma(x), |v| \le M_0, \end{cases} \tag{17}$$

where $M_0 > \max\{M^*, \|\sigma'\|_\infty, \|\gamma'\|_\infty\}$;

$$\Lambda(x, u) = \begin{cases} \Phi_p(u) - \Phi_p(\sigma(x)), & \text{if } u < \sigma(x), \\ 0, & \text{if } \sigma(x) \le u \le \gamma(x), \\ \Phi_p(u) - \Phi_p(\gamma(x)), & \text{if } u > \gamma(x), \end{cases}$$
$$Q(x, u) = \begin{cases}]-\infty, h_1(x)], & \text{if } u < \sigma(x), \\ \mathbb{R}, & \text{if } \sigma(x) \le u \le \gamma(x), \\ [h_2(x), +\infty[, & \text{if } u > \gamma(x). \end{cases} \tag{18}$$

We set $H_1(x, u, v) = H(x, \varrho(x, u, v)) \cap Q(x, u)$. For $u \in [\sigma(x), \gamma(x)]$ and all $|v| < M_0$, we have $H_1(x, u, v) = H(x, u, v)$. Moreover, for almost all $x \in \Pi$, all $u, v \in \mathbb{R}$, and all $w \in H(x, u, v)$, we have $|w| \le \psi_r(t)$ with $r = \max\{M_0, \|\sigma\|_\infty, \|\gamma\|_\infty\}$. For every $z \in W^{1,p}(0, \alpha)$, we set

$$\widehat{\Lambda}(z)(.) = \Lambda(., z(.)), \tag{19}$$

where $\widehat{\Lambda}$ is the Nemitsky operator corresponding to Λ. Then, we define Z: $W(0, \alpha) \longrightarrow \mathscr{P}_{nwkc}(L^q(\Pi))$ by

$$Z(z) = S^q_{H_1(.,z(.),z'(.))} + \widehat{\Lambda}(u). \tag{20}$$

□

Proposition 10. *If hypothesis* (H_H) *hold, then* Z *is usc from* $W^{1,p}(0, \alpha)$ *into* $L^q(\Pi)_w$ *(by* $L^q(\Pi)_w$*, we denote the Lebesgue space* $L^q(\Pi)$ *furnished with the weak topology).*

Proof. See the proof of Proposition 3.7 of Bader-Papageorgiou [11]. □

Let us introduce the set D and the operators Γ: $D \subseteq L^p(\Pi) \longrightarrow L^q(\Pi)$ and E: $L^p(\Pi) \longrightarrow \mathbb{R}$ defined, respectively, by

$$
\begin{aligned}
D &= \left\{ z \in C^1(\Pi) \colon \Phi(\Theta(.)z') \in W^{1,q}(0,\alpha), \Theta(0)z'(0) \in G_1(z(0)) - \Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)) \right\},\\
\Gamma(z)(.) &= -(\Phi(\Theta(.)z'(.)))' \quad \text{for all } z \in D,\\
E(z) &= \begin{cases} \displaystyle\int_0^\alpha J(x, z(x))\mathrm{d}x, & \text{if } J(., z(.)) \in L^q(\Pi),\\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{21}
$$

Let $\widehat{E}$ be the restriction of E to the set $W^{1,p}(0,\alpha)$. We have,

$$\partial\widehat{E}(z) = \partial E(z) = S^q_{A(z(.))} \quad \forall z \in W^{1,p}(0,\alpha), \tag{22}$$

where E is lower semicontinuous, proper, and convex (see Barbu [21]) and $W^{1,p}(0,\alpha)$ is a reflexive Banach space. Then, $\partial\widehat{E}$ is a maximal monotone map.

Proposition 11. *Suppose that hypotheses* (H_Φ), (H_Θ), (H_G), (H_A), *and* (H_H) *are satisfied. Then,* $\Gamma + \partial E\colon D \subseteq L^p(\Pi) \longrightarrow L^q(\Pi)$ *is maximal monotone.*

Proof. Let $f \in L^q(\Pi)$. Let us consider the following nonlinear boundary value problem:

$$
\begin{cases} -(\Phi(\Theta(x)z'(x)))' + \Phi_p(z(x)) + A(z(x)) \ni f(x) \ a.e \text{ on } \Pi = [0,\alpha],\\ \Theta(0)z'(0) \in G_1(z(0)) - \Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)). \end{cases}
\tag{23}
$$

The problem (23) has a single solution $z \in C^1(\Pi)$. To demonstrate this, consider the following problem:

$$
\begin{cases} -(\Phi(\Theta(x)z'(x)))' + \Phi_p(z(x)) + A(z(x)) \ni f(x) \ a.e \text{ on } \Pi = [0,\alpha],\\ z(0) = c, z(x) = d, \end{cases}
\tag{24}
$$

where $v, w \in \mathbb{R}$. Let us set $\mu(x) = (1 - (x/\alpha))c + (x/\alpha)d$. We have, $\gamma(0) = c$ and $\gamma(\alpha) = d$. Then, let y be the function defined by $y(x) = z(x) - \mu(x)$. That means that $z(x) = y(x) + \mu(x)$. By replacing $z(x)$ by its expression as a function of y in (24), we obtain the following problem:

$$
\begin{cases} -(\Phi(\Theta(x))(y'(x) + \gamma'(x)))' + \Phi_p(y(x) + \mu(x)) + A(y(x) + \mu(x)) \ni f(x) \ a.e \text{ on } \Pi = [0,\alpha],\\ y(0) = y(\alpha) = 0. \end{cases}
\tag{25}
$$

To study (25), let us consider the nonlinear operator $\Xi\colon W_0^{1,p}(\Pi) \longrightarrow W^{-1,q}(\Pi)$ defined by

$$\langle \Xi(y), z\rangle_0 = \int_0^\alpha \Phi(\Theta(x))(y'(x) + \gamma'(x))z'(x)dx + \int_0^\alpha \left(\Phi_p(y(x) + \mu(x))\right) z(x)dx, \forall y, z \in W_0^{1,p}(\Pi), \tag{26}$$

where Ξ is strictly monotone and demicontinuous (see the proof of Proposition 3.10 of Behi-Adjé-Goli [4]). Whence, Ξ is maximal monotone.

For all $z \in W_0^{1,p}(\Pi)$, we have

$$\begin{aligned}\langle \Xi(z), z\rangle_0 &= \int_0^\alpha \Phi(\Theta(x)(z'(x)+\gamma'(x)))z'(x)dx + \int_0^\alpha \Phi_p(z(x)+\mu(x))z(x)\mathrm{dx}\\ &= \int_0^\alpha \frac{\Phi(\Theta(x)(z'(x)+\gamma'(x)))\Theta(x)(z'(x)+\gamma'(x))}{\Theta(x)}\mathrm{dx} - \int_0^\alpha \Phi(\Theta(x)(z'(x)+\gamma'(x)))\gamma'(x)\mathrm{dx}\\ &\quad + \int_0^\alpha \Phi_p(z(x)+\mu(x))(z(x)+\mu(x))\mathrm{dx} - \int_0^\alpha \Phi_p(z(x)+\mu(x))\mu(x)\mathrm{dx}\\ &\geq \frac{1}{M}\int_0^\alpha \Phi(\Theta(x)(z'(x)+\gamma'(x)))\Theta(x)(z'(x)+\gamma'(x))dx - \frac{|w-v|}{T}\int_0^\alpha \left|\Phi(\Theta(x)(z'(x)+\gamma'(x)))\right|\mathrm{dx}\\ &\quad + \int_0^\alpha \Phi_p(z(x)+\mu(x))(z(x)+\mu(x))\mathrm{dx} - \max_\Pi\{|\mu(x)|\}\int_0^\alpha \left|\Phi_p(z(x)+\mu(x))\right|\mathrm{dx}.\end{aligned} \tag{27}$$

Using the hypotheses $(H_\Phi)(b)$ and (H_Θ), it follows

$$\begin{aligned}\langle \Xi(z), z\rangle_0 &\geq \frac{d_1}{M}\int_0^\alpha \left|z'(x)+\gamma'(x)\right|^p\mathrm{dx} - \eta_2\int_0^\alpha |z(x)+\mu(x)|^{p-1}\mathrm{dx}\\ &\quad + \int_0^\alpha |z(x)+\mu(x)|^p\mathrm{dx} - \eta_1, \text{for some } \eta_1, \eta_2 > 0.\end{aligned} \tag{28}$$

Therefore, there exist some $d_4, d_5, d_6 > 0$ such that for all $u \in W_0^{1,p}(\Pi)$ such that

$$\langle \Xi(z), z\rangle_0 \geq d_4\|z+\mu\|^p - d_5\|z+\mu\|^{p-1} - d_6. \tag{29}$$

So, Ξ is coercive.

Let $\widehat{E}_0$ be the restriction of $E(.+\mu)$ to the set $W^{1,p}(0,\alpha)$. $\widehat{E}_0$ is a proper, semicontinuous, convex, functional map and $\partial\widehat{E}_0\colon W_0^{1,p}(0,\alpha) \longrightarrow P_{nwkc}(W^{-1,q}(0,\alpha))$ is a maximal monotone map. Moreover, for all $z \in W_0^{1,p}(0,\alpha)$ and all,

$$(v, v_0) \in \partial\widehat{E}_0(z) \times \partial\widehat{E}_0(0) = \partial E(z+\mu) \times \partial E_0(\mu), \tag{30}$$

and we have,

$$\langle v, z\rangle_0 = \langle v - v_0, z\rangle_0 + \langle v_0, z\rangle_0 \geq \langle v_0, z\rangle_0 \geq -d_7\|z\| \quad \text{for some } d_7 > 0. \tag{31}$$

Then, from (29) and (31), we obtain

$$\langle \Xi(z) + v, z\rangle_0 \geq d_4\|z+\gamma\|^p - d_5\|z+\gamma\|^{p-1} - d_7\|z\| - d_6. \tag{32}$$

Therefore, $\Xi + \partial\widehat{E}_0$ is weakly coercive. Also, since Ξ and $\partial\widehat{E}_0$ are maximal monotone maps, with Ξ defined on all $W_0^{1,p}(0,\alpha)$, $\Xi + \partial\widehat{E}_0$ is maximal monotone. Moreover, Ξ is surjective (because Ξ is maximal monotone and weakly coercive). Then, $\Xi + \partial\widehat{E}_0$ is surjective. Whence, there exists $y \in W_0^{1,p}(0,\alpha)$ such that $\Xi(y) + \partial\widehat{E}_0(y) \ni f$. Since $\Xi + \partial\widehat{E}_0$ is strictly monotone, y is unique. It follows that y is the unique solution of problem (25). Then, $u = y + \mu \in C^1(\Pi)$ is the unique solution of problem (24). We can define the solution map $\zeta\colon \mathbb{R}\times\mathbb{R} \longrightarrow C^1(\Pi)$ which assigns to each pair (c,d) the unique solution of the problem (24). Let $K\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$ be defined by

$$\begin{aligned}K(c,d) = (&-\Phi((\Theta(0)\zeta(c,d)'(0)),\\ &\cdot \Phi(\Theta(\alpha)\zeta(c,d)'((\alpha)).\end{aligned} \tag{33}$$

As in the proof of the Proposition 3.10 of Béhi-Adjé-Goli [4] or Proposition 13 of [3], we can show that K is monotone, continuous, and weakly coercive. We infer that K is surjective. Set $\overline{G}_1 = \Phi\circ G_1$ and $\overline{G}_1 = \Phi\circ G_1$. We define $G\colon \mathbb{R}\times\mathbb{R} \longrightarrow P(\mathbb{R}\times\mathbb{R})$ by $G(c,d) = (\overline{G}_1(c), \overline{G}_2(d))$, for all $(c,d) \in \mathbb{R}\times\mathbb{R}$. Let $\Omega\colon \mathbb{R}\colon \mathbb{R}\colon P(\mathbb{R}\times\mathbb{R})$ be defined by

$$\Omega(c,d) = K(c,d) + G(c,d) \quad \text{for all } (c,d) \in \mathbb{R}\times\mathbb{R}. \tag{34}$$

By Corollary 2.7, p.36 of [22], previous arguments on K and hypothesis (H_G), we deduce that Ω is surjective. Then, there exists $(s, t) \in \mathbb{R}\times\mathbb{R}$ such that $(0, 0) \in \theta(s,t)$. Whence, $(\Theta(0)z'(0), \Theta(\alpha)z'(\alpha)) \in G(z(0), z(\alpha))$. Thus, $z_0 = \sigma(s,t)$ is the single solution of problem (23).

We consider B: $L^p(\Pi) \longrightarrow L^q(\Pi)$ as the operator defined by

$$B(z)(.) = \Phi_p(z(.)), \tag{35}$$

where B is maximal monotone because Φ_p is continuous and monotone. Given any choice of f in $L^q(0,\alpha)$, arguments above show that $\Gamma + B + \partial E$ is surjective. We deduce that $\Gamma + \partial E$: $D \subseteq L^p(\Pi) \longrightarrow L^q(\Pi)$ is maximal monotone. □

As a result, $\Gamma + B + \partial E$ is surjective and strictly monotone. Thus, $\Psi = \Gamma + B + \partial E$: $L^q(\Pi) \longrightarrow D \subseteq W^{1,p}(0,\alpha)$ is well-defined, single-valued, and maximal monotone.

Proposition 12. *If hypotheses (H_Φ) and (H_G) hold, then Ψ: $L^q(\Pi) \longrightarrow D \subseteq W^{1,p}(0,\alpha)$ is completely continuous.*

Proof. Let $(v_n)_{n\geq 1}$ be a sequence which converge weakly to v in $L^q(\Pi)$. As in the proof of Proposition 14 of [3], we can show that $\Psi(v_n) \longrightarrow \Psi(v_n)$ in $W^{1,p}(\Pi)$. Therefore, the operator Ψ is completely continuous. □

2.2. Existence Results. We introduce the functional interval

$$\Delta = [\sigma,\gamma] = \left\{z \in W^{1,p}(0,\alpha):\ \sigma(x) \leq z(x) \leq \gamma(x) \quad \text{for all } x \in \Pi\right\}. \tag{36}$$

We consider the operator τ: $W^{1,p}(0,\alpha) \longrightarrow W^{1,p}(0,\alpha)$ defined by

$$\tau(z)(x) = \begin{cases} \sigma(x), & \text{if } z(x) < \sigma(x),\\ z(x), & \text{if } \sigma(x) \leq z(x) \leq \gamma(x),\\ \gamma(x), & \text{if } z(x) > \gamma(x). \end{cases} \tag{37}$$

We see that τ is bounded and is continuous.

Let $w \in \Delta$. We consider the following auxiliary boundary problem:

$$\begin{cases} (\Phi(\Theta(x)z'(x)))' \in A(z(x)) + H_1(x, z(x), z'(x)) + \Lambda(x, z(x)) \\ + g(w(x)) - M[\tau(z(x))]^l + Mw^l(x) \ a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)). \end{cases} \tag{38}$$

Proposition 13. *If the hypotheses (H_0), (H_Θ), (H_H), (H_Φ), and (H_g) hold, then problem (38) has a solution $z \in C^1(\Pi) \cap \Delta$.*

Proof. Let Z_1: $W^{1,p}(0,\alpha) \longrightarrow L^q(\Pi)$ be the nonlinear operator defined by

$$Z_1(z) = -Z(z) + B(\tau(z)) - M[\tau(z)]^l + \widehat{g}(w) + Mw^l, \forall z \in W^{1,p}(0,\alpha), \tag{39}$$

where $\widehat{g}(w)(.) = g(w(.)) \in L^\infty(0,\alpha)$. From the Proposition 10 and the continuity of the operators B and τ, we infer that Z_1 is usc from $W^{1,p}(0,\alpha)$ into $L^q(\Pi)_w$. Let T be the set defined by

$$T = \left\{z \in W^{1,p}(0,\alpha):\ z \in \nu(\Psi \circ Z_1) \text{ with } \nu \in (0,1)\right\}. \tag{40}$$

Suppose that $z \in T$. Then, $(1/\nu)z \in (\Psi \circ Z_1)(z)$. It follows that

$$\Gamma\left(\frac{1}{\nu}z\right) + \partial E\left(\frac{1}{\nu}z\right) + \lambda\widehat{\Phi}\left(\frac{1}{\nu}z\right) + B\left(\frac{1}{\nu}z\right) \ni f \quad \text{for some } f \in Z_1(z). \tag{41}$$

Whence, for some $z \in \partial E(1/\nu z)$, we have

$$\left\langle \Gamma\left(\frac{1}{\nu}z\right), z\right\rangle_p + \langle v, z\rangle_p + \left\langle B\left(\frac{1}{\nu}z\right), z\right\rangle_p = \langle f, z\rangle_p, \quad (42)$$

where $\langle . \rangle_p$ denotes the duality brackets between $L^p(\Pi)$ and $L^q(\Pi)$. By integration by parts, we obtain

$$\begin{aligned}\left\langle \Gamma\left(\frac{1}{\nu}z\right), z\right\rangle_p &= -\int_0^\alpha \left(\Phi\left(\Theta(x)\frac{z'(x)}{\nu}\right)\right)' z(x)\mathrm{d}x \\ &= \Phi\left(\Theta(0)\frac{z'(0)}{\nu}\right)z(0) - \Phi\left(\Theta(\alpha)\frac{z'(\alpha)}{\nu}\right)z(\alpha) + \int_0^\alpha \Phi\left(\Theta(x)\frac{z'(x)}{\nu}\right)z'(x)\mathrm{d}x,\end{aligned} \quad (43)$$

where $1/\nu u \in D$ implies that $\Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha))$. G_1 and G_2 are maximal monotone, Φ is monotone, and $0 \in G_1(0), 0 \in G_2(0)$. We deduce that

$$\Phi\left(\Theta(0)\frac{z'(0)}{\nu}\right)z(0) - \Phi\left(\Theta(\alpha)\frac{z'(\alpha)}{\nu}\right)z(\alpha) \geq 0. \quad (44)$$

Using hypotheses $(H_\Phi)b)$, (H_Θ), (43), and (44), we obtain

$$\left\langle \Gamma\left(\frac{1}{\nu}z\right), z\right\rangle_p \geq \frac{d_1 m^{p-1}}{\nu^{p-1}}\|z'\|_p^p. \quad (45)$$

For all $(v, g_0) \in \partial E(1/\nu z) \times \partial E(0)$,

$$\langle v, z\rangle_p = \langle v - g_0, z\rangle_p + \langle g_0, z\rangle_p \geq \langle g_0, z\rangle_p \geq -d_{14}\|z\|_p \quad \text{for some } d_{14} > 0, \quad (46)$$

because $\nu > 0$ and $\partial E(0)$ is bounded. Also, for all $u \in G$,

$$\left\langle B\left(\frac{1}{\nu}z\right), z\right\rangle_p \geq \frac{1}{\nu^{p-1}}\|z\|_p^p. \quad (47)$$

Furthermore, hypotheses on H_1 and τ imply

$$\langle f, z\rangle_p \leq \|f\|_q\|z\| \leq d_{15}\|z\| \quad \text{for some } d_{15} > 0. \quad (48)$$

Using (44)–(48) in (42), we deduce that

$$\min\left\{\frac{1}{\nu^{p-1}}, \frac{d_1 m^{p-1}}{\nu^{p-1}}\right\}\left(\|z\|\|_p^p + \|z'\|\|_p^p\right) \leq d_{11}\|z\|. \quad (49)$$

Then,

$$\|z\| \leq d_{16} \quad \text{for some } d_{16} > 0. \quad (50)$$

Whence, the set T is bounded. As a result, it follows that $\Psi \circ Z_1$ maps bounded sets into relatively compact sets. Thus, by Theorem 2, we obtain $u \in W^{1,p}(0,\alpha)$ such that $z \in \Psi \circ Z_1(z)$. Then, we have

$$\begin{cases} (\Phi(\Theta(x)z'(x)))' = j(x) + h(x) + \Phi_p\big((\tau(z(x))) - \Phi_p(z(x))z(x)\big) - \Phi_p(z(x))\tau \\ + \Lambda(x, z(x)) + g(w(x)) - M[\tau(z(x))]^l + Mw^l(x) \text{ a.e on } \Pi = [0,\alpha], \\ \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)), \end{cases} \quad (51)$$

with $j \in S^q_{A(z(.))}$ and $f \in S^q_{H_1(.,z(.),z'(.)))}$. Also, by definition, a function $\sigma \in C^1(\Pi)$ is said to be a lower solution of problem (1), if there exist $j_1 \in S^q_{A(\sigma(.))}, h_1 \in S^q_{H(.,\sigma(.),\sigma'(.))}$ such that

$$\begin{cases} (\Phi(\Theta(x)\sigma'(x)))' \geq j_1(x) + h_1(x) + g(\sigma(x)) \; a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)\sigma'(0) \in G_1(\sigma(0)) + \mathbb{R}_+, -\Theta(\alpha)\sigma'(\alpha) \in G_2(\sigma(\alpha)) + \mathbb{R}_+. \end{cases} \tag{52}$$

Then, as in the proof of the Proposition 4.1 of [4] or Proposition 9 of [12], we show that any solution u of (1) belongs to Δ. □

Theorem 14. *If the hypotheses (H_A), (H_Θ), (H_H), (H_G), and (H_g) hold, then problem (1) has a solution $z \in C^1(\Pi)$.*

Proof. We will use Theorem 2 to establish the proof of this theorem. Let us set $U = W^{1,p}(0,\alpha)$. U is a separable, reflexive, ordered Banach space. $\Delta = [\sigma,\gamma] \subseteq W^{1,p}(0,\alpha)$ and $S: \Delta \longrightarrow \mathscr{P}(\Delta)\backslash\varnothing$ is the solution multifunction for the auxiliary problem (10). Then, for every $w \in \Delta$, $S(w)$ is subset of Δ solutions of problem (10). From Proposition 13, we know that $S(w) \neq \varnothing$ and $S(w) \subseteq \Delta$. Then, $S(\Delta) \neq \varnothing$. Moreover, for all $w \in \Delta$, it follows from the proof of the Proposition 13 that $S(w)$ is a weakly closed part of $W^{1,p}(0,\alpha)$ and $S(\Delta)$ is bounded part of $W^{1,p}(0,\alpha)$. Now, let us check points (i) and (ii) in Theorem 2. Suppose that $w = \sigma$. Then, by Proposition 13, $S(\sigma) \neq \varnothing$ and $S(\sigma) \subseteq \Delta$. It follows that if $z \in S(\sigma)$, $\sigma \leq z$. So, (i) is verified.

It remains to verify statement (ii) of Theorem 2. Let $w_1, w_2 \in \Delta$, $w_1 \leq w_2$, and $z_1 \in S(w_1)$ with $w_1 \leq z_1$. $z_1 \in S(w_1)$ implies that $\Lambda(x, z_1(x)) = 0$, for almost all $x \in \Pi$ and $\tau(z_1) = z_1$. $p_{M_0}: \mathbb{R} \longrightarrow \mathbb{R}$ is the map defined by

$$p_{M_0}(u) = \begin{cases} -M_0, & \text{if } u < M_0, \\ z(x), & \text{if } |u| \leq M_0, \\ M_0, & \text{if } M_0 \leq u. \end{cases} \tag{53}$$

We have, $\varrho(x, z_1(x), z_1'(x)) = (\tau(z_1(x)), p_M(\tau'(z_1)(x)) = (z_1(x), p_M(z_1'(x)))z_1(x)), p_M(\tau'(z_1)(x)) = (z_1(x), p_M(z_1'(x)))\tau$. Then, $H_1(x, z_1(x), z_1'(x)) = H(x, z_1(x), z_1'(x))$ for all $x \in \Pi$. Whence, for all $w_1 \in S(\Delta)$, the auxiliary problem becomes

$$\begin{cases} (\Phi(\Theta(x)z_1'(x))' \in A(z_1(x)) + H_1(x, z_1(x), z_1'(x)) + \Lambda(x, z_1(x)) \\ + g(w_1(x)) - M[\tau(z_1(x))]^l + Mw_1^l(x) \; a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)z_1'(0) \in G_1(z(0)), -\Theta(x)z_1'(\alpha) \in B_2(z(\alpha)). \end{cases} \tag{54}$$

Since $w_1 \leq w_2$, by hypothesis (H_g), we have

$$g(w_1) + M[w_1(\alpha)]^l \geq g(w_2) + M[w_2(x)]^l \quad \text{for all } x \in \Pi. \tag{55}$$

Using (55) in (54), we obtain

$$\begin{cases} (\Phi(\Theta(x)z_1'(x)))' \geq j_1(x) + h_1(x) \\ + g(w_2(t)) - M[\tau(z_1(t))]^l + Mw_2^l(x) \; a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)z_1'(0) \in G_1(z(0)), -\Theta(\alpha)z_1'(\alpha) \in G_2(z(\alpha)), \end{cases} \tag{56}$$

for some $j_1 \in S^q_{A(z(.))}$ and some $h_1 \in H_1(., z(.), z'(.))$. It follows that z_1 is the lower solution of the boundary value problem.

$$\begin{cases} (\Phi(\Theta(x)z'(x)))' \in A(z(x)) + H_1(x, z(x), z'(x)) + \Lambda(x, z(x)) \\ + g(w_2(x)) - M[\tau(z(x))]^l + Mw_2^l(x) \; a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)). \end{cases} \tag{57}$$

Furthermore, we recall that γ is an upper solution of (1). Then, by definition,

$$\begin{cases} (\Phi(\Theta(x)\gamma'(x)))' \le j_2(x) + h_2(x) + g(\gamma(x)) \text{ a.e on } \Pi = [0,\alpha], \\ \Theta(0)\gamma'(0) \in G_1(\gamma(0)) - \mathbb{R}_+, -\Theta(\alpha)\gamma'(\alpha) \in G_2(\gamma(\alpha)) - \mathbb{R}_+, \end{cases} \quad (58)$$

for some $j_2 \in S^q_{A(\gamma(.))}$ and some $h_1 \in H_1(., \gamma(.), \gamma'(.))$. Since $w_2 \le \gamma$, we use hypothesis (H_g), and we obtain

$$g(w_2(x)) + M[w_2(x)]^l \ge g(\gamma(x)) + M[\gamma(x)]^l \quad \text{for all } x \in \Pi. \quad (59)$$

Using (59) in (57), we obtain

$$\begin{cases} (\Phi(\Theta(x)\gamma'(x)))' \le j_2(x) + h_2(x) + g(w_2(x)) + M[w_2(x)]^l \\ - M[\gamma(x)]^l \ a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)\gamma'(0) \in G_1(\gamma(0)) - \mathbb{R}_+, -\Theta(\alpha)\gamma'(\alpha) \in G_2(\gamma(\alpha)) - \mathbb{R}_+. \end{cases} \quad (60)$$

It follows γ is an upper solution of (57). So, z_1 and γ are ordered lower and upper solutions of (57), respectively. Then, using the same arguments as in the auxiliary problem (38), we obtain a solution $z_2 \in C^1(\Pi)$ of (1) such that $z_1 \le z_2 \le \gamma$. □

2.3. Existence of Extremal Solutions

Theorem 15. *If the hypotheses* (H_0) (H_A), (H_Θ), (H_H), (H_G), *and* (H_g) *hold, then problem (1) has some extremal solutions in the order interval* $\Delta = [\sigma, \gamma]$.

Proof. By making a few modifications to the proof of Theorem 13 of [12] in relation to the above arguments, we can easily establish the existence of extremal solutions of problem (1) in the functional interval $[\sigma, \gamma]$. □

2.4. Example and the Periodic Problem

2.4.1. Example. Let us consider the following problem:

$$\begin{cases} \left(\dfrac{\sqrt{2 + \left(1 + e^{(p-1)x}|z'(x)|^{p-1}\right)^2}}{1 + e^{(p-1)x}|z'(x)|^{p-1}} e^{(p-1)x}|z'(x)|^{p-2}z'(x) \right)' \\ = \dfrac{2x}{T} - 1 + 2(p-1)2^{p-1}x^{p-1}e^{(p-1)x}(1+x)[z(x)]^{p-1} + \max\left\{2x, x^2\right\}[z'(x)]^{p-1} \\ + \operatorname{sgn}(z(x)) \ a.e \text{ on } \Pi = [0,\alpha], \\ \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)). \end{cases} \quad (61)$$

Here,

$$\begin{aligned}
\Phi(\Theta(x)u) &= \frac{\sqrt{2+\left(1+e^{(p-1)x}|u|^{p-1}\right)^2}}{1+e^{(p-1)x}|u|^{p-1}}e^{(p-1)x}|u|^{p-2}u, \text{with } \Theta(x) = e^x,\\
h(x) &= 2(p-1)2^{p-1}x^{p-1}e^{(p-1)x}(1+x)[u]^{p-1} + \max\left\{2x, x^2\right\}[v]^{p-1},\\
j(x) &= \frac{2x}{T} - 1,\\
g(u) &= \operatorname{sgn}(u) = \begin{cases} 1, & \text{if } u > 0,\\ -1, & \text{if } u < 0. \end{cases}
\end{aligned} \tag{62}$$

We have, $j \in S^q_{A(u)}$, $h \in S^q_{H(.,u,v)}$, where $A(u) = \partial|u| = \begin{cases} 1, & \text{if } u > 0,\\ [-1,1], & \text{if } u = 0,\\ -1, & \text{if } x < 0. \end{cases}$ Suppose that for $i = 1, 2, G_i = \partial I_C$, where I_C is the indicator function of C, a closed interval of real numbers containing 0. Then, G_1 and G_2 are maximal monotone maps such that $0 \in G_1 \cap G_2$. If $G_1(u) = G_2(u) = \{0\}$, then (61) becomes a homogeneous Dirichlet problem. If $G_1(u) = G_2(u) = \mathbb{R}$, then (61) becomes a homogeneous Neumann problem. If for $i = 1, 2$, $G_i(u) = 1/\lambda_i u, \lambda_i > 0$, then (61) becomes a Sturm–Liouvile problem.

If $p \in 2\mathbb{N}^*$, for the cases of Dirichlet and Neumann problems, we show that σ and γ defined by $\sigma(x) = -1$ and $\gamma(x) = x^2 + 1$ are well-ordered lower and upper solutions of (61). It follows that the problem admits a solution and extremal solutions in the functional interval $[\sigma, \gamma]$.

In a general view, if hypothesis (H_0) is satisfied, the problem (61) admits a solution and extremal solutions in the functional interval $[\sigma, \gamma]$ because hypotheses (H_A), (H_H), (H_Φ), and (H_G) are satisfied.

Suppose that $A(u) = \begin{cases} \mathbb{R}_+, & \text{if } u = 0,\\ 0, & \text{if } u < 0,\\ \varnothing, & \text{if } u > 0. \end{cases}$ Then, problem (1) becomes the following variational inequality:

$$\begin{cases} (\Phi(\Theta(x)z'(x)))' \leq h(x) + g(z(x)) \ a.e \text{ on } \Pi = [0,\alpha],\\ \Theta(0)z'(0) \in G_1(z(0)), -\Theta(\alpha)z'(\alpha) \in G_2(z(\alpha)), \end{cases} \tag{63}$$

where $f \in S^q_{H(.,(z(.),z'(.)))}$. Thus, our results stay true for this kind of problems.

2.4.2. Periodic Problem. Our method stays true for the following periodic problem:

$$\begin{cases} (\Phi(\Theta(x)z'(x)))' \in A(z(x)) + H(x, z(x), z'(x)) + g(z(x)) \ a.e \text{ on } \Pi = [0,\alpha],\\ z(0) = z(\alpha), \Theta(0)z'(0) = \Theta(\alpha)z'(\alpha). \end{cases} \tag{64}$$

Indeed, set

$$D = \left\{z \in C^1(\Pi)\colon \Phi\left(\Theta(.)z'\right) \in W^{1,q}(0,\alpha), z(0) = z(x), \Theta(0)z'(0) = \Theta(\alpha)z'(\alpha).\right\}, \tag{65}$$

and consider the nonlinear operator $\Gamma + \partial E$: $D \subseteq L^p(\Pi) \longrightarrow L^q(\Pi)$ is defined by

$$\Gamma(z)(.) = -(\Phi(\Theta(.)z'(.)))' \quad \text{for all } z \in D. \tag{66}$$

To establish that the operator $\Gamma + \partial E$ is maximal monotone, consider the following auxiliary problem:

$$\begin{cases} -(\Phi(\Theta(x)z'(x)))' + \Phi_p(z(x)) + A(z(x)) \ni f(x) \ a.e \text{ on } \Pi = [0,\alpha], \\ z(0) = z(x), \Theta(0)z'(0) = \Theta(\alpha)z'(\alpha). \end{cases} \tag{67}$$

We replace the auxiliary problem (23) by the following nonhomogeneous Dirichlet problem:

$$\begin{cases} -(\Phi(\Theta(x)z'(x)))' + \Phi_p(z(x)) + A(z(x)) \ni f(x) \ a.e \text{ on } \Pi = [0,\alpha], \\ z(0) = z(x) = a, \end{cases} \tag{68}$$

where $a \in \mathbb{R}$. Setting $y(x) = z(x) - a$, problem (68) becomes the following homogeneous Dirichlet problem:

$$\begin{cases} -(\Phi(\Theta(x)y'(x)))' + \Phi_p(y(x)+a) + A(y(x)+a) \ni f(x) \ a.e \text{ on } \Pi = [0,\alpha], \\ y(0) = y(\alpha) = 0. \end{cases} \tag{69}$$

Then, the nonlinear operator $\Xi\colon W_0^{1,p}(\Pi) \longrightarrow W^{-1,q}(\Pi)$ is defined by

$$\langle \Xi(y), z\rangle_0 = \int_0^\alpha \Phi(\Theta(x))(y'(x))z'(x)dx + \int_0^\alpha \left(\Phi_p(+a)\right)z(x)dx, \quad \forall y, z \in W_0^{1,p}(\Pi). \tag{70}$$

Arguing as in the proof of Proposition 11, we show that $\Gamma + \partial E$ is strictly monotone, demicontinuous, and coercive. Whence, $\Gamma + \partial E$ is strictly monotone and surjective. Then, there exists a unique $y \in W_0^{1,p}((0,\alpha))$ such that $\Xi(y) + \partial\widehat{E}_0(y) \ni f$ which is the unique solution of problem (68). Then, $z = y + a \in C^1(\Pi)$ is the unique solution of the problem (69). We can define the solution map $\zeta\colon \mathbb{R}\times\mathbb{R} \longrightarrow C^1(\Pi)$ which assigns to each pair (c,d) the unique solution of the problem (24). Let $K\colon D_K \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$ be defined by

$$K(c,d) = \left(0, \Phi(\Theta(x))\zeta((c,d)'(\alpha)) - \Phi(\Theta(0)\zeta(c,d)'(0))\right), \tag{71}$$

where $D_K = \{(a,a), a \in \mathbb{R}\}$. Arguing as in the proof of Proposition 11, we show that K is monotone, continuous, and coercive, and $z = \sigma(a,a)$ is the unique solution of (67). We deduce that $\Psi = \Gamma + B + \partial E\colon L^q(\Pi) \longrightarrow D \subseteq W^{1,p}(0,\alpha)$ is well-defined, single-valued, maximal monotone and completely continuous (from $L^q(\Pi)$ into $L^p(\Pi)$).

Finally, with slight modifications to the rest of the arguments, we can establishe the existence of solutions and extremal solutions for the periodic problem.

3. Conclusion

In this paper, we have studied two second-order nonlinear differential inclusions containing a nonhomogeneous $\Phi-$ Laplacian operator and variational inequalities. One is subject to multivalued boundary conditions encompassing the classical Dirichlet, Neumann, and Sturm–Liouville boundary conditions, and the other is subject to periodic boundary conditions. To study these problems, we have used a method that combine the lower and upper solutions methods, the analysis of multifunctions, the theory of monotone operators, and a fixed point theorem for reflexive Banach spaces. We have obtained results showing the existence of solutions and extremal solutions when the lower and upper solutions are well ordered. We have also demonstrated the applications of our results using some examples [23, 24].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that Droh Arsène Béhi is the main author of this article. Each of the authors Assohoun Adjé and Konan Charles Etiennes Goli contributed to the revision of the article at the following three levels: (1) correction of typing errors in the manuscript and provision of more recent bibliographical references; (2) help in providing more details in certain proofs, in particular, concerning the proofs of equation (29) and the periodic case; and (3) advice on improving the introduction to the referees' reports.

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