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Research Article

Study of Hybrid Problems under Exponential Type Fractional-Order Derivatives

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In this investigation, we develop a theory for the hybrid boundary value problem for fractional differential equations subject to three-point boundary conditions, including the antiperiodic hybrid boundary condition. On suggested problems, the third-order Caputo–Fabrizio derivative is the fractional operator applied. In this regard, the corresponding hybrid fractional integral equation is obtained by the Caputo–Fabrizio operator's properties with the Green function's aid. Then, we apply Dhage's nonlinear alternative to the Schaefer type to prove the existence results. Finally, two examples are provided to confirm the validity of our main results.

1. Introduction

Many researchers have concentrated on looking into the solutions of nonlinear differential equations that incorporate many fractional differential operators to develop the field of fractional calculus, such as Riemann–Liouville, Caputo, and Hilfer (see [1–4]). However, these operators have a power law kernel and have limited ability to mimic physical problems. A novel fractional derivative (FD) method known as Caputo and Fabrizio (CF) [5] was created in 2015 to get around this problem. Its lack of a solitary kernel makes it useful for simulating a particular class of real-world problems that adhere to the exponential decay law. It was attempted to be used by certain researchers to solve various equations (see [6–11]). We also point out here some important results that dealt with the analytical solution of some fractional differential equations (FDEs) by using the optimal auxiliary function method [12], the Extended Direct Algebraic Method, [13] the Laplace transform decomposition method, and the variational iteration transform method [14].

For further information on the existence of solutions to FDEs of the hybrid kind, see [15, 16]. For example, Dhage and Lakshmikantham [15] have investigated the following hybrid classical.

$$\begin{cases} \dfrac{d}{d\varkappa}\left(\dfrac{\varphi(\varkappa)}{Z(\varkappa,\varphi(\varkappa))}\right) = f(\varkappa,\varphi(\varkappa)), \quad 0 \le \varkappa < T, \\ \varphi(\varkappa_0) = \varphi_0 \in \mathbb{R}. \end{cases} \tag{1}$$

Following the similar approach of [15], Zhao et al. in [16] expanded the analysis of hybrid (1) to the following hybrid FDE of the Riemann–Liouville type.

$$\begin{cases} {}^{\mathrm{RL}}\mathscr{D}_{0^+}^{\beta}\left(\dfrac{\varphi(\varkappa)}{Z(\varkappa,\varphi(\varkappa))}\right) = f(\varkappa,\varphi(\varkappa)), & 0 \leq \varkappa < T, \\ \varphi(0) = 0, & 0 < \beta < 1. \end{cases} \tag{2}$$

Recent reviews of several results of hybrid-type FDEs can be found in [17–24]. In this research, we investigate the existence of the solution to the fractional boundary value problem (BVP) hybrid-type:

$$\begin{cases} {}^{\mathrm{CF}}\mathscr{D}_{0}^{\beta+2}\dfrac{\varphi(\varkappa)}{Z(\varkappa,\varphi(\varkappa))} = \mathrm{f}(\varkappa,\varphi(\varkappa)), \varkappa \in \mho \coloneqq [0,1], \\ \dfrac{\varphi(\varkappa)}{Z(\varkappa,\varphi(\varkappa))}\Bigg|_{\varkappa=0} + \dfrac{\varphi(\varkappa)}{Z(\varkappa,\varphi(\varkappa))}\Bigg|_{\varkappa=1} = 0, \\ \varphi'(0) = \varphi''(0) = 0, \end{cases} \tag{3}$$

where $2 < \beta + 2 < 3$, ${}^{CF}\mathscr{D}_0^{\beta+2}$ is the Caputo–Fabrizio FD of order $\beta + 2$, and $\mathrm{f}\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$, $Z\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R} \times \{0\}$ are continuous.

To our knowledge, the fractional order $\beta + 2 \in (2,3)$ under Caputo–Fabrizio FD for the hybrid-type FDE with hybrid boundary conditions has not been studied before in the literature. Additionally, because this hybrid BVP is broad, several fractional dynamical systems can be considered as special cases.

Setting $Z(\varkappa,\varphi(\varkappa)) \equiv 1$ as a constant function, the hybrid problem (3) in this instance will be reduced to the following problem.

$$\begin{cases} {}^{\mathrm{CF}}\mathscr{D}_{0}^{\beta+2}\varphi(\varkappa) = f(\varkappa,\varphi(\varkappa)), & \varkappa \in \mho, \\ \varphi(\varkappa)|_{\varkappa=0} + \varphi(\varkappa)|_{\varkappa=1} = 0, \\ \varphi'(0) = \varphi''(0) = 0, \end{cases} \tag{4}$$

where $2 < \beta + 2 < 3$, which is also not studied in the literature.

The third-order Caputo–Fabrizio fractional derivative will be the subject of our interest in analyzing nonlinear hybrid equations. Our analysis of this work focuses on establishing some essential theorems regarding the existence of solutions for hybrid BVPs (3) and (4) by employing Dhage's nonlinear Schaefer-type alternative.

The structure of the essay is as follows. In Section 2, we will mention certain lemmas that provide Caputo–Fabrizio FDs and fixed point theory. The solution formulation for hybrid BVP (3) is obtained in Section 3. Then, we present our key results together with their proofs. The paper's major results are shown by two examples in Section 4 before we complete a section of conclusions.

2. Related Results

Here are the definitions and notations we will be using in this study.

Let $\mho \coloneqq [0,1]$, $\mathbb{R}$ constantly represent real space, and $\mathscr{X} \coloneqq \mathscr{C}(\mho,\mathbb{R})$ be the space of all continuous functions on $\mho$. Define the supremum norm $\|\cdot\|$ in $\mathscr{X}$ by

$$\|\varphi\|_{\mathscr{X}} = \sup_{\varkappa\in\mho}|\varphi(\varkappa)|, \tag{5}$$

and a multiplication in $\mathscr{X}$ by $(\varphi\psi)(\varkappa) = \varphi(\varkappa)\psi(\varkappa)$. Obviously, $\mathscr{X}$ is a Banach algebra with above norm and multiplication in it.

Definition 1 (see [5]). The Caputo–Fabrizio FD of order β for the function φ is expressed as follows:

$${}^{\mathrm{CF}}\mathscr{D}_{0}^{\beta}\varphi(\varkappa) = \frac{\mathscr{N}(\beta)}{1-\beta}\int_0^{\varkappa} e^{-\lambda_\beta(\varkappa-\tau)}\varphi'(\tau)\mathrm{d}\tau, \quad \varkappa \geq 0, \tag{6}$$

where $\lambda_\beta = \beta/1-\beta$ and $\mathscr{N}(\beta)$ is a normalization constant with $\mathscr{N}(0) = \mathscr{N}(1) = 1$.

The corresponding integral can be written as

$${}^{\mathrm{CF}}\mathscr{I}_{0}^{\beta}\varphi(\varkappa) = \frac{1-\beta}{\mathscr{N}(\beta)}\varphi(\varkappa) + \frac{\beta}{\mathscr{N}(\beta)}\int_0^{\varkappa}\varphi(\tau)\mathrm{d}\tau, \quad 0 < \beta < 1. \tag{7}$$

Lemma 2 (see [5, 17]). *If* $n \geq 1$ *and* $0 \leq \beta \leq 1$, *then* ${}^{CF}\mathscr{D}_0^{\beta+n}\varphi = {}^{CF}\mathscr{D}_0^{\beta}(D^n\varphi)$.

Lemma 3 (see [5]). *If* $\varphi^{(k)} = 0$ *for* $k = 1,2,3,\ldots,n$, *then* ${}^{CF}\mathscr{D}_0^{\beta}(D^n\varphi) = D^n\,{}^{CF}\mathscr{D}_0^{\beta}\varphi$, *where D is the classical derivation.*

Lemma 4 (see [5]). *Let* $\beta \in (0,1)$ *and* $\psi(0) = 0$. *Then the problem* ${}^{CF}\mathscr{D}_0^{\beta}\varphi(\varkappa) = \psi(\varkappa)$ *has unique solution given by*

$$\varphi(\varkappa) = \varphi(0) + \frac{1-\beta}{\mathscr{N}(\beta)}\psi(\varkappa) + \frac{\beta}{\mathscr{N}(\beta)}\int_0^{\varkappa}\varphi(\tau)\mathrm{d}\tau. \tag{8}$$

Definition 5 (see [24]). Let $n \in \mathbb{N}$ and $\varphi^{(k)} \in H^1(0,1)$. Then Caputo–Fabrizio FD of order β and n is expressed as

$${}^{CF}\mathscr{D}_0^{\beta+n}\varphi(\varkappa) = \frac{\mathscr{N}(\beta)}{1-\beta}\int_0^{\varkappa} e^{-\lambda_\beta(\varkappa-\tau)}\varphi^{(n+1)}(\tau)\mathrm{d}\tau. \tag{9}$$

Also,

$$\left({}^{\mathrm{CF}}\mathscr{D}_0^{\beta}\right)^{(n)}\varphi(\varkappa) = D^n\left({}^{\mathrm{CF}}\mathscr{D}_0^{\beta}\varphi(\varkappa)\right) = \frac{\mathscr{N}(\beta)}{1-\beta}D^n\int_0^{\varkappa} e^{-\lambda_\beta(\varkappa-\tau)}\varphi'(\tau)\mathrm{d}\tau, \tag{10}$$

where D is the classical derivative.

Lemma 6 (see [24]). *Let $n \in \mathbb{N}$ and $\beta \in (0, 1)$. Then*

$$\left({}^{\mathrm{CF}}\mathscr{D}_0^{\beta}\right)^{(n)} \varphi(\varkappa) = {}^{\mathrm{CF}}\mathscr{D}_0^{\beta+n} \varphi(\varkappa) + e^{-\lambda_\beta \varkappa} \frac{\mathscr{N}(\beta)^n}{1-\beta}{}_{i=1} \left(\lambda_\beta\right)^{n-i} \varphi^{(i)}(\varkappa). \tag{11}$$

Lemma 7 (see [24]). *Let $n \in \mathbb{N}$, $\beta \in (0, 1)$ and $g \in H^1(0, 1)$. Then the solution of the problem ${}^{CF}\mathscr{D}_0^{\beta+n}\varphi(\varkappa) = g(\varkappa)$ is given by*

$$\varphi(\varkappa) = \frac{1-\beta}{\mathscr{N}(\beta)}\mathscr{I}^n g(\varkappa) + \frac{\beta}{\mathscr{N}(\beta)}\mathscr{I}^{n+1} g(\varkappa) + \varphi(0) + \varkappa\varphi'(0) + \varkappa^2\frac{\varphi''(0)}{2!} + \cdots + \varkappa^n\frac{\varphi^{(n)}(0)}{n!}. \tag{12}$$

In particular, if $n = 2$, we have

$$\varphi(\varkappa) = \frac{1-\beta}{\mathscr{N}(\beta)}\mathscr{I}^2 g(\varkappa) + \frac{\beta}{\mathscr{N}(\beta)}\mathscr{I}^3 g(\varkappa) + \varphi(0) + \varkappa\varphi'(0) + \varkappa^2\frac{\varphi''(0)}{2!}. \tag{13}$$

Theorem 8 (see [25]). *Let $\mathscr{X}$ be a Banach algebra $\mathscr{X}$. For some $\epsilon \in \mathbb{R}^+$ consider an open ball $V_\epsilon(0)$ and a closed ball $\overline{V}_\epsilon(0)$ in $\mathscr{X}$. Assume that two operators $\mathcal{O}_1\colon \mathscr{X} \longrightarrow \mathscr{X}$ and $\mathcal{O}_2\colon V_\epsilon(0) \longrightarrow \mathscr{X}$ satisfy the following conditions: (i) $\mathcal{O}_1$ is an operator including the Lipschitzian property with a Lipschitz constant L^*; (ii) $\mathcal{O}_2$ has the complete continuity property; (iii) $L^* M^* < 1$, $M^* = \|\mathcal{O}_2(V_\epsilon(0))\|_{\mathscr{X}} = \sup\{\|\mathcal{O}_2 z\|_{\mathscr{X}}\colon z \in V_\epsilon(0)\}$. Then either*

(a) There is a solution in $V_\epsilon(0)$ for the operator equation $\mathcal{O}_1 z \mathcal{O}_2 z = z$, or

(b) There is an element $v \in \mathscr{X}$ with $\|v\|_{\mathscr{X}} = r$ such that $\lambda\mathcal{O}_1 v\mathcal{O}_2 v = v$, for some $0 < \lambda < 1$.

3. Main Results

The Caputo–Fabrizio problems (3) and (4) are the subject of some qualitative analysis in this section.

Lemma 9. *Let $2 < \beta + 2 < 3$ $(n = 2)$. Assume that $\varphi \longrightarrow \varphi(\varkappa)/\mathrm{Z}(\varkappa, \varphi(\varkappa))$ is increasing in $\mathbb{R}$, for each $\varkappa \in \mho$ and $\mathrm{g} \in H^1(0, 1)$. Then, φ is a solution of the problem (3) given by*

$$\begin{cases} {}^{\mathrm{CF}}\mathscr{D}_0^{\beta+2}\dfrac{\varphi(\varkappa)}{\mathrm{Z}(\varkappa, \varphi(\varkappa))} = \mathrm{g}(\varkappa), \varkappa \in \mho \coloneqq [0, 1], \\[2ex] \varphi'(0) = \varphi''(0) = 0, \\[2ex] \left.\dfrac{\varphi(\varkappa)}{\mathrm{Z}(\varkappa, \varphi(\varkappa))}\right|_{\varkappa=0} + \left.\dfrac{\varphi(\varkappa)}{\mathrm{Z}(\varkappa, \varphi(\varkappa))}\right|_{\varkappa=1} = 0, \end{cases} \tag{14}$$

if and only if

$$\varphi(\varkappa) = \mathrm{Z}(\varkappa, \varphi(\varkappa))\int_0^1 G(\varkappa, \tau)\mathrm{g}(\tau)\mathrm{d}\tau, \tag{15}$$

where

$$G(\varkappa, \tau) = \begin{cases} \dfrac{1-\beta}{\mathscr{N}(\beta)}(\varkappa - \tau) + \dfrac{\beta}{2\mathscr{N}(\beta)}(\varkappa - \tau)^2; & 0 \le \varkappa \le \tau \le 1, \\[2ex] \dfrac{1-\beta}{\mathscr{N}(\beta)}(\varkappa - \tau) + \dfrac{\beta}{2\mathscr{N}(\beta)}(\varkappa - \tau)^2 - \mathrm{Z}_{0,\varphi}\left[\dfrac{1-\beta}{\mathscr{N}(\beta)}(1 - \tau) + \dfrac{\beta}{2\mathscr{N}(\beta)}(1 - \tau)^2\right]; & 0 \le \tau \le \varkappa \le 1, \end{cases} \tag{16}$$

and $Z_{0,\varphi} \coloneqq Z(0, \varphi(0))/1 + Z(0, \varphi(0)) > 0$. Here, $G(\varkappa, \tau)$ is called the Green function of BVP (14).

Proof. Let φ be a solution function for the hybrid problem (14). Then by Lemma 7, we have

$$\varphi(\varkappa) = \mathrm{Z}(\varkappa, \varphi(\varkappa))\left[\frac{1-\beta}{\mathscr{N}(\beta)}\mathscr{I}^2 g(\varkappa) + \frac{\beta}{\mathscr{N}(\beta)}\mathscr{I}^3 g(\varkappa) + \varphi(0) + \varkappa\varphi'(0) + \varkappa^2\frac{\varphi''(0)}{2!}\right]. \tag{17}$$

It follows from $\varphi'(0) = \varphi''(0) = 0$ that

$$\varphi(\varkappa) = Z(\varkappa, \varphi(\varkappa))\left(\frac{1-\beta}{\mathcal{N}(\beta)}\mathcal{I}^2 g(\varkappa) + \frac{\beta}{\mathcal{N}(\beta)}\mathcal{I}^3 g(\varkappa) + \varphi(0)\right). \tag{18}$$

By using the boundary condition $\varphi(0)/Z(0,\varphi(0)) + \varphi(1)/Z(1,\varphi(1)) = 0$, and

$$\varphi(1) = Z(1, \varphi(1))\left(\frac{1-\beta}{\mathcal{N}(\beta)}\mathcal{I}^2 g(1) + \frac{\beta}{\mathcal{N}(\beta)}\mathcal{I}^3 g(1) + \varphi(0)\right), \tag{19}$$

we have

$$\frac{\varphi(0)}{Z(0,\varphi(0))} = -\left(\frac{1-\beta}{\mathcal{N}(\beta)}\mathcal{I}^2 g(1) + \frac{\beta}{\mathcal{N}(\beta)}\mathcal{I}^3 g(1) + \varphi(0)\right), \tag{20}$$

which implies

$$\varphi(0) = -\frac{Z(0,\varphi(0))}{1 + Z(0,\varphi(0))}\left(\frac{1-\beta}{\mathcal{N}(\beta)}\mathcal{I}^2 g(1) + \frac{\beta}{\mathcal{N}(\beta)}\mathcal{I}^3 g(1)\right). \tag{21}$$

Replacing $\varphi'(0)$ in (18), we obtain

$$\begin{aligned}\varphi(\varkappa) &= Z(\varkappa, \varphi(\varkappa))\\ &\quad\cdot\left(\frac{1-\beta}{\mathcal{N}(\beta)}\mathcal{I}^2 g(\varkappa) + \frac{\beta}{\mathcal{N}(\beta)}\mathcal{I}^3 g(\varkappa) - Z_{0,\varphi}\left[\frac{1-\beta}{\mathcal{N}(\beta)}\mathcal{I}^2 g(1) + \frac{\beta}{\mathcal{N}(\beta)}\mathcal{I}^3 g(1)\right]\right),\end{aligned} \tag{22}$$

which implies

$$\begin{aligned}\varphi(\varkappa) &= Z(\varkappa, \varphi(\varkappa))\\ &\quad\cdot\left(\frac{1-\beta}{\mathcal{N}(\beta)}\int_0^{\varkappa} g(\tau)(\varkappa-\tau)d\tau - Z_{0,\varphi}\frac{1-\beta}{\mathcal{N}(\beta)}\int_0^1 g(\tau)(1-\tau)d\tau + \frac{\beta}{2\mathcal{N}(\beta)}\int_0^{\varkappa} g(\tau)(\varkappa-\tau)^2 d\tau - Z_{0,\varphi}\frac{\beta}{2\mathcal{N}(\beta)}\int_0^1 g(\tau)(1-\tau)^2 d\tau\right)\\ &= Z(\varkappa, \varphi(\varkappa))\left(\int_0^{\varkappa}\left(\frac{1-\beta}{\mathcal{N}(\beta)}(\varkappa-\tau) + \frac{\beta}{2\mathcal{N}(\beta)}(\varkappa-\tau)^2\right)g(\tau)d\tau - Z_{0,\varphi}\int_0^1\left(\frac{1-\beta}{\mathcal{N}(\beta)}(1-\tau) + \frac{\beta}{2\mathcal{N}(\beta)}(1-\tau)^2\right)g(\tau)d\tau\right)\\ &= Z(\varkappa, \varphi(\varkappa))\int_0^1 G(\varkappa,\tau)g(\tau)d\tau.\end{aligned} \tag{23}$$

On the other hand, one can determine if the given function $\varphi(\varkappa)$ is a solution to the problem by performing some calculations.

Now, we need following assumptions on Z and f.

(H1) There exists bounded function $\vartheta_Z\colon \mho \longrightarrow \mathbb{R}_+$ such that

$|Z(\varkappa,\varphi) - Z(\varkappa,\overline{\varphi})| \le \vartheta_Z(\varkappa)|\varphi - \overline{\varphi}|, \varkappa \in \mho, \varphi, \overline{\varphi} \in \mathbb{R}$.

(H2) There exist $\delta_f\colon \mho \longrightarrow \mathbb{R}_+$ and let $\Upsilon\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuously nondecreasing function such that

$|f(\varkappa,\varphi)| \le \delta_f(\varkappa)\Upsilon(|\varphi|), \varkappa \in \mho, \varphi \in \mathbb{R}$.

(H3) There exists $\varepsilon > 0$ such that

$$\varepsilon \ge \frac{Z_0\mathscr{H}}{1 - \vartheta\mathscr{H}}, \tag{24}$$

where $Z_0 = \sup_{\varkappa\in\mho}|Z(\varkappa,0)|$, $\vartheta := \|\vartheta_Z\| = \sup_{\varkappa\in\mho}|\vartheta_Z(\varkappa)|$, $\mathscr{H} = ((1/\mathcal{N}(\beta)) + (\beta Z_{0,\varphi}/2\mathcal{N}(\beta)))\delta\Upsilon(\|\varphi\|)$, and $\delta := \|\delta_f\| = \sup_{\varkappa\in\mho}|\delta_f(\varkappa)|$. □

Theorem 10. *Suppose that* $Z\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ *and* $f\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous. Also (H1)–(H3) hold. If* $\vartheta\mathscr{H} < 1$*, then the hybrid problem (3) has at least one solution defined on* $\mho$.

Proof. Define the set $\overline{\mathscr{V}}_{\varepsilon}(0) = \{\varphi \in \mathscr{X}\colon \|\varphi\|_{\mathscr{X}} \leq \varepsilon\}$, where ε satisfies (H3). Certainly, $\overline{\mathscr{V}}_{\varepsilon}(0) \subseteq \mathscr{X}$. By Lemma 7 and by (15), we define two operators $\mathscr{O}_1, \mathscr{O}_2\colon \overline{\mathscr{V}}_{\varepsilon}(0) \longrightarrow \mathscr{X}$ by

$$\begin{aligned} &\mathscr{O}_1\varphi(\varkappa) = Z(\varkappa, \varphi(\varkappa)), \quad \varkappa \in \mho, \text{and} \\ &\mathscr{O}_2\varphi(\varkappa) = \int_0^1 G(\varkappa, \tau) f(\tau, \varphi(\tau)) d\tau, \quad \varkappa \in \mho. \end{aligned} \tag{25}$$

The operator equation $\mathscr{O}_1\varphi\mathscr{O}_2\varphi = \varphi$ is satisfied by the function $\varphi \in \mathscr{X}$ as a solution function for the hybrid BVP (3). To show this, we establish the existence of such a solution using assumptions of Theorem 8.

Claim 1: $\mathscr{O}_1$ is Lipschitzian on $\mathscr{X}$.

Indeed, for $\varphi, \overline{\varphi} \in \overline{\mathscr{V}}_{\varepsilon}(0) \subset \mathscr{X}$ and by (H1), we obtain

$$\left|\mathscr{O}_1\varphi(\varkappa) - \mathscr{O}_1\overline{\varphi}(\varkappa)\right| = |Z(\varkappa, \varphi(\varkappa)) - Z(\varkappa, \overline{\varphi}(\varkappa))| \leq \vartheta_Z(\varkappa)|\varphi(\varkappa) - \overline{\varphi}(\varkappa)|, \quad \varkappa \in \mho. \tag{26}$$

Then, we take the supremum over $\mho$ to obtain

$$\|\mathscr{O}_1\varphi - \mathscr{O}_1\overline{\varphi}\|_{\mathscr{X}} \leq \vartheta\|\varphi - \overline{\varphi}\|_{\mathscr{X}}. \tag{27}$$

Thus, $\mathscr{O}_1\colon \overline{\mathscr{V}}_{\varepsilon}(0) \longrightarrow \mathscr{X}$ is Lipschitzian on $\mathscr{X}$ with Lipschitz constant ϑ.

Claim 2: $\mathscr{O}_2\colon \overline{\mathscr{V}}_{\varepsilon}(0) \longrightarrow \mathscr{X}$ is completely continuous.

We start by showing that $\mathscr{O}_2$ is continuous on $\overline{\mathscr{V}}_{\varepsilon}(0)$. Let $\{\varphi_n\}_{n\geq 1}$ be a convergence sequence in $\overline{\mathscr{V}}_{\varepsilon}(0)$ such that $\varphi_n \longrightarrow \varphi \in \overline{\mathscr{V}}_{\varepsilon}(0)$, as $n \longrightarrow \infty$. Then by Lebesgue's dominated convergence theorem [26], we have

$$\begin{aligned} \lim_{n\longrightarrow\infty} \mathscr{O}_2\varphi_n(\varkappa) &= \int_0^1 G(\varkappa, \tau) \lim_{n\longrightarrow\infty} f(\tau, \varphi_n(\tau)) d\tau \\ &= \int_0^1 G(\varkappa, \tau) f(\tau, \varphi(\tau)) d\tau \\ &= \mathscr{O}_2\varphi(\varkappa), \end{aligned} \tag{28}$$

for all $\varkappa \in \mho$. Therefore, $\mathscr{O}_2\varphi_n \longrightarrow \mathscr{O}_2\varphi$, as $n \longrightarrow \infty$ which proves that $\mathscr{O}_2$ is continuous on $\overline{\mathscr{V}}_{\varepsilon}(0)$.

Next, we show that $\mathscr{O}_2$ is uniformly bounded on $\overline{\mathscr{V}}_{\varepsilon}(0)$. Note that

$$\max_{\varkappa\in\mho}|G(\varkappa, \tau)| \leq \left|\frac{1-\beta}{\mathscr{N}(\beta)}\right| + \left|\frac{\beta}{2\mathscr{N}(\beta)}\right| + Z_{0,\varphi}\left|\frac{-(1-\beta)}{\mathscr{N}(\beta)}\right| + Z_{0,\varphi}\left|\frac{-\beta}{2\mathscr{N}(\beta)}\right| < \frac{1}{\mathscr{N}(\beta)} + \frac{\beta Z_{0,\varphi}}{2\mathscr{N}(\beta)}. \tag{29}$$

Let $\varphi \in \overline{\mathscr{V}}_{\varepsilon}(0)$. Then by (H2), we obtain

$$\begin{aligned} |\mathscr{O}_2\varphi(\varkappa)| &\leq \int_0^1 |G(\varkappa, \tau)||f(\tau, \varphi(\tau))| d\tau \\ &\leq \left(\frac{1}{\mathscr{N}(\beta)} + \frac{\beta Z_{0,\varphi}}{2\mathscr{N}(\beta)}\right) \int_0^1 \delta_f(\tau)\Upsilon(|\varphi(\tau)|) d\tau \\ &\leq \left(\frac{1}{\mathscr{N}(\beta)} + \frac{\beta Z_{0,\varphi}}{2\mathscr{N}(\beta)}\right) \int_0^1 \|\delta_f\| \Upsilon(\|\varphi\|) d\tau \\ &\leq \left(\frac{1}{\mathscr{N}(\beta)} + \frac{\beta Z_{0,\varphi}}{2\mathscr{N}(\beta)}\right) \delta\Upsilon(\varepsilon) \coloneqq \varepsilon^*. \end{aligned} \tag{30}$$

Thus, $\|\mathscr{O}_2\varphi\| \leq \varepsilon^*$, for all $\varphi \in \overline{\mathscr{V}}_{\varepsilon}(0)$. This consequence shows that $\mathscr{O}_2(\overline{\mathscr{V}}_{\varepsilon}(0))$ is uniformly bounded set on $\mathscr{X}$.

Finally, we prove that the set $\mathscr{O}_2(\overline{\mathscr{V}}_{\varepsilon}(0))$ is equicontinuous in $\mathscr{X}$.

Let $\varkappa_1, \varkappa_2 \in \mho$ with $\varkappa_1 \leq \varkappa_2$, and $\varphi \in \overline{\mathscr{V}}_{\varepsilon}(0)$. Then

$$\begin{aligned} &\left|\mathscr{O}_2\varphi(\varkappa_2) - \mathscr{O}_2\varphi(\varkappa_1)\right| \\ &= \left|\int_0^{\varkappa_2} \left(\frac{1-\beta}{\mathscr{N}(\beta)}(\varkappa_2 - \tau) + \frac{\beta}{2\mathscr{N}(\beta)}(\varkappa_2 - \tau)^2\right) f(\tau, \varphi(\tau)) d\tau\right. \end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\varkappa_1}\left(\frac{1-\beta}{\mathcal{N}(\beta)}(\varkappa_1-\tau)+\frac{\beta}{2\mathcal{N}(\beta)}(\varkappa_1-\tau)^2\right)\mathrm{f}(\tau,\varphi(\tau))\mathrm{d}\tau\Bigg|\\
&\leq \int_0^{\varkappa_1}\left(\frac{1-\beta}{\mathcal{N}(\beta)}(\varkappa_2-\varkappa_1)+\frac{\beta}{2\mathcal{N}(\beta)}\left[(\varkappa_2-\tau)^2-(\varkappa_1-\tau)^2\right]\right)|\mathrm{f}(\tau,\varphi(\tau))|\mathrm{d}\tau\\
&+\int_{\varkappa_1}^{\varkappa_2}\left(\frac{1-\beta}{\mathcal{N}(\beta)}(\varkappa_2-\tau)+\frac{\beta}{2\mathcal{N}(\beta)}(\varkappa_2-\tau)^2\right)|\mathrm{f}(\tau,\varphi(\tau))|\mathrm{d}\tau\\
&\leq\left[\frac{1-\beta}{\mathcal{N}(\beta)}(\varkappa_2-\varkappa_1)\varkappa_1+\frac{\beta}{2\mathcal{N}(\beta)}\varkappa_2\varkappa_1(\varkappa_2-\varkappa_1)\right]\delta\Upsilon(\varepsilon)\\
&+\left(\frac{1-\beta}{\mathcal{N}(\beta)}\left[\varkappa_2(\varkappa_2-\varkappa_1)+\frac{\varkappa_1^2-\varkappa_2^2}{2}\right]+\frac{\beta}{2\mathcal{N}(\beta)}\left[\frac{\varkappa_2^3-\varkappa_1^3}{3}+\varkappa_2\varkappa_1(\varkappa_1-\varkappa_2)\right]\right)\delta\Upsilon(\varepsilon)\\
&\leq\left[\frac{1-\beta}{\mathcal{N}(\beta)}(\varkappa_2-\varkappa_1)\varkappa_1+\frac{\beta}{2\mathcal{N}(\beta)}\varkappa_2\varkappa_1(\varkappa_2-\varkappa_1)\right]\delta\Upsilon(\varepsilon)\\
&+\left(\frac{1-\beta}{\mathcal{N}(\beta)}\left[\varkappa_2(\varkappa_2-\varkappa_1)\right]+\frac{\beta}{2\mathcal{N}(\beta)}\left[\frac{\varkappa_2^3-\varkappa_1^3}{3}\right]\right)\delta\Upsilon(\varepsilon).
\end{aligned}
\tag{31}
$$

We notice that the right side of the inequality converges to zero as $\varkappa_1 \longrightarrow \varkappa_2$. Thus,

$$\left|\mathcal{O}_2\varphi(\varkappa_2)-\mathcal{O}_2\varphi(\varkappa_1)\right| \longrightarrow 0, \text{as } \varkappa_1 \longrightarrow \varkappa_2. \tag{32}$$

So, $\mathcal{O}_2$ is equicontinuous on $\mathcal{X}$.

As a result, we conclude that the operator $\mathcal{O}_2$ possesses the completely continuous property on $\mathcal{X}$ using the Arzelà–Ascoli theorem.

Claim 3: Assumption (iii) of Theorem 8 is satisfied, i.e.,

$$M=\|\mathcal{O}_2(\overline{\mathcal{V}}_\varepsilon(0))\|_{\mathcal{X}}=\sup\left\{\left|\mathcal{O}_2\varphi\right|:\ \varphi\in\overline{\mathcal{V}}_\varepsilon(0)\right\}=\left(\frac{1}{\mathcal{N}(\beta)}+\frac{\beta Z_{0,\varphi}}{2\mathcal{N}(\beta)}\right)\delta\Upsilon(\|\varphi\|)=\mathcal{H}. \tag{33}$$

With $L^* = \vartheta$, we obtain $L^*\delta < 1$, and as a result, one can see that all assumptions of Theorem 8 hold to both $\mathcal{O}_1$ and $\mathcal{O}_2$. Hence, either condition (a) or condition (b) of Theorem 8 holds. Let us assume that φ satisfies the operator equation $\varphi = \lambda\mathcal{O}_1\varphi\mathcal{O}_2\varphi$, for some $\lambda \in (0,1)$. So, $\|\varphi\| = \varepsilon$. Now, one can write

$$
\begin{aligned}
|\varphi(\varkappa)| &\leq \lambda\left|\mathcal{O}_1\varphi(\varkappa)\right|\left|\mathcal{O}_2 y(\varkappa)\right|\\
&\leq \lambda|Z(\varkappa,\varphi(\varkappa))|\int_0^1|G(\varkappa,\tau)\mathrm{f}(\tau,\varphi(\tau))|\mathrm{d}\tau\\
&\leq [|Z(\varkappa,\varphi(\varkappa))-Z(\varkappa,0)|+|Z(\varkappa,0)|]\left(\frac{1}{\mathcal{N}(\beta)}+\frac{\beta Z_{0,\varphi}}{2\mathcal{N}(\beta)}\right)\|\delta_{\mathrm{f}}\|\,\Upsilon(\|\varphi\|)\\
&\leq [\vartheta\|\varphi\|+Z_0]\left(\frac{1}{\mathcal{N}(\beta)}+\frac{\beta Z_{0,\varphi}}{2\mathcal{N}(\beta)}\right)\delta\Upsilon(\|\varphi\|)\\
&=\vartheta\|\varphi\|\mathcal{H}+Z_0\mathcal{H},
\end{aligned}
\tag{34}
$$

which implies $\|\varphi\| \leq Z_0\mathcal{H}/1-\vartheta\mathcal{H}$. Hence,

$$\varepsilon \leq \frac{Z_0 \mathscr{H}}{1 - \vartheta \mathscr{H}}, \tag{35}$$

which contradicts condition (24). This proves that the requirement (ii) of Theorem 8 is unsatisfied. Consequently, condition (a) of Theorem 8 is satisfied, and the fractional hybrid BVP (3) has a solution on $\overline{\mathscr{V}}_\varepsilon(0)$. □

3.1. Particular Case. We present now a special case corresponding to a fractional differential equation under the Caputo–Fabrizio operator for which the existence of a solution can be proved. In the case $Z(\varkappa, \varphi(\varkappa)) \equiv 1$, the hybrid fractional BVP (3) reduces to

$$\begin{cases} {}^{\mathrm{CF}}\mathscr{D}_0^{\beta+2} \varphi(\varkappa) = \mathrm{f}(\varkappa, \varphi(\varkappa)), & \varkappa \in \mho := [0, 1], \\ \varphi(\varkappa)\,|_{\varkappa=0} + \varphi(\varkappa)\,|_{\varkappa=1} = 0, \\ \varphi'(0) = \varphi''(0) = 0, \end{cases} \tag{36}$$

where $2 < \beta + 2 < 3$.

Its Green function is

$$G(\varkappa, \tau) = \begin{cases} \dfrac{1-\beta}{\mathscr{N}(\beta)}(\varkappa - \tau) + \dfrac{\beta}{2\mathscr{N}(\beta)}(\varkappa - \tau)^2; & 0 \leq \varkappa \leq \tau \leq 1, \\ \dfrac{1-\beta}{\mathscr{N}(\beta)}(\varkappa - \tau) + \dfrac{\beta}{2\mathscr{N}(\beta)}(\varkappa - \tau)^2 - \dfrac{1}{2}\left[\dfrac{1-\beta}{\mathscr{N}(\beta)}(1 - \tau) + \dfrac{\beta}{2\mathscr{N}(\beta)}(1 - \tau)^2\right]; & 0 \leq \tau \leq \varkappa \leq 1. \end{cases} \tag{37}$$

The corresponding solution of problem (36) is

$$\varphi(\varkappa) = \int_0^1 G(\varkappa, \tau)\mathrm{f}(\tau, \varphi(\tau))\mathrm{d}\tau. \tag{38}$$

Theorem 11. *Suppose that $Z\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ and $f\colon \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous. Also (H1)–(H3) hold. If $\vartheta\mathscr{H} < 1$, then the hybrid problem (36) has at least one solution defined on $\mho$.*

Proof. By setting $Z(\varkappa, \varphi(\varkappa)) \equiv 1$ in Theorem 10, and following the same proof procedures in Theorem 10, we obtain the desired result. □

4. Examples

Here, we give two examples to demonstrate the outcomes attained.

$$\begin{cases} {}^{\mathrm{CF}}\mathscr{D}_0^{9/4}\left(\dfrac{\varphi(\varkappa)}{\varkappa|\varphi(\varkappa)|/1 + |\varphi(\varkappa)| + 10}\right) = \dfrac{\varkappa}{100}\cos\varphi, \varkappa \in [0, 1], \\ \left.\dfrac{\varphi(\varkappa)}{\varkappa|\varphi(\varkappa)|/1 + |\varphi(\varkappa)| + 10}\right|_{\varkappa=0} + \left.\dfrac{\varphi(\varkappa)}{\varkappa|\varphi(\varkappa)|/1 + |\varphi(\varkappa)| + 10}\right|_{\varkappa=1} = 0, \\ \varphi'(0) = \varphi''(0) = 0, \end{cases} \tag{39}$$

where $\beta = 1/4$, $n = 2$, $\mathrm{f} \in \mathscr{C}(\mho \times \mathbb{R}, \mathbb{R})$ defined by $\mathrm{f}(\varkappa, \varphi) = (\sin(\varkappa)/100)\varphi$, and $Z \in \mathscr{C}(\mho \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ defined by $Z(\varkappa, \varphi) = \varkappa|\varphi|/1 + |\varphi| + 10$ with $Z_{0,\varphi} = 10 > 0$. It is clear that $|Z(\varkappa, \varphi) - Z(\varkappa, \psi)| = |(\varkappa|\varphi|/1 + |\varphi|) - (\varkappa|\psi|/1 + |\psi|)| \leq \varkappa|\varphi - \psi|$, for $\varkappa \in \mho$, $\varphi, \psi \in \mathbb{R}$ and $|\mathrm{f}(\varkappa, \varphi)| \leq \varkappa/100$, for $\varkappa \in \mho$, $\varphi \in \mathbb{R}$.

Example 1. Consider the following hybrid FDE:

Thus, $\vartheta_Z(\varkappa) = \varkappa$, $\delta_f(\varkappa) = \varkappa/100$ and $\Upsilon(|\varphi|) = 1$. Hence, $\vartheta = \sup_{\varkappa\in\mho}|\varkappa| = 1$, $\delta = \sup_{\varkappa\in\mho}|\varkappa/100| = 1/100$. By given data, we get $\mathscr{H}\vartheta = 0.022\,5 < 1$. Therefore, Theorem 10 implies that there is at least one solution on [0, b1] for the fractional hybrid BVP (39).

$$\begin{cases} {}^{\mathrm{CF}}\mathscr{D}_0^{11/5}\varphi(\varkappa) = \dfrac{1}{10+\varkappa}\left(\dfrac{1}{9}|\varphi(\varkappa)| + \dfrac{1}{6}\sin\varphi(\varkappa) + \dfrac{1}{18}\right), \quad \varkappa \in [0,1], \\ \varphi(0) + \varphi(1) = 0, \\ \varphi'(0) = \varphi''(0) = 0, \end{cases} \tag{40}$$

where $\beta = 1/5$, $n = 2$, $f \in \mathscr{C}(\mho \times \mathbb{R}, \mathbb{R})$ defined by $f(\varkappa,\varphi) = 1/10 + \varkappa(1/9|\varphi| + 1/6\sin\varphi + 1/18)$. It is clear that $|f(\varkappa,\varphi)| \leq 1/10(1/9|\varphi| + 2/9)$, for $\varkappa \in \mho$, $\varphi \in \mathbb{R}$. Hence, $\delta = 1/10$ and $\Upsilon(|\varepsilon|) = (1/9)|\varepsilon| + (2/9)$. Choose $\varepsilon = 1/2$, and we get $\mathscr{H} = 0.029167 < 1$. Therefore, Theorem 11 implies that there is at least one solution on [0, 1] for the fractional hybrid BVP (40).

Example 2. Consider the following FDE:

5. Conclusions

Many researchers have been interested in the Caputo–Fabrizio FD due to its appearance in a variety of applications and its nonlocal and nonsingular kernel of convolution type. Through this fractional operator, our work has produced a theory for the hybrid BVP for third-fractional order differential equations with antiperiodic hybrid boundary conditions. Indeed, we have obtained the equivalent integral equation for the problem at hand with the help of Green's function. Then, we have proved the existence results using Dhage's nonlinear alternative to the Schaefer type and by the minimum of hypotheses. Finally, two examples were given to support the theoretical findings. Obviously, the order of the fractional derivative α determines the degree of the differential equation. For $0 < \beta < 1$, it is noted that BVPs of integer order 3 can be derived from fractional differential equations of order $(\beta + 2)$. However, not all FDEs can be converted from fractional equations to integer ones. In our future work, we intend to investigate the Ulam–Hyers stability for the current problem, and several classes of nonlinear fractional equations can be extended to other types of equations with used operators in our work. Also, one can study the reported problem with mixed boundary conditions when generalized to FD.

Data Availability

No datasets were generated or analyzed during this study.

Disclosure

This work was carried out as part of our duties at Hodeidah University.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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