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Research Article

Analysis of Prey-Predator Scheme in Conjunction with Help and Gestation Delay

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This paper presents a three-dimensional continuous time dynamical system of three species, two of which are competing preys and one is a predator. We also assume that during predation, the members of both teams of preys help each other and the rate of predation of both teams is different. The interaction between prey and predator is assumed to be governed by a Holling type II functional response and discrete type gestation delay of the predator for consumption of the prey. In this work, we establish the local asymptotic stability of various equilibrium points to understand the dynamics of the model system. Different conditions for the coexistence of equilibrium solutions are discussed. Persistence, permanence of the system, and global stability of the positive interior equilibrium solution are discussed by constructing suitable Lyapunov functions when the gestation delay is zero, and there is no periodic orbit within the interior of the first quadrant of state space around the interior equilibrium. As we introduced time delay due to the gestation of the predator, we also discuss the stability of the delayed model. It is observed that the existence of stability switching occurs around the interior equilibrium point as the gestation delay increases through a certain critical threshold. Here, a phenomenon of Hopf bifurcation occurs, and a stable limit cycle corresponding to the periodic solution of the system is also observed. This study reveals that the delay is taken as a bifurcation parameter and also plays a significant role for the stability of the proposed model. Computer simulations of numerical examples are given to explain our proposed model. We have also addressed critically the biological implications of our analytical findings with proper numerical examples.

1. Introduction

In population dynamics, prey-predator coordination contributes a decisive responsibility through the last few decades [1, 2]. The dynamical relationship between predators and their preys has been recognized as an important topic in theoretical ecology since the discovery of the famous Lotka–Volterra equation. In ecosystem, prey-predator relationship contributes an imperative role. During the first World War, Lotka and Volterra symbolized mathematical appearance of prey-predator system [2, 3]. Since then, a huge number of research works have been carried out by following their mathematical expression. Hou [4] considered the permeance for general Lotka–Volterra model along with time delay, cooperation, and competition. Pal et al. [5] studied one-prey and one-predator harvesting system with the imprecise biological parameters. The nonautonomous Lotka–Volterra competition model is presented by Ahmed [6] and May [7] who discussed some simple mathematical models along with some complicated dynamics. There are a few approaches to signify prey-predator relations, for instance competition [8] and cooperation [3].

The stability and bifurcation investigation of prey-predator structure are determined by the functional response. In modelling the prey-predator structure, functional response makes a crucial contribution. There are several categories of functional responses in the subsistence literature [1, 2]. Holling category I functional response is categorized mathematically via a straight line through the origin [9, 10]. In the same way, the mathematical expression of Holling type II function is specified by $ex(f + x)^{-1}$ where e, x, and f have their respective meanings [9, 11–13].

A huge number of ecologists have premeditated prey-predator scheme by means of Holling category I functional response. In analysing the ecosystem, researchers have traced more information on two-dimensional prey-predator structure used for an elongated time. Food chain dynamics was discussed by Kuznetsov et al. [14] and Li et al. [15]. Srinivasu et al. [16] studied the consequences of assigning additional food on the dynamical system. Bandyopadhyay et al. [17] elucidated the dynamics of autotroph-herbivore ecosystem along with nutrient recycling. Theoretical ecologists were avoiding three or more dimensional species model system for an elongated time. It is mainly because higher dimensional models incorporate a greater number of differential equations which make it tricky to study the model structure. However, in a real ecosystem, higher dimensional models are very much imperative. Consequently, especially three-dimensional models are becoming more significant in different branches of ecology and ecosystem. Erbe et al. deals with the three-dimensional food chain model where mutual interference among predators and time delay due to gestation are proposed [18]. Fredman et al. presented a competition model involving three species [19]. The dynamical behavior of mussel and fish population is explained by Gazi et al. [20], and Maiti et al. [21] discussed the tritrophic food chain system with discrete time lag. Maiti et al. [22] extended the work and studied the effectiveness of biocontrol of pests in tea plants. Pal et al. [23] studied the influence of uncertainties in a food chain system. Pal et al. [24] presented a one predator and two prey systems by using fuzzy number and interval as biological parameters. The dynamical behavior of a one predator and two prey systems along with predator harvesting is studied by Gakkhar et al. [25].

Stage-structure-based prey-predator demonstrations by way of gestation time lag due to adulthood of the species are ornately discussed by numerous researchers. Bifurcation analysis of predator-prey models with the time lag is elucidated by Pal et al. [26] and Zhang [27]. Prey-predator system with discrete time lag and harvesting of the predator species is studied by Misra et al. [28], and stage-structured system of prey-predator with time lag for gestation is presented by Bandyopadhyay et al. [29]. Freedman et al. [30] presented Gauss prey-predator system including mutual interference and gestation time lag. This type of depiction is ended by inserting time delay in the differential equation. Generally, when predator species munch through the prey species, alteration of prey biomass into predator biomass is not instant. This necessitated some time lag for the alteration. Consequently, prey-predator-based holdup coordination is very much indispensable in mathematical ecology.

Naik et al. [31] recently introduced a two-dimensional discrete time chemical model with the subsistence of its fixed points; along with this, the flip and generalised flip bifurcations are identified for this system. The 1- and 2-parameter bifurcations of discrete time predator-prey model with the mixed functional response are discussed by Naik et al. [32]. Naik et al. [33] investigated the complex dynamical aspects of discrete-time Bazykin–Berezovskaya predator-prey system along with strong Allee effect.

A three-dimensional prey-predator model where two prey groups help each other from the predator group is discussed by Elettreby [3] and Tripathi et al. [34, 35] extended the work adding the competitive interaction among prey groups when there is no predator group present.

Following the works of Elettreby [3] and Tripathi [34, 35], in this contemporary circumstance, we deem a three-dimensional prey-predator (two prey teams and one predator team) structure with help and discrete type gestation delay of the predator. Holling type II functional rejoinder is used for interface amid prey squads and predator squad. In nonappearance of predator, the prey teams fight with apiece other for widespread food wherewithal. Once more, when prey teams are assaulted by the predator, then two prey species help each other for defensing them from predator. Also, after chomp through the prey, the escalation of predator species is not immediate, and it requires some time insulate for the exchange.

In this paper, we have discussed a three-dimensional predator-prey model with logistic equation where the prey species are competing with each other for the essential elements, e.g., food and space, and also, two teams of prey species are helping each other at the time of predation. To the best of our knowledge, all these above factors, at the same time, have not yet used.

Rest of the paper is presented in the following manner: research gaps are presented in Section 2. Mathematical portrayal of our projected structure is carried out in Section 3. Section 4 presents the positivity, boundedness, and permanence of our planned model. Behavior of the model in nonappearance of delay is described in Section 5. Behavior of our planned model in presence of delay is depicted through Section 6. Numerical illustrations through graphical staging are presented in Section 7. General discussion about our proposed model system is conducted in Section 8. Concluding remark is delivered in Section 9.

2. Research Gaps

Different types of prey-predator models along with different types of factors are analysed by many researchers. To discuss their work in a simplified way, we have presented a table which briefly explains the work carried out till now. In this table, the comparative discussion has been carried out in a tabular form which gives a quick overview about the research gaps. We have categorised the work on six main terms, viz., competition, mutualism, time delay, one predator-two preys, logistic equation, and Holling type-II functional response. From Table 1, it is clear that all the

Table 1: Summary of the research gap based on literature review.

Authors	Competition	Mutualism	Time delay	One predator-two preys	Logistic equation	Holling type-II functional response
Kar and Batabyal [36]	✓	×	✓	✓	✓	✓
Reddy et al. [37]	✓	×	✓	✓	✓	×
Xie and Xu [38]	×	×	✓	×	✓	✓
Zhang et al. [39]	✓	×	✓	✓	✓	✓
Arifah and Krisnawan [40]	×	×	✓	×	✓	✓
Savitri et al. [41]	✓	×	×	×	✓	×
Savitri et al. [42]	×	×	✓	✓	✓	✓
Mondal and Samnata [43]	✓	×	✓	✓	✓	✓
Naji and Majeed [44]	✓	×	✓	✓	×	×
Ikbal [45]	✓	×	×	✓	✓	✓
Present paper	✓	✓	✓	✓	✓	✓

factors have never been used at the same time by any researchers, but all these conditions are used in our proposed model.

3. Mathematical Portrayal of the Model Structure

Our anticipated mathematical model is supported in the subsequent suppositions:

(i) In nonexistence of the predator, both the preys are budding logistically

(ii) In nonattendance of the predator, two teams of preys fight with each other for widespread wherewithal

(iii) Two teams of preys are plateful themselves for the fortification from their attackers

(iv) Prey populace augmentation rate is abridged due to the consequence of predation which is deliberate by a term comparative to the prey and predator populations

(v) Predator's death is cropped up due to nonappearance of any prey teams

(vi) There might be antagonism amid the predator individuals due to insufficient quantity of food supply

(vii) For the development of predators, a time lag l is assumed

(viii) Our wished-for model is reserved only by two preys and one widespread predator

According to the suppositions (i)–(viii), our anticipated model structure can be articulated mathematically in the subsequent approach.

$$\begin{aligned}
\frac{\mathrm{d}X_p(t)}{\mathrm{d}t} &= \zeta_1 X_p(t)\left(1-\frac{X_p(t)}{\xi_1}\right) - \zeta_2\frac{X_p(t)}{\mu+\mu_1 X_p(t)}X_r(t) - \zeta_3 X_p(t)X_q(t) + \zeta_4 X_p(t)X_q(t)X_r(t),\\
\frac{\mathrm{d}X_q(t)}{\mathrm{d}t} &= \varkappa_1 X_q(t)\left(1-\frac{X_q(t)}{\xi_2}\right) - \varkappa_2\frac{X_q(t)}{\rho+\rho_1 X_q(t)}X_r(t) - \varkappa_3 X_p(t)X_q(t) + \varkappa_4 X_p(t)X_q(t)X_r(t),\\
\frac{\mathrm{d}X_r(t)}{\mathrm{d}t} &= -\phi_1 X_r(t) - \phi_2 X_r^2(t) + \phi_3\frac{X_p(t-l)}{\mu+\mu_1 X_p(t-l)}X_r(t) + \phi_4\frac{X_q(t-l)}{\rho+\rho_1 X_q(t-l)}X_r(t),
\end{aligned} \tag{1}$$

by means of the preliminary conditions

$$\begin{aligned}
X_p(\theta) &= \psi_1(\theta)\geq 0, \quad \theta\in[-l,0);\ \psi_1(0)>0,\\
X_q(\theta) &= \psi_2(\theta)\geq 0, \quad \theta\in[-l,0);\ \psi_2(0)>0,\\
X_r(\theta) &= \psi_3(\theta)\geq 0,
\end{aligned} \tag{2}$$

where $X_p(t)$ denotes the population density of the first prey, $X_q(t)$ denotes that of the second prey, and $X_r(t)$ denotes the population density of predator species; ξ_1 and ξ_2 are environmental carrying capacity of prey species X_p and X_q, respectively; ζ_1 and $\varkappa_1$ put up intrinsic augmentation rates of X_p and X_q correspondingly; ζ_2 and $\varkappa_2$ correspond to the per capita decrease rate of X_p and X_q correspondingly; μ and ρ give the environment defense for the species X_p and X_q, respectively; μ_1 and ρ_1 stand for the effect of handing time for predators; ζ_3, $\varkappa_3$ and ζ_4, $\varkappa_4$ denote the competition rates in the absence of predator species and cooperation coefficients for the prey species X_p and X_q, respectively; ϕ_1, ϕ_2, ϕ_3, and ϕ_4 stand for natural death rate of predator; X_r, density dependence rate of the predator, exchange rate of X_p and X_q into new offspring of predator species, respectively; $l\geq 0$ designates the requisite time taken by the prey species to become an adult. Finally, we consider that the coefficients ζ_1, ζ_2, ζ_3, ζ_4, $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, $\varkappa_4$, ϕ_1, ϕ_2, ϕ_3, ϕ_4, ξ_1, ξ_2, μ, ρ, μ_1, and ρ_1 are all positive numbers.

Our wished-for mathematical model is fine fitted for the group of Gazelles and Zebra which serve as two teams of prey and their attacker Tiger or Lion play the task of predator.

4. Positivity, Boundedness, and Permanence of Our Anticipated Model

The intention of this segment is to confer about the positivity, boundedness, and permanence of our anticipated representation (1) amid preliminary settings (2). To ascertain the affirmed behavior of our wished-for model, we state a foremost Lemma.

Lemma 1. *Under the conditions* $\alpha > 0$, $\beta > 0$, *and* $\mathrm{d}\Omega/\mathrm{d}t \le (\ge)\Omega(t)(\beta - \alpha\Omega(t))$, $\Omega(t_0) > 0$, *afterward* $\lim_{t\longrightarrow\infty}\sup \Omega(t) \le \beta/\alpha\,(\lim_{t\longrightarrow\infty}\inf \Omega(t) \ge \beta/\alpha)$.

Consequent Hale [46] and Jordan [47], we affirm the subsequent theorem.

Theorem 2. *The coefficients* $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \phi_1, \phi_2, \phi_3, \phi_4, \xi_1, \xi_2, \mu, \rho, \mu_1$, *and* ρ_1 *are bounded positive quantities. Subsequently, the model structure (1) has a sole solution on* $[0, +\infty)$ *by means of opening conditions (2).*

Theorem 3. *Solutions of the model scheme (1) through preliminary conditions (2) are always greater than zero for all positive values of* t.

Proof. We stumble on the fact that the right hand side of the model system (1) is absolutely continuous in addition to locally Lipschitzian on the space of continuous functions. Therefore, the solution $(X_p(t), X_q(t), X_r(t))$ of (1) by way of primary conditions (2) subsists and is unique on $[0, \zeta)$ for all $\zeta \in (0, +\infty)$. Equation of one of the model systems (1) gives

$$X_p(t) = X_p(0)\exp\left\{-\int_0^t\left(\zeta_1 X_p(s)\xi_1^{-1} - \zeta_1 + \zeta_2 X_r(s)\left(\mu + \mu_1 X_p(s)\right)^{-1} + \zeta_3 X_q(s) - \zeta_4 X_q(s)X_r(s)\right)\mathrm{d}s\right\} > 0. \tag{3}$$

Next equation of the model scheme (1) provides

$$X_q(t) = X_q(0)\exp\left\{-\int_0^t\left(\varkappa_1 X_q(s)\xi_2^{-1} - \varkappa_1 + \varkappa_2 X_r(s)\left(\rho + \rho_1 X_q(s)\right)^{-1} + \varkappa_3 X_p(s) - \varkappa_4 X_p(s)X_r(s)\right)\mathrm{d}s\right\} > 0. \tag{4}$$

In the same way, the last equation of the model structure (1) affords

$$X_r(t) = X_r(0)\exp\left\{-\int_0^t\left(\phi_1 + \phi_2 X_r(s) - \phi_3 X_p(s-l)\left(\mu + \mu_1 X_p(s-l)\right)^{-1} - \phi_4 X_q(s-l)\left(\rho + \rho_1 X_q(s-l)\right)^{-1}\right)\mathrm{d}s\right\} > 0, \tag{5}$$

which concludes the proof of the theorem. □

Theorem 4. *Let* $Q = \left\{(X_p, X_q, X_r)\colon X_p \in [0, Z_1], X_q \in [0, Z_2] \text{ and } X_r \in [0, Z_3]\right\}$ *where* $Z_1 = \max\left\{X_p(0), \xi_1\right\}$, $Z_2 = \max\left\{X_q(0), \xi_2\right\}$, $Z_3 = \max\{X_r(0), \xi_3\}$, *and* $\xi_3 = (\phi_3 Z_1\rho + \phi_4 Z_2\mu - \phi_1\mu\rho)(\phi_2\mu\rho)^{-1}$. *As a result of that, Q is invariantly positive.*

Proof. If it is assumed that $(X_p(0), X_q(0), X_r(0)) \in Q$, then $(X_p(t), X_q(t), X_r(t))$ is always positive. If we can prove that $X_p(t) \le Z_1$, $X_q(t) \le Z_2$, and $X_r(t) \le Z_3$, then it is clear that $(X_p(t), X_q(t), X_r(t)) \in Q$ for all values of t greater than or equal to zero. In the first attempt, we try to prove that $X_p(t) \le Z_1$. For avoiding the population outburst, it is assumed that the assistance term is dominated by competition amid prey species as well as interface among prey X_p and predator species X_r. Using the said deliberation and positive values of X_p, X_q, and X_r, the first equation of the model scheme (1) bestows that

$$\frac{\mathrm{d}X_p(t)}{\mathrm{d}t} \le \zeta_1 X_p(t)\left(1 - X_p(t)\xi_1^{-1}\right). \tag{6}$$

From (6), we get $X_p(t) = \max\left\{X_p(0), \xi_1\right\} = Z_1$ for all values of greater than or equal to zero. Again, next equation of (1) bestows that

$$\frac{\mathrm{d}X_q(t)}{\mathrm{d}t} \le \varkappa_1 X_q(t)\left(1 - X_q(t)\xi_2^{-1}\right). \tag{7}$$

Again from (7), we have $X_q(t) = \max\left\{X_q(0), \xi_2\right\} = Z_2$ for all values greater than or equal to zero. Also, the third equation of (1) bestows that

$$\frac{dX_r(t)}{dt} \le X_r(t)\left(\phi_3 Z_1 \mu^{-1} + \phi_4 Z_2 \rho^{-1} - \phi_1 - \phi_2 X_r(t)\right). \quad (8)$$

Using (8), we have $X_r(t) \le \max\{X_r(0), \xi_3\} = Z_3$ for all values of greater than or equal to zero, where $\xi_3 = (\phi_3 Z_1 \rho + \phi_4 Z_2 \mu - \phi_1 \mu\rho)(\phi_2 \mu\rho)^{-1}$. This completes the proof. □

Theorem 5. *If the conditions* $\zeta_2\xi_3\mu^{-1} + \zeta_3\xi_2 < \zeta_1$, $\varkappa_3\xi_3 \rho^{-1} + \varkappa_3\xi_1 < \varkappa_1$, *and* $\zeta_2\phi_3 f_1\xi_1 N + \zeta_1\phi_4 f_2\xi_2 M > \zeta_1\zeta_2\phi_1 MN$ *are contented, then the model scheme (1) is permanent.* M, N, f_1, *and* f_2 *are defined in the proof given underneath.*

Proof. Due to amply bulky t, equation (6) provides $X_p(t) \in (0, \xi_1)$. Furthermore, for amply bulky t, equations (7) and (8) provide $X_q(t) \in (0, \xi_2)$ and $X_r(t) \in (0, \xi_3)$. As X_p, X_q, and X_r are positive, the first equation of (6) furnishes

$$\begin{aligned}\frac{dX_p(t)}{dt} &= X_p(t)\left(\zeta_1 - \zeta_1 X_p(t)\xi_1^{-1} - \zeta_2 X_r(t)\left(\mu + \mu_1 X_p(t)\right)^{-1} - \zeta_3 X_q(t) + \zeta_4 X_q(t) X_r(t)\right)\\ &\ge X_p(t)\left(\zeta_1 - \zeta_1 X_p(t)\xi_1^{-1} - \zeta_2 k_3 \mu^{-1} - \zeta_3\xi_2\right)\\ &= X_p(t)\left(f_1 - \zeta_1 X_p(t)\xi_1^{-1}\right),\end{aligned} \quad (9)$$

due to amply bulky t, where $f_1 = \zeta_1 - \zeta_2\xi_3\mu^{-1} - \zeta_3\xi_2$. If $f_1 > 0$, i.e., $\zeta_2 k_3\mu^{-1} + \zeta_3\xi_2 < \zeta_1$ is satisfied, then Lemma 1 provides

$$\lim_{t\longrightarrow+\infty} \inf X_p(t) \ge f_1\xi_1\zeta_1^{-1} \equiv \theta_1. \quad (10)$$

Thus, for any arbitrary value of $\epsilon_1 > 0$, there exists a number $X_1\,(>0)$ such that $X_p(t) \ge \theta_1 - \epsilon_1$ for all values of t greater than X_1. In a similar fashion, second equation of (1) provides

$$\begin{aligned}\frac{dX_q(t)}{dt} &= X_q(t)\left(\varkappa_1 - \varkappa_1\xi_2^{-1} X_q(t) - \varkappa_2\left(\rho + \rho_1 X_q(t)\right)^{-1} X_r(t) - \varkappa_3 X_p(t) + \varkappa_4 X_p(t) X_r(t)\right)\\ &\ge X_q(t)\left(\varkappa_1 - \varkappa_1\xi_2^{-1} X_q(t) - \varkappa_2 k_3\rho^{-1} - \varkappa_3\xi_1\right)\\ &= X_q(t)\left(f_2 - \varkappa_1\xi_2^{-1} X_q(t)\right),\end{aligned} \quad (11)$$

for sufficiently large t, where $f_2 = \varkappa_1 - \varkappa_2\xi_3\rho^{-1} - \varkappa_3\xi_1$. If $f_2 > 0$, i.e., $\varkappa_2\xi_3\rho^{-1} + \varkappa_3\xi_1 < \varkappa_1$ is satisfied, then Lemma 1 gives

$$\lim_{t\longrightarrow+\infty} \inf X_q(t) \ge f_2\xi_2\varkappa_1^{-1} \equiv \theta_2. \quad (12)$$

Hence, for any arbitrary $\epsilon_2 > 0$, there exists a number $X_2\,(>0)$ such that $X_q(t) \ge \theta_2 - \epsilon_2$ for all values of t greater than X_2. Again, last equation of (1) provides

$$\begin{aligned}\frac{dX_r(t)}{dt} &= \left(-\phi_1 - \phi_2 X_r(t) + \phi_3\left(\mu + \mu_1 X_p(t)\right)^{-1} X_p(t) + \phi_4\left(\rho + \rho_1 X_q(t)\right)^{-1} X_q(t)\right) X_r(t)\\ &\ge X_r(t)\left(-\phi_1 - \phi_2 X_r(t) + \phi_3\left(\mu + \mu_1 X_p(t)\right)^{-1}\left(f_1\xi_1\zeta_1^{-1} - \epsilon_1\right) + \phi_4\left(\rho + \rho_1 X_q(t)\right)^{-1}\left(f_2\xi_2\varkappa_1^{-1} - \epsilon_2\right)\right).\end{aligned} \quad (13)$$

As an upshot of arbitrary $\epsilon_1 > 0$ and $\epsilon_2 > 0$, the above differential inequality can be expressed as

$$\frac{dX_r(t)}{dt} \ge X_r(t)\left(\phi_3 f_1\xi_1\zeta_1^{-1} M^{-1} + \phi_4 f_2\xi_2\varkappa_1^{-1} N^{-1} - \phi_1 - \phi_2 X_r(t)\right) = X_r(t)\left(f_3 - \phi_2 X_r(t)\right), \quad (14)$$

where $M = \mu + \mu_1 \xi_1$, $N = \rho + \rho_1 \xi_2$, and $f_3 = \phi_3 f_1 \xi_1 \zeta_1^{-1} M^{-1} + \phi_4 f_2 \xi_2 \varkappa_1^{-1} N^{-1} - \phi_1$. If the condition $f_3 > 0$, i.e., $\phi_3 f_1 \xi_1 \zeta_1^{-1} M^{-1} + \phi_4 f_2 \xi_2 \varkappa_1^{-1} N^{-1} > \phi_1$ is fulfilled, then Lemma 1 provides

$$\lim_{t \longrightarrow +\infty} \inf X_r(t) \geq \frac{1}{\phi_2}\left[\phi_3 f_1 \xi_1 \zeta_1^{-1} M^{-1} + \phi_4 f_2 \xi_2 \varkappa_1^{-1} N^{-1} - \phi_1\right] \equiv \theta_3, \tag{15}$$

for sufficiently large t. Also, from inequalities (6)–(8), together with Lemma 1, we have

$$\lim_{t \longrightarrow +\infty} \sup X_p(t) \leq \xi_1, \lim_{t \longrightarrow +\infty} \sup X_q(t) \leq \xi_2 \text{ and } \lim_{t \longrightarrow +\infty} \sup X_r(t) \leq \xi_3. \tag{16}$$

Now, choosing $\eta_1 = \min(\theta_1, \theta_2, \theta_3)$ and $\eta_2 = \max(\theta_1, \theta_2, \theta_3)$, we obtain the permanence of the system (1). □

5. Model Structure with Nonappearance of Time Lag

Model system (1) captures the subsequent structure in nonappearance of time lag l

$$\begin{aligned}
\frac{dX_p(t)}{dt} &= \zeta_1 X_p(t)\left(1 - \frac{X_p(t)}{\xi_1}\right) - \zeta_2 \frac{X_p(t)}{\mu + \mu_1 X_p(t)} X_r(t) - \zeta_3 X_p(t) X_q(t) + \zeta_4 X_p(t) X_q(t) X_r(t),\\
\frac{dX_q(t)}{dt} &= \varkappa_1 X_q(t)\left(1 - \frac{X_q(t)}{\xi_2}\right) - \varkappa_2 \frac{X_q(t)}{\rho + \rho_1 X_q(t)} X_r(t) - \varkappa_3 X_p(t) X_q(t) + \varkappa_4 X_p(t) X_q(t) X_r(t),\\
\frac{dX_r(t)}{dt} &= -\phi_1 X_r(t) - \phi_2 X_r^2(t) + \phi_3 \frac{X_p(t)}{\mu + \mu_1 X_p(t)} X_r(t) + \phi_4 \frac{X_q(t)}{\rho + \rho_1 X_q(t)} X_r(t),
\end{aligned} \tag{17}$$

together with preliminary stipulations

$$X_p(0) > 0, X_q(0) > 0 \text{ as well as } X_r(0) > 0. \tag{18}$$

Remark 6. If $\xi_1 = \varkappa_1/\varkappa_3$ is considered, then Γ_4 and Γ_2 are equal. Again if $\xi_2 = \zeta_1/\zeta_3$ is satisfied, then Γ_4 and Γ_3 are the same.

5.1. Subsistence of Equilibrium Points and Local Stability Investigation. The probable equilibrium points are specified underneath:

(*i*) $\Gamma_1(0,0,0)$ (*ii*) $\Gamma_2(\xi_1,0,0)$ (*iii*) $\Gamma_3(0,\xi_2,0)$ (*iv*) Γ_4 $(\overline{X}_p, \overline{X}_q, 0)$ (*v*) $\Gamma_5(0, X_q', X_r')$ (*vi*) $\Gamma_6(X_p'', 0, X_r'')$ (*vii*) Γ_7 (X_p^*, X_q^*, X_r^*).

It is palpable that the equilibrium points Γ_1, Γ_2, and Γ_3 subsist forever. Our only task is to authenticate the existence of lingering equilibrium points.

5.1.1. Subsistence of Γ_4. By solving the first two linear simultaneous equations

$$\zeta_1 - \zeta_1 \overline{X}_p \xi_1^{-1} - \zeta_3 \overline{X}_q = 0, \varkappa_1 - \varkappa_1 \xi_2^{-1} \overline{X}_q - \varkappa_3 \overline{X}_p = 0, \tag{19}$$

we get $\overline{X}_p = \xi_1 \varkappa_1 (\zeta_3 \xi_2 - \zeta_1)(\zeta_3 \varkappa_3 \xi_1 \xi_2 - \zeta_1 \varkappa_1)^{-1}$ and $\overline{X}_q = \xi_2 \zeta_1 (\varkappa_3 \xi_1 - \varkappa_1)(\zeta_3 \varkappa_3 \xi_1 \xi_2 - \zeta_1 \varkappa_1)^{-1}$. Therefore, Γ_4 exists provided $\zeta_3 \xi_2 > \zeta_1$ and $\varkappa_3 \xi_1 > \varkappa_1$.

5.1.2. Subsistence of Γ_5. Consider two nonlinear equations

$$\varkappa_1 - \varkappa_1 \xi_2^{-1} X_q' - \varkappa_2 \left(\rho + \rho_1 X_q'\right)^{-1} X_r' = 0, \tag{20}$$

$$-\phi_1 - \phi_2 X_r' + \phi_4 \left(\rho + \rho_1 X_q'\right)^{-1} X_r' = 0. \tag{21}$$

From (21), we get

$$X_q' = \frac{\rho(\phi_1 + \phi_2 X_r')}{\phi_4 - \rho(\phi_1 + \phi_2 X_r')}. \tag{22}$$

Putting the value of X_q' in (20), we have

$$A_1 X_r'^3 + A_2 X_r'^2 + A_3 X_r' + A_4 = 0, \tag{23}$$

where $A_1 = \rho_1^2 \phi_2^2 \varkappa_2 \xi_2 > 0$, $A_2 = 2\rho_1 \varkappa_2 \xi_2 \phi_2 (\rho_1 \phi_1 - \phi_4)$, $A_3 = \rho\rho_1 \varkappa_1 \xi_2 \phi_1 \phi_4 + \rho^2 \varkappa_1 \phi_1 \phi_2 + \varkappa_2 \xi_2 (\rho_1 \phi_1 - \phi_4)^2 > 0$, and $A_4 = \rho \varkappa_1 \phi_4 (\phi_1 (\rho_1 \xi_2 + \rho) - \xi_2 \phi_4)$. For the positive and unique solution of the (23), the stipulations specified underneath must be satisfied.

$$A_2 > 0 \text{ and } A_4 < 0 \text{ i.e., if } \rho_1\phi_1 > \phi_4 \text{ and } \phi_1(\rho_1\xi_2 + \rho) < \xi_2\phi_4. \tag{24}$$

Therefore, the equilibrium point subsists Γ_5 if the above supposed stipulations (24) are fulfilled.

Remark 7. In the same way, we easily prove that equilibrium point Γ_5 subsists under the conditions $\mu_1\phi_1 > \phi_3$ and $\phi_1(\mu_1\xi_1 + \mu) > \phi_3\xi_1$.

5.1.3. Subsistence of Γ_7. Clearly, (X_p^*, X_q^*, X_r^*) is achieved by solving the set of nonlinear simultaneous equations prearranged by

$$\zeta_1 - \zeta_1\xi_1^{-1}X_p^* - \zeta_2\left(\mu + \mu_1X_p^*\right)^{-1}X_r^* - \zeta_3X_q^* + \zeta_4X_q^*X_r^* = 0, \tag{25}$$

$$\varkappa_1 - \varkappa_1\xi_2^{-1}X_q^* - \varkappa_2\left(\rho + \rho_1X_q^*\right)^{-1}X_r^* - \varkappa_3X_p^* - \varkappa_4X_p^*X_r^* = 0, \tag{26}$$

$$\phi_1 + \phi_2X_r^* - \phi_3\left(\mu + \mu_1X_p^*\right)^{-1}X_p^* - \phi_4\left(\rho + \rho_1X_q^*\right)^{-1}X_q^* = 0. \tag{27}$$

Solving (25) and (27), we have

$$g\left(X_p^*, X_q^*\right) \equiv \varkappa_2\left(\rho + \rho_1X_q^*\right)^{-1}M\left(X_p^*, X_q^*\right) + X_p^*\left(\varkappa_3 - \varkappa_4M\left(X_p^*, X_q^*\right)\right) - \varkappa_1\left(1 - X_q^*\xi_2^{-1}\right), \tag{28}$$

where $M(X_p^*, X_q^*) = \{\zeta_1(1 - X_p^*\xi_1^{-1}) - \zeta_3X_q^*\}\{\mu + \mu_1X_p^*\}(\zeta_2 - \zeta_4X_q^*(\mu + \mu_1X_p^*))^{-1}$. From (28), if $X_p^* \longrightarrow 0$, then $X_q^* \longrightarrow X_{qa}^*$, where

$$L_1X_{\text{qa}}^{*3} + L_2X_{\text{qa}}^{*2} + L_3X_{\text{qa}}^* + L_4 = 0, \tag{29}$$

where $L_1 = \zeta_4\varkappa_1\rho_1\mu$, $L_2 = \zeta_4\varkappa_1\mu(\rho - \xi_2\rho_1)$, $L_3 = \xi_2(\zeta_3\varkappa_2 - \zeta_4\varkappa_1\mu\rho) + \zeta_2\varkappa_1(\rho_1\xi_2 - \rho)$, and $L_4 = \xi_2(\zeta_2\varkappa_1\rho - \zeta_1\varkappa_2\mu)$. As $L_1 > 0$; therefore, the (29) has a positive solution if $L_2 > 0$, $L_3 > 0$, and $L_4 < 0$.

From (28), one can obtain

$$\frac{dX_q^*}{dX_p^*} = -\frac{\partial g/\partial X_p^*}{\partial g/\partial X_q^*} = n_1m_1^{-1}\,(\text{say}). \tag{30}$$

It is evident that $dX_q^*/dX_p^* = n_1m_1^{-1} > 0$ if either

$$n_1 > 0 \text{ and } m_1 > 0 \text{ or } n_1 < 0 \text{ and } m_1 < 0 \tag{31}$$

From (26), we calculate X_r^* and substitute it in (27), and we acquire

$$\kappa_1\left(X_p^*, X_q^*\right) \equiv \phi_1 + \phi_2M\left(X_p^*, X_q^*\right) - \phi_3X_p^*\left(\mu + \mu_1X_p^*\right)^{-1} - \phi_4X_q^*\left(\rho + \rho_1X_q^*\right)^{-1}. \tag{32}$$

From (32), one can observe that, when $X_p^* \longrightarrow 0$, then $X_q^* \longrightarrow X_{\text{qb}}^*$, where

$$C_1X_{\text{qb}}^{*2} + C_2X_{\text{qb}}^* + C_3 = 0, \tag{33}$$

where $C_1 = (\zeta_4\phi_1 + \zeta_3\phi_3)\rho_1\mu - \zeta_4\phi_4\mu$, $C_2 = (\zeta_4\phi_1 + \zeta_3\phi_2)\mu\rho + \zeta_4\phi_4 - \zeta_2\rho(\phi_1 + \phi_2)$, and $C_3 = -\zeta_2\rho(\phi_1 + \phi_2)$. Since $C_3 < 0$, the (33) has a positive solution if $C_1 > 0$ and $C_2 > 0$.

From (32), one can obtain that

$$\frac{dX_q^*}{dX_p^*} = -\frac{\partial\kappa_1/\partial X_p^*}{\partial\kappa_1/\partial X_q^*} = -n_2m_2^{-1}\,(\text{say}). \tag{34}$$

It is obvious that $dX_q^*/dX_p^* < 0$ if either

$$n_2 > 0 \text{ and } m_2 > 0 \text{ or } n_2 < 0 \text{ and } m_2 < 0. \tag{35}$$

Therefore, the meeting point of (28) and (29) is unique. Also, the conditions (31) and (35) and the inequality $X_{\text{qa}}^* < X_{\text{qb}}^*$ are fulfilled. Again, by placing the values of X_p^* and X_q^* in (27), we have achieved the value of X_r^*. So, the subsistence of positive inner equilibrium point Γ_7 is verified.

At present, we are in the situation to talk about the local stability behavior of the model structure (17) at each proposed equilibrium points.

Theorem 8. *Nature of equilibrium point $\Gamma_1(0,0,0)$ is saddle point.*

Proof. At the equilibrium point $\Gamma_1(0,0,0)$, the variational matrix of the model structure (17) has obtained the form

$$J_{\Gamma_1} = \begin{bmatrix} \zeta_1 & 0 & 0 \\ 0 & \varkappa_1 & 0 \\ 0 & 0 & -\phi_1 \end{bmatrix}. \tag{36}$$

The eigenvalues of J_{Γ_1} are $\lambda_1 = \zeta_1$, $\lambda_2 = \varkappa_1$, and $\lambda_3 = -\phi_1$ and $\zeta_1 > 0$, $\varkappa_1 > 0$, and $\phi_1 > 0$. Therefore, $\Gamma_1(0,0,0)$ is the saddle point in nature in conjunction with unstable manifold in X_p and X_q directions, respectively, and stable manifold in the X_r direction. □

Theorem 9. *If the conditions $\xi_1 > \varkappa_1/\varkappa_3$ and $\phi_3\xi_1 < \phi_1(\mu + \mu_1\xi_1)$ are satisfied, then axial equilibrium point $\Gamma_2(\xi_1,0,0)$ is stable in nature.*

Proof. At the equilibrium point $\Gamma_2(\xi_1,0,0)$, the variational matrix J_{Γ_2} of the model structure (17) takes the form

$$J_{\Gamma_2} = \begin{bmatrix} -\zeta_1 & -\zeta_3\xi_1 & -\zeta_2\xi_1(\mu+\mu_1\xi_1)^{-1} \\ 0 & \varkappa_1 - \varkappa_3\xi_1 & 0 \\ 0 & 0 & -\phi_1 + \phi_3\xi_1(\mu+\mu_1\xi_1)^{-1} \end{bmatrix}. \quad (37)$$

The eigenvalues of J_{Γ_2} are $\lambda_1 = -\zeta_1 < 0$, $\lambda_2 = \varkappa_1 - \varkappa_3\xi_1$, and $\lambda_3 = \phi_3\xi_1(\mu+\mu_1\xi_1)^{-1} - \phi_1$. Now if $\lambda_2 < 0$ and $\lambda_3 < 0$, i.e., if $\xi_1 > \varkappa_1/\varkappa_3$ and $\phi_3\xi_1 < \phi_1(\mu+\mu_1\xi_1)$, then $\Gamma_2(\xi_1, 0, 0)$ is stable in nature. □

Theorem 10. *$\Gamma_3(0,\xi_2,0)$ is stable if $\xi_2 > \zeta_1/\zeta_3$ and $\phi_4\xi_2 < \phi_1(\rho+\rho_1\xi_2)$*

Proof. In the same way as above, this theorem can be proved. □

Remark 11. From the previous three theorems, ecologically it can be interpreted that the co-operating coefficients ζ_4 and $\varkappa_4$ do not give any participation for establishing the stability behavior of Γ_1, Γ_2, and Γ_3. The intercompetition coefficients $\varkappa_3$ and ζ_3 and the interference coefficients μ_1 and ρ_1 provide positive effect on the stability behavior of Γ_2 and Γ_3 correspondingly.

Theorem 12. *The predator-free equilibrium point $\Gamma_4(\overline{X}_p, \overline{X}_q, 0)$ is stable if the conditions $\phi_3\overline{X}_p(\mu+\mu_1\overline{X}_p)^{-1} + \phi_4\overline{X}_q(\rho+\rho_1\overline{X}_q)^{-1} < \phi_1$ and $\zeta_1\phi_1 > \zeta_3\varkappa_3\xi_1\xi_2$ are contented.*

Proof. Analogous to Γ_4, one of eigenvalues of the matrix J_{Γ_4} is specified by $\phi_3\overline{X}_p(\mu+\mu_1\overline{X}_p)^{-1} + \phi_4\overline{X}_q(\rho+\rho_1\overline{X}_q)^{-1} - \phi_1$. Since J_{Γ_4} is a 3×3 matrix, the remaining two eigenvalues are the solutions of the following equation:

$$\lambda^2 + a_{11}\lambda + a_{12} = 0, \quad (38)$$

where $a_{11} = \xi_1^{-1}\zeta_1\overline{X}_p + \xi_2^{-1}\varkappa_1\overline{X}_q > 0$ and $a_{12} = (\xi_1\xi_2)^{-1}\overline{X}_p\overline{X}_q(\zeta_1\varkappa_1 - \zeta_3\varkappa_3\xi_1\xi_2)$. As $a_{11} > 0$, according to Routh–Hurwitz criterion [1], the (38) has negative real part solutions if $a_{12} > 0$, i.e., $\zeta_1\varkappa_1 > \zeta_3\varkappa_3\xi_1\xi_2$. Hence, if the conditions $\phi_3\overline{X}_p(\mu+\mu_1\overline{X}_p)^{-1} + \phi_4\overline{X}_q(\rho+\rho_1\overline{X}_q)^{-1} < \phi_1$ and $\zeta_1\phi_1 > \zeta_3\varkappa_3\xi_1\xi_2$ are contented, then Γ_4 is stable in nature.

At $\Gamma_5(0, X_q', X_r')$, matrix J_{Γ_5} takes the form

$$J_{\Gamma_5} = \begin{bmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}, \quad (39)$$

where $\beta_{11} = \zeta_1 - \zeta_2\mu^{-1}X_r' - \zeta_3X_q' + \zeta_4X_q'X_r'$, $\beta_{12} = 0$, $\beta_{13} = 0$, $\beta_{21} = (\varkappa_4X_r' - \varkappa_3)X_q'$, $\beta_{22} = \varkappa_2\rho_1X_q'X_r'(\rho+\rho_1X_q')^{-2} - \varkappa_1\xi_2^{-1}X_q'$, $\beta_{23} = -\varkappa_2(\rho+\rho_1X_q')^{-1}X_q'$, $\beta_{31} = \phi_3\mu^{-1}X_r'$, $\beta_{32} = \rho\phi_4(\rho+\rho_1X_q')^{-2}X_r'$, and $\beta_{33} = -\phi_2X_r'$.

Therefore, the characteristic equation of J_{Γ_5} is

$$\lambda^3 + \Pi_1\lambda^2 + \Pi_2\lambda + \Pi_3 = 0, \quad (40)$$

where $\Pi_1 = -(\beta_{11}+\beta_{22}+\beta_{33})$, $\Pi_2 = \beta_{11}\beta_{33} + \beta_{11}\beta_{22} + \beta_{22}\beta_{33} - \beta_{23}\beta_{32}$, and $\Pi_3 = \beta_{11}\beta_{23}\beta_{32} - \beta_{11}\beta_{22}\beta_{33}$.

Therefore, using Routh–Hurwitz criterion [1, 2], we say that the solutions of the characteristic (40) has real component with less than zero iff

$$\Pi_1 > 0, \Pi_3 > 0 \text{ as well as } \Pi_1\Pi_2 - \Pi_3 > 0. \quad (41)$$

□

Theorem 13. *If the provision (41) is fulfilled, then $\Gamma_5(0, X_q', X_r')$ is stable in nature.*

Over again, at the point $\Gamma_6(X_p'', 0, X_r'')$, J_{Γ_6} obtains the form

$$J_{\Gamma_6} = \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ 0 & \eta_{22} & 0 \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix}, \quad (42)$$

where $\eta_{11} = \zeta_2\mu_1X_p''X_r''(\mu+\mu_1X_p'')^{-2} - \zeta_1\xi_1^{-1}X_p''$, $\eta_{12} = (\zeta_4X_r'' - \zeta_3)X_p''$, $\eta_{13} = -\varkappa_2(\mu+\mu_1X_p'')^{-1}X_p''$, $\eta_{21} = 0$, $\eta_{22} = \varkappa_1 - \varkappa_2\rho^{-1}X_r'' - \varkappa_3X_p''$, $\eta_{23} = 0$, $\eta_{31} = \phi_3\mu(\mu+\mu_1X_p'')^{-2}X_r''$, $\eta_{32} = \phi_4\rho^{-1}X_r''$, and $\eta_{33} = -\phi_2X_r''$.

Therefore, the characteristic equation of J_{Γ_6} is

$$\lambda^3 + G_1\lambda^2 + G_2\lambda + G_3 = 0, \quad (43)$$

where $G_1 = -(\eta_{11}+\eta_{22}+\eta_{33})$, $G_2 = \eta_{11}\eta_{22} + \eta_{11}\eta_{33} + \eta_{22}\eta_{33} - \eta_{13}\eta_{31}$, and $G_3 = \eta_{13}\eta_{31}\eta_{22} - \eta_{11}\eta_{22}\eta_{33}$.

Using Routh–Hurwitz condition [1, 2], we obtain the solutions of the characteristic (43) has nonpositive real component iff

$$G_1 > 0, G_3 > 0 \text{ as well as } G_1G_2 - G_3 > 0. \quad (44)$$

Theorem 14. *If the stipulation (44) is fulfilled, then $\Gamma_6(X_p'', 0, X_r'')$ is stable in nature.*

Finally, at $\Gamma_7(X_p^*, X_q^*, X_r^*)$, J_{Γ_7} takes the form

$$J_{\Gamma_7} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}, \quad (45)$$

where $\delta_{11} = X_p^*(\zeta_2\mu_1(\mu+\mu_1X_p^*)^{-2}X_r^* - \zeta_1\xi_1^{-1})$, $\delta_{12} = X_p^*(\zeta_4X_r^* - \zeta_3)$, $\delta_{13} = X_p^*(\zeta_4X_q^* - (\mu+\mu_1X_p^*)^{-1}\zeta_2)$, $\delta_{21} = X_q^*(\zeta_4X_r^* - \varkappa_3)$, $\delta_{22} = X_q^*(\varkappa_2\rho_1(\rho+\rho_1X_q^*)^{-2}X_r^* - \varkappa_1\xi_2^{-1})$, $\delta_{23} = X_q^*(\varkappa_4X_p^* - \varkappa_2(\rho+\rho_1X_q^*)^{-1})$, $\delta_{31} = \phi_3\mu(\mu+\mu_1X_p^*)^{-2}X_r^*$, $\delta_{32} = \varkappa_4\rho(\rho+\rho_1X_q^*)^{-2}X_r^*$, and $\delta_{33} = -\phi_2X_r^*$.

Therefore, the characteristic equation of J_{Γ_7} is

$$\lambda^3 + U_1\lambda^2 + U_2\lambda + U_3 = 0, \quad (46)$$

where $U_1 = -(\delta_{11}+\delta_{22}+\delta_{33})$, $U_2 = \delta_{11}\delta_{22} + \delta_{11}\delta_{33} + \delta_{22}\delta_{33} - \delta_{12}\delta_{21} - \delta_{13}\delta_{31} - \delta_{23}\delta_{32}$ and $U_3 = \{(\delta_{11}\delta_{23}\delta_{32} + \delta_{12}\delta_{21}\delta_{33} + \delta_{13}\delta_{22}\delta_{31}) - (\delta_{11}\delta_{22}\delta_{33} + \delta_{12}\delta_{23}\delta_{31} + \delta_{13}\delta_{22}\delta_{31})\}$.

Again using Routh–Hurwitz condition [1, 2], we obtain the solutions of the characteristic (46) has nonpositive real component iff

$$U_1 > 0, U_3 > 0 \text{ and } U_1 U_2 - U_3 > 0. \quad (47)$$

Theorem 15. *If the conditions (47) are contented, then inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is stable in nature.*

5.2. Stability in Global Perspective. The current section provides stability performance of the model structure (17) at $\Gamma_7(X_p^*, X_q^*, X_r^*)$ in a global point of view.

Theorem 16. *The conditions (A.1) and (A.2) imply the global stability behavior of the equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$.*

Proof. See Appendix. □

6. Model Analysis due to Time Lag

Due to time lag ($\neq 0$), stability nature of our wished-for replica structure (1) at $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is offered in the contemporary part. At $\Gamma_7(X_p^*, X_q^*, X_r^*)$, the system (1) has the characteristic equation as given in the following equation:

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 + (\lambda\Omega_4 + \Omega_5)e^{-\lambda l} = 0, \quad (48)$$

where $\Omega_1 = \Delta - \Sigma - \Theta$, $\Omega_2 = \Theta(\Sigma - \Delta) - \Sigma\Delta - \Phi\Psi$, $\Omega_3 = (\Sigma\Theta - \Phi\Psi)\Delta$, $\Omega_4 = (\Gamma\Pi + \Omega\Xi)$, $\Omega_5 = -(\Sigma\Gamma\Pi + \Phi\Gamma\Xi + \Omega\Psi\Pi + \Omega\Theta\Xi)$, $\Sigma = \zeta_2\mu_1 X_p^* X_r^*(\mu + \mu_1 X_p^*)^{-2} - \zeta_1 X_p^* \xi_1^{-1}$, $\Phi = (\zeta_3 X_p^* - \zeta_4 X_p^* X_r^*)$, $\Omega = (\zeta_2 X_p^*(\mu + \mu_1 X_p^*)^{-1} - \zeta_4 X_p^* X_q^*)$, $\Psi = (\varkappa_3 X_q^* - \varkappa_4 X_q^* X_r^*)$, $\Theta = \varkappa_2\rho_1 X_q^* X_r^*(\rho + \rho_1 X_q^*)^{-2} - \varkappa_1 X_q^* \xi_2^{-1}$, $\Gamma = (\varkappa_2 X_q^*(\rho + \rho_1 X_q^*)^{-1} - \varkappa_4 X_p^* X_q^*)$, $\Delta = \phi_2 X_r^*$, $\Xi = \phi_3 \mu X_r^*(\mu + \mu_1 X_p^*)^{-2}$, and $\Pi = \phi_4 \rho X_r^*(\rho + \rho_1 X_r^*)^{-2}$.

Let $\lambda = ik$, $(k > 0)$ be a solution of (48). Therefore, it is evident that

$$-\mathrm{i}k^3 - \Omega_1 k^2 + i\Omega_2 k + \Omega_3 + (\mathrm{i}k\Omega_4 + \Omega_5)e^{-\mathrm{i}kl} = 0. \quad (49)$$

Separating real and imaginary parts of (49), we attain

$$-\Omega_1 k^2 + \Omega_3 = -\Omega_5 \cos kl - \Omega_4 k \sin kl \text{ and } -k^3 + \Omega_2 k = -\Omega_4 k \cos kl + \Omega_5 \sin kl. \quad (50)$$

Adding both squared equations of (50), the subsequent equation is obtained

$$k^6 + Q_1 k^4 + Q_2 k^2 + Q_3 = 0, \quad (51)$$

where $Q_1 = \Omega_1^2 - 2\Omega_2$, $Q_2 = \Omega_2^2 - \Omega_4^2 - 2\Omega_1\Omega_3$, and $Q_3 = \Omega_3^2 - \Omega_5^2$. Therefore, the sole positive root k_+^2 of (51) is obtained under the conditions $Q_1 > 0, Q_2 > 0$, and $Q_3 < 0$. So, we dig up a pair of imaginary solutions $\pm ik_+$ of (48). The value of l is gettable by substituting the value of k_+^2 in (50). The idiom of l is specified by

$$l_j^+ = \frac{1}{k_+}\arccos\left[\left\{\Omega_4 k_+^4 + (\Omega_1\Omega_5 - \Omega_2\Omega_4)k_+^2 - \Omega_3\Omega_5\right\}\left\{\Omega_5^2 + \Omega_4^2 k_+^2\right\}^{-1}\right] + \frac{2j\pi}{k_+}, \quad j = 0, 1, 2, \ldots. \quad (52)$$

Next, Lemma is followed by the over argument.

Lemma 17. *The couple of imaginary solutions of (48) is attained for $l = l_0^+$.*

Theorem 18. *Assume l_i^+ is defined by (52), also $\Gamma_7(X_p^*, X_q^*, X_r^*)$ subsists. $Q_1 > 0$, $Q_2 > 0$, and $Q_3 < 0$ as well as the conditions (47) are satisfied. If l increases through zero, then there exists a value of l say l_0^+ for which $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is asymptotically stable for $0 < l < l_0^+$, and it becomes unstable when $l > l_0^+$. In addition to that, for $l = l_0^+$ (where $l_0^+ = l_j^+$ for $j = 0$) at the point $\Gamma_7(X_p^*, X_q^*, X_r^*)$, the structure (1) experiences a Hopf bifurcation.*

Proof. For $l = 0$, by the stipulation (47), the inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is stable in nature. Therefore, using Butler's lemma [30], we obtain that the inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ remains stable under the condition $l < l_0^+$. Our main intention is to show that the value of $d(\mathrm{Re}\lambda)/dl|_{l=l_0^+, k=k_+}$ is always greater than zero, which implies that when $l > l_0^+$, our proposed structure has the slightest one positive eigenvalue with positive real component. Depending on the above discussion, we conclude that the conditions of Hopf bifurcation are fulfilled along with expected periodic solution. Differentiating both sides of (48) with regard to l, we achieve

$$\left[3\lambda^2 + 2\Omega_1\lambda + \Omega_2 + \Omega_4 e^{-\lambda l} - l\{\Omega_4\lambda + \Omega_5\}e^{-\lambda l}\right]\frac{d\lambda}{dl} = \lambda(\Omega_4\lambda + \Omega_5)e^{-\lambda l}. \quad (53)$$

From (53), we get

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}l}\right)^{-1} = \frac{2\lambda^3 + \Omega_1\lambda^2 - \Omega_3}{-\lambda^2\left(\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3\right)} - \frac{\Omega_5}{\lambda^2\left(\Omega_4\lambda + \Omega_5\right)} - \frac{l}{\lambda}, \tag{54}$$

$$\begin{aligned}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}l}\right)^{-1}_{\lambda=ik_+} &= \frac{-2ik_+^3 - \Omega_1 k_+^2 - \Omega_3}{k_+^2\left(-ik_+^3 - \Omega_1 k_+^2 + i\Omega_2 k_+ + \Omega_3\right)} + \frac{\Omega_5}{k_+^2\left(i\Omega_4 k_+ + \Omega_5\right)} + i\frac{l}{k_+} \\ &= \frac{\left(\Omega_1 k_+^2 + \Omega_3 + 2ik_+^3\right)\left\{\left(\Omega_1 k_+^2 - \Omega_3\right) - i\left(k_+^3 - \Omega_2 k_+\right)\right\}}{k_+^2\left\{\left(\Omega_1 k_+^2 - \Omega_3\right)^2 + \left(k_+^3 - \Omega_2 k_+\right)^2\right\}} + \frac{\Omega_5}{k_+^2\left(\Omega_5^2 + \Omega_4^2 k_+^2\right)} + i\frac{l}{k_+}.\end{aligned} \tag{55}$$

Thus, $\operatorname{sign}\{\mathrm{d}(\mathrm{Re}\lambda)/\mathrm{d}l\}_{\lambda=ik_+} = \operatorname{sign}\left\{\mathrm{Re}\left\{(\mathrm{d}\lambda/\mathrm{d}l)^{-1}|_{\lambda=ik_+}\right\}\right\}$. After some algebraic manipulations from (55), we get

$$\begin{aligned}\operatorname{sign}\left\{\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}l}\right\}_{\lambda=ik_+} &= \frac{1}{k_+^2}\left[\frac{\left(\Omega_1^2 - 2\Omega_2\right)k_+^2 + 2k_+^6}{\Omega_5^2 + \Omega_4^2 k_+^2} + \frac{\Omega_5^2 - \Omega_3^2}{\Omega_5^2 + \Omega_4^2 k_+^2}\right] \\ &= \frac{1}{k_+^2}\left[\frac{Q_1 k_+^2 + 2k_+^6}{\Omega_5^2 + \Omega_4^2 k_+^2} - \frac{Q_3}{\Omega_5^2 + \Omega_4^2 k_+^2}\right].\end{aligned} \tag{56}$$

Hence, $(\mathrm{d}(\mathrm{Re}\lambda)/\mathrm{d}l)|_{l=l_0^+,k=k_+} > 0$ if Q_1 is greater than zero and Q_3 is less than zero. Hence, the transversality stipulation is fulfilled and Hopf bifurcation arose at $k = k_+$ and $l = l_0^+$. Thus, the proof of the theorem is completed. □

6.1. Assessment of the Time Lag Length to Safeguard Stability. To estimate the time lag length to protect the stability of the model structure (1), we initialized the system concerning its inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$. The initial model structure is prearranged underneath

$$\left.\begin{aligned}\frac{\mathrm{d}z_1}{\mathrm{d}t} &= \xi_{11}z_1 + \xi_{12}z_2 + \xi_{13}z_3, \\ \frac{\mathrm{d}z_2}{\mathrm{d}t} &= \xi_{21}z_1 + \xi_{22}z_2 + \xi_{23}z_3, \\ \frac{\mathrm{d}z_3}{\mathrm{d}t} &= \xi_{33}z_3 + \eta_{31}z_1(t-l) + \eta_{32}z_2(t-l),\end{aligned}\right\} \tag{57}$$

where $z_1(t) = X_p(t) + X_p^*$, $z_2(t) = X_q(t) + X_q^*$, and $z_3(t) = X_r(t) + X_r^*$.

If we apply Laplace transform on both sides of (57), we eventually gain

$$\begin{aligned}(r - \xi_{11})\overline{z_1}(r) &= \xi_{12}\overline{z_2}(r) + \xi_{13}\overline{z_3}(r) + z_1(0), \\ (r - \xi_{22})\overline{z_2}(r) &= \xi_{21}\overline{z_1}(r) + \xi_{23}\overline{z_3}(r) + z_2(0), \\ (r - \xi_{33})\overline{z_3}(r) &= \eta_{31}e^{-rt}\overline{z_1}(r) + \eta_{31}e^{-rt}\sigma_1(r) + \eta_{32}e^{-rt}\overline{z_2}(r) + \eta_{32}e^{-rt}\sigma_2(r) + z_3(0).\end{aligned} \tag{58}$$

Here, $\sigma_1(r) = \int_{-l}^{0} e^{-rt} z_1(t)dt$ and $\sigma_2(r) = \int_{-l}^{0} e^{-rt} z_2(t)dt$. The Laplace transform of $z_1(t)$, $z_2(t)$, and $z_3(t)$ is expressed by $\overline{z_1}(r)$, $\overline{z_2}(r)$, and $\overline{z_3}(r)$ correspondingly.

Using Nyquist theorem [26] and from [18], the local asymptotic stability stipulations of the inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ can be articulated in the subsequent form.

$$\operatorname{Im} R(\mathrm{i}w_0) > 0, \tag{59}$$

$$\operatorname{Re} R(\mathrm{i}w_0) = 0. \tag{60}$$

Here, $(r) = r^3 + \Omega_1 r^2 + \Omega_2 r + \Omega_3 + (r\Omega_4 + \Omega_5)e^{-rl}$. The smallest positive solution of the (60) is w_0.

In the previous section, we detect that in nonappearance of time lag, inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is stable. Then, by Bulter's lemma [18], we have adequately tiny >0, and all eigenvalues will be negative real components. Also, when l augments through zero, one can assure that there are no eigenvalues with positive real component that bifurcates from infinity.

In this current circumstance, stipulations (59) and (52) bestow

$$\Omega_2 w_0 - w_0^3 > \Omega_5 \sin w_0 l - \Omega_4 w_0 \cos w_0 l, \tag{61}$$

$$\Omega_3 - \Omega_1 w_0^2 = -\Omega_5 \cos w_0 l - \Omega_4 w_0 \sin w_0 l. \tag{62}$$

If the conditions (61) and (62) are fulfilled concurrently, then these stipulations furnish the sufficient stipulations for assurance stability. We shall utilize them to get an estimate on the length of delay. To estimate the length of time lag, we shall use these stipulations. Our objective is to specify the upper bound w_+ of w_0 which is independent of l in such a way that (61) holds for all values of w where $w \in [0, w_+]$. Therefore, for a particular value of w say w_0, we can redraft (62) as follows:

$$\Omega_1 w_0^2 = \Omega_3 + \Omega_5 \cos w_0 l + \Omega_4 w_0 \sin w_0 l. \tag{63}$$

Maximizing $\Omega_3 + \Omega_5 \cos w_0 l + \Omega_4 w_0 \sin w_0 l$ subject to $|\sin w_0 l| \le 1$ and $|\cos w_0 l| \le 1$, we obtain

$$\Omega_1 w_0^2 \le \Omega_3 + |\Omega_5| + |\Omega_4| w_0. \tag{64}$$

Hence, if

$$w_+ = \frac{1}{2\Omega_1}\left[|\Omega_4| + \sqrt{\Omega_4^2 + 4\Omega_1(\Omega_3 + |\Omega_5|)}\right], \tag{65}$$

it is clear from (65) that $w_0 \le w_+$.

Again (55) provides

$$w_0^2 < \Omega_2 + \Omega_4 \cos w_0 l - \frac{\Omega_5}{w_0} \sin w_0 l. \tag{66}$$

Also, for $l = 0$, the inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is stable as well as (66) holds due to adequately tiny $l > 0$. Replacing (63) into (66) gives

$$(\Omega_1\Omega_4 - \Omega_5)(1 - \cos w_0 l) + \left(\Omega_4 w_0 + \frac{\Omega_1\Omega_5}{w_0}\right)\sin w_0 l < \Omega_1\Omega_2 + \Omega_1\Omega_4 - \Omega_3 - \Omega_5. \tag{67}$$

The bounds of w_0 provides

$$(\Omega_1\Omega_4 - \Omega_5)(1 - \cos w_0 l) = (\Omega_1\Omega_4 - \Omega_5)2\sin^2\left(\frac{w_0 l}{2}\right) \le \frac{1}{2}|\Omega_1\Omega_4 - \Omega_5| w_+^2 l^2 \tag{68}$$

and

$$\left(\Omega_4 w_0 + \frac{\Omega_1\Omega_5}{w_0}\right)\sin w_0 l \le \left(\Omega_4 w_+^2 + \Omega_1\Omega_5\right) l. \tag{69}$$

Now, from (67)–(69), we get

$$l_1 l^2 + l_2 l < l_3, \tag{70}$$

where $l_1 = 1/2|\Omega_1\Omega_4 - \Omega_5| w_+^2$, $l_2 = (\Omega_4 w_+^2 + \Omega_1\Omega_5)$, and $l_3 = \Omega_1\Omega_2 + \Omega_1\Omega_4 - \Omega_3 - \Omega_5$.

Hence, if

$$l_+ = \frac{1}{2l_1}\left[-l_2 + \sqrt{l_2^2 + 4l_1 l_3}\right], \tag{71}$$

then stability is preserved for $0 \le l < l_+$.

From the above discussed outcomes, the next theorem is followed.

Theorem 19. *If the time lag l satisfies the inequality $0 < l < l_+$, then the model structure (1) is locally asymptotically stable where l_+ is provided in (71).*

7. Numerical Verifications

Numerical verification of analytical findings is very much important from a practical view point. This verification is not possible without the help of a computer software like MATLAB and Mathematica. In this current section, we have mainly verified the analytical finding by graphical presentation. Authentication of analytical finding of the model

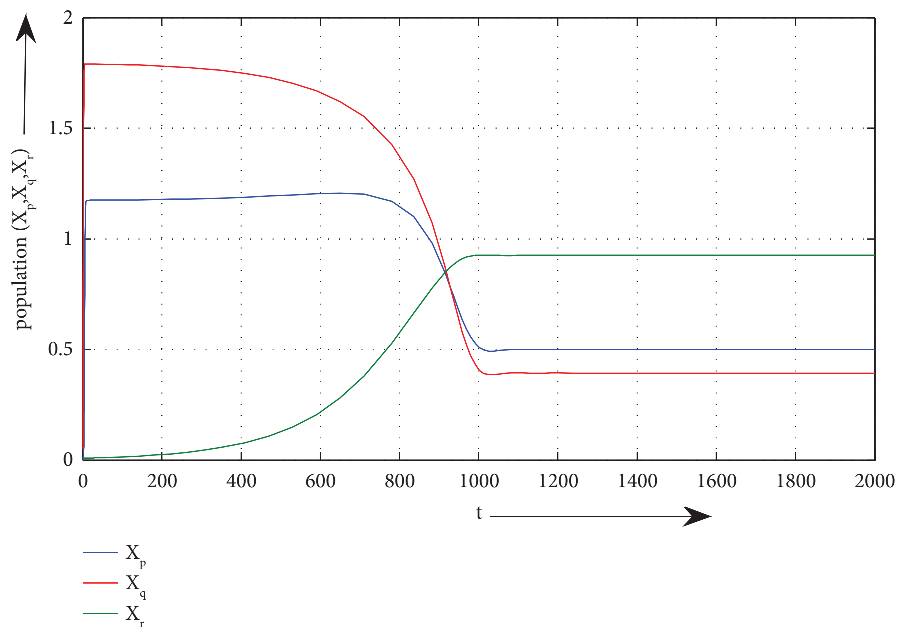

Figure 1: Time series plot of X_p, X_q, X_r with $X_p(0) = 0.001, X_q(0) = 0.05, X_r(0) = 0.008$.

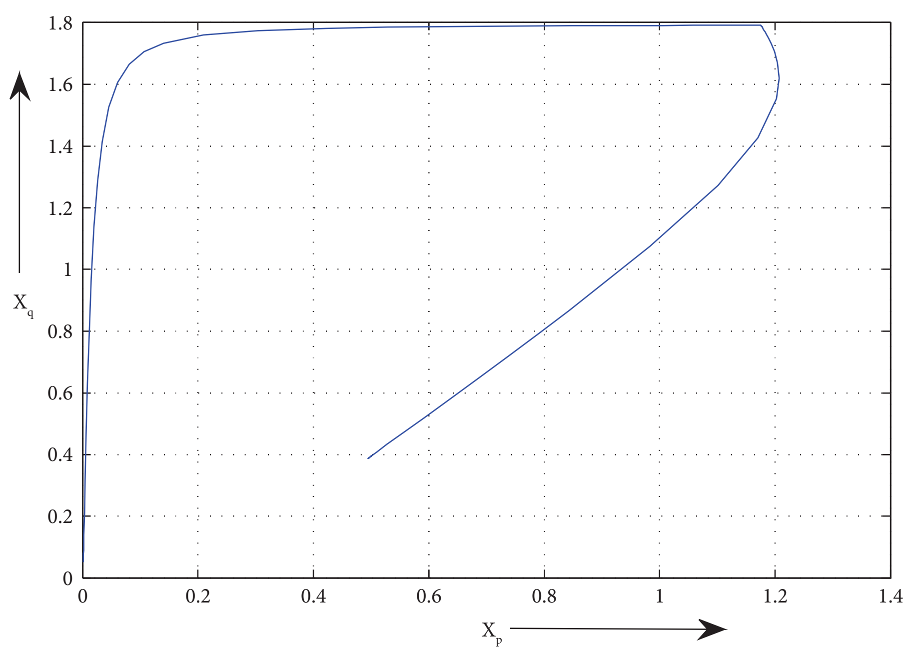

Figure 2: X_pX_q plane projection of the solution with $X_p(0) = 0.001$, $X_q(0) = 0.05$, $X_r(0) = 0.008$.

structure (1) is very much significant from a realistic standpoint also.

For the model structure (17), the values of the biological parameters and initial populations are taken in the subsequent way $\zeta_1 = 1.6$, $\zeta_2 = 0.1$, $\zeta_3 = 0.02$, $\zeta_4 = 0.842$, $\varkappa_1 = 2.2$, $\varkappa_2 = 0.1$, $\varkappa_3 = 0.004$, $\varkappa_4 = 0.842$, $\phi_1 = 0.002$, $\phi_2 = 0.0005$, $\phi_3 = 0.0001$, $\phi_4 = 0.0002$, $\xi_1 = 1.2$, $\xi_2 = 1.8$, $\mu = 0.07$, $\rho = 0.04$, $\mu_1 = 0.01$, $\rho_1 = 0.01$, and $(X_p(0), X_q(0), X_r(0)) = (0.001, 0.05, 0.008)$.

For our setting parameter values, the inner equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*)$ is equal to $(0.499629, 0.394704, 0.924954)$. Also, for this set of parameter values, all the species persist and we get a nontrivial equilibrium point $\Gamma_7(X_p^*, X_q^*, X_r^*) = (0.499629, 0.394704, 0.924954)$. At Γ_7 $(0.499629, 0.394704, 0.924954)$, conditions of Theorem 15 are fulfilled as $U_1 = 0.877854 > 0$, $U_3 = 0.0019832 > 0$, and $U_1U_2 - U_3 = 0.0499693 > 0$. Hence, $(0.499629, 0.394704, 0.924954)$ is locally asymptotically stable.

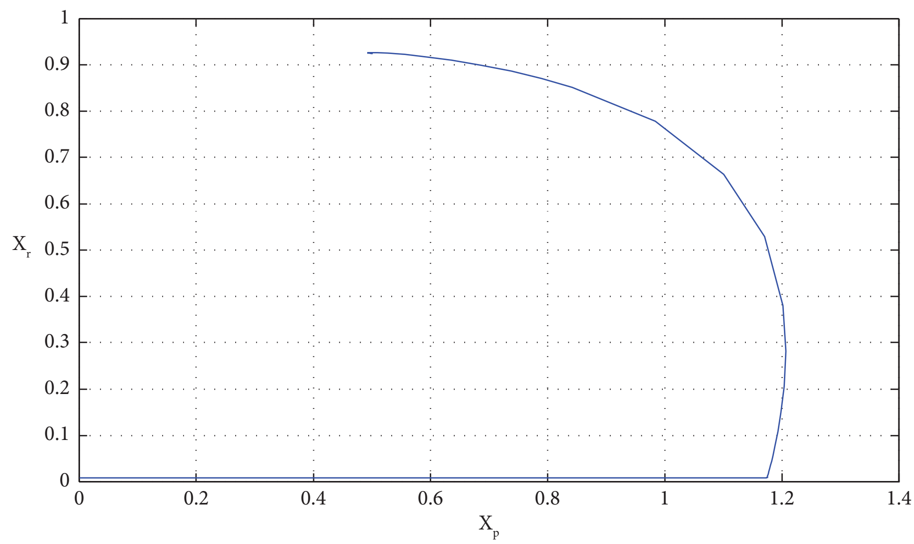

FIGURE 3: X_pX_r plane projection of the solution with $X_p(0) = 0.001$, $X_q(0) = 0.05$, $X_r(0) = 0.008$.

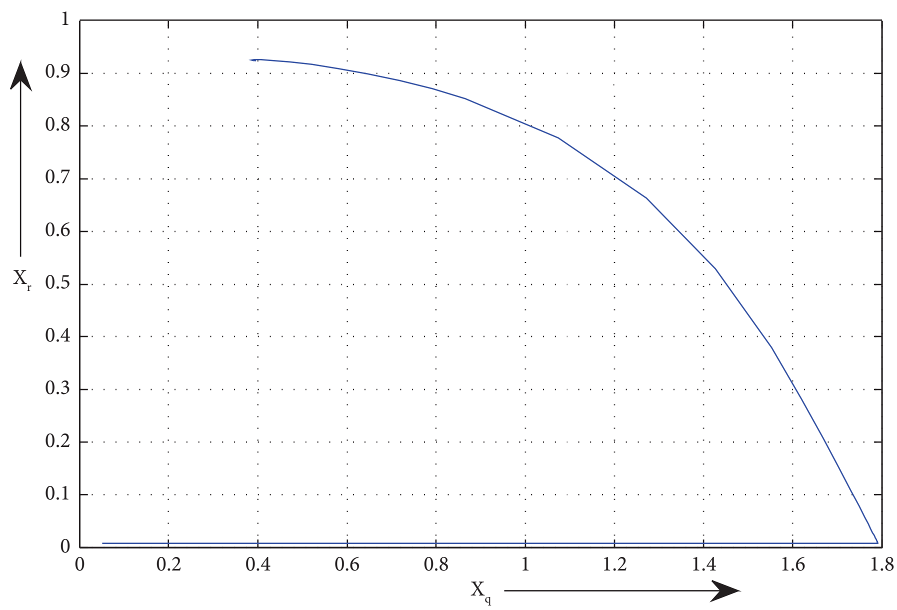

FIGURE 4: X_qX_r plane projection of the solution with $X_p(0) = 0.001$, $X_q(0) = 0.05$, $X_r(0) = 0.008$.

Figure 1 depicts that starting with an initial condition $(0.001, 0.05, 0.008)$, the populations reach their respective stable situation X_p^*, X_q^*, and X_r^* in a limited period of time.

As per our expectation, we monitor from Figure 1 that as the predator species steadily enlarges and both the prey species steadily diminishes and finally after a limited period of time, the population system comes to a steady-state situation.

Figures 2, 3, and 4 present the X_pX_q plane, X_pX_r plane, and X_qX_r plane protrusions of the system (17) correspondingly.

Figure 2 illustrates that in the X_pX_q projection, the trajectory starting with the initial condition $(0.001, 0.05, 0.008)$ converges to the inner equilibrium point Γ_7. Similarly, Figures 3 and 4 portray the X_pX_r plane and X_qX_r plane projection, respectively. In Figures 3 and 4, the trajectory starting with the initial condition $(0.001, 0.05, 0.008)$ converges to the inner equilibrium point Γ_7.

Next, we investigate for the delay model (1). It is a well-known fact that if a model structure is stable in nonattendance of time lag $(l = 0)$, it is not assured that the system remains stable in the occurrence of time lag $(l \neq 0)$. Let us choose the parametric values of the same system as stated above. Now, for these choices of parameters, Theorem 18 and Lemma 17 assured that (51) has a sole positive solution $k_+ = 0.0417958$ and (52) gives the critical value $l_0^+ = 23.7602$. Using Theorem 18 and Figures 5(a), 5(b) and 6(a), 6(b), it is monitored that when $l < 23.7602$, then the inner equilibrium point $\Gamma_7(0.499629, 0.394704, 0.924954)$ exhibits asymptotically stable behavior.

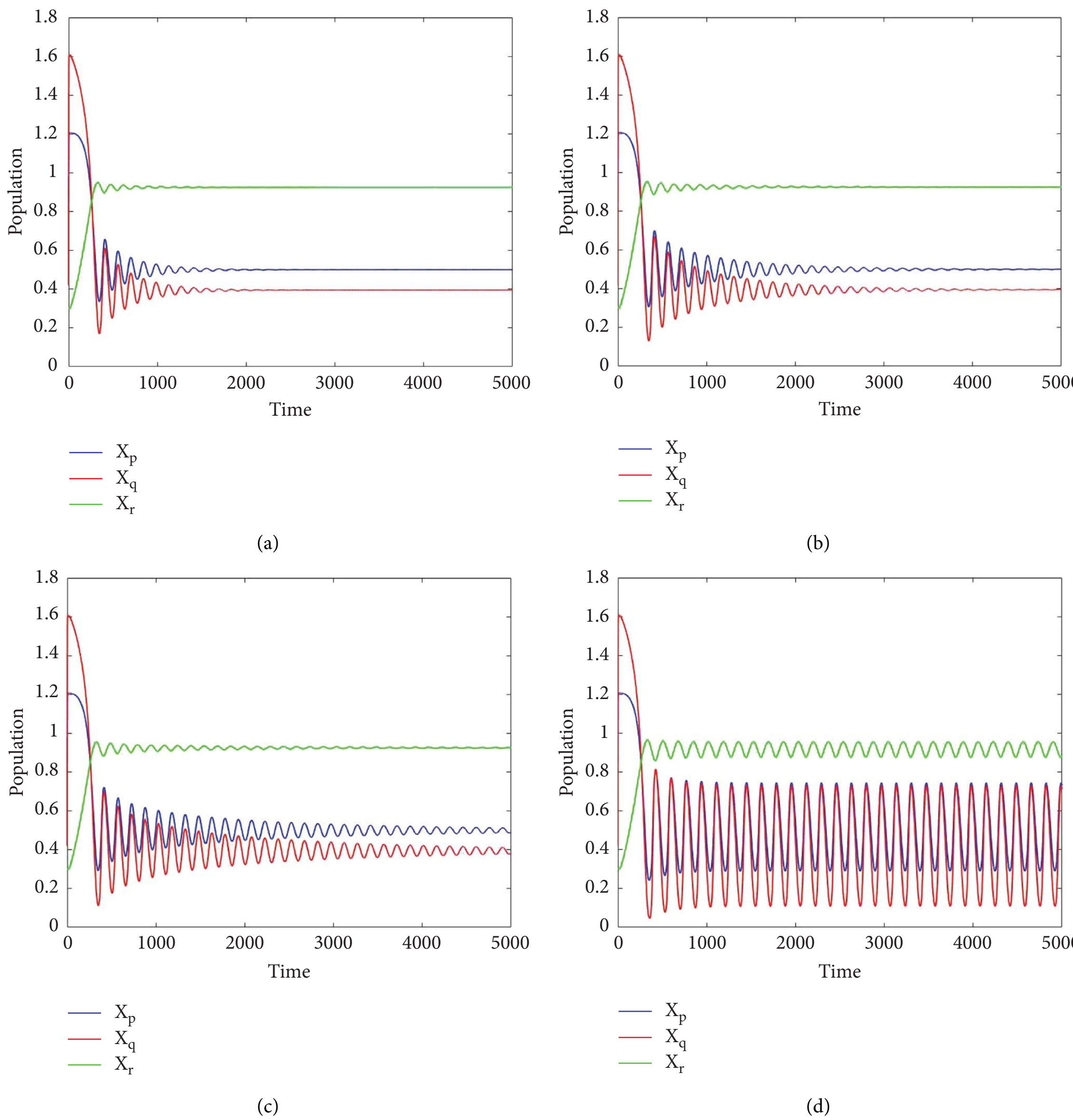

Figure 5: Time series of the model (1) with $X_p(0) = 0.51, X_q(0) = 0.42, X_r(0) = 0.3$. (a) Stable behavior of X_p, X_q, X_r for $l = 21 < l_0^+ = 23.7602$, (b) stable behavior of X_p, X_q, X_r for $l = 23 < l_0^+ = 23.7602$, (c) unstable behavior of X_p, X_q, X_r for $l = 24 > l_0^+ = 23.7602$, and (d) unstable behavior of X_p, X_q, X_r for $l = 28 > l_0^+ = 23.7602$.

Figure 5 depicts that as $l < 23.7602$, all the species of the population system converges to their respective stable state levels also as $l > 23.7602 = l_0^+$ and if all the other parameter values are kept the same as stated above, then a delayed model structure becomes unstable. Again from Figure 6, the phase space diagram is portrayed.Therefore, when we augment the values of l above the critical value l_0^+, the population system exhibits growing oscillatory behavior. From Figures 5 and 6, the change in the stability behavior of this system is clearly visible. When the value of l is slightly higher than its critical value l_0^+, the stable equilibrium point becomes unstable.

Therefore, we may conclude that keeping other parameters fixed, if we take $l > l_0^+$, then $\Gamma_7(0.499629, 0.394704, 0.924954)$ becomes unstable and exhibits Hopf bifurcation, and a bifurcating periodic solution is noticed around $\Gamma_7(0.499629, 0.394704, 0.924954)$.

8. Results and Discussions

The current paper studied a three-dimensional prey-predator co-operative structure along with gestational time lag of the predator species. In the ecosystem, there exist many species who lived in a crowd and cooperate themselves by distributing similar territory. As the species sharing the same territory, depending on proper circumstances, the grouped populations may co-operate sometimes, and they may also sometimes compete with themselves. Sea anemone and the clown fish are the good examples of the grouping population. A proper predator-prey structure can be formed

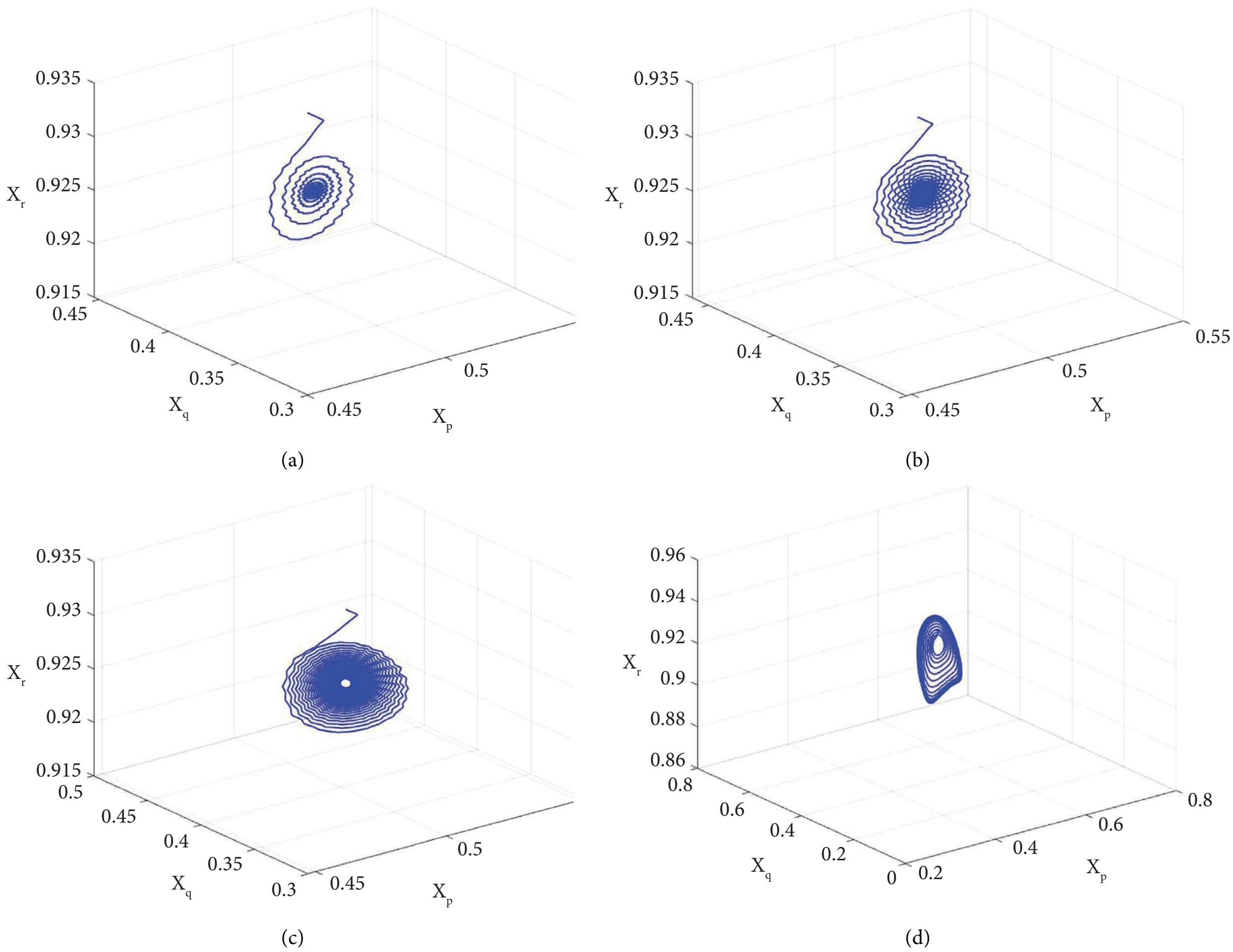

FIGURE 6: Phase portrait of the model (1) with $X_p(0) = 0.51$, $X_q(0) = 0.42$, $X_r(0) = 0.93$. (a) Stable behavior of X_p, X_q, X_r for $l = 21 < l_0^+ = 23.7602$, (b) stable behavior of X_p, X_q, X_r for $l = 23 < l_0^+ = 23.7602$, (c) unstable behavior of X_p, X_q, X_r for $l = 24 > l_0^+ = 23.7602$, and (d) unstable behavior of X_p, X_q, X_r for $l = 28 > l_0^+ = 23.7602$.

with the help of these categories of grouping species. For the existence of a particular species, the most essential elements are food and space. Therefore, the prey species are competing with each other for such kind of general assets. On the other hand, predator species cropped the prey population at a fixed rate due to their survival. Also, the alteration of prey biomass to predator biomass is not instantaneous; it needs some time lag for alteration. Motivated by these facts, in this current paper, we create and investigate dissimilar behaviors of a prey-predator time lag model structure consisting of two groups of contending as well as supportive preys and one group of predators.

The existences of different equilibrium points of the model structure (1) and their stabilities are pointed out carefully. Global stability behavior of the inner equilibrium point is addressed properly. We finally observe that when the delay parameter $l < l_0^+$ (critical vale of l), the stability nature of the inner equilibrium point becomes unstable and exhibits Hopf bifurcations. Our analytical findings are properly illustrated graphically through Figures 1 to 6 correspondingly. Time-series plot of X_p, X_q, X_r, X_pX_q plane projection, X_pX_r plane projection, and X_qX_r plane projection is described, and the stable and unstable phase space diagrams are illustrated in the above figures. Also, it is demonstrated that the phase portrait of the model is stable for $l < 23.7602$ and unstable for $l > 23.7602$.

9. Conclusions

Finally, we conclude that the whole of our proposed delayed model structure is supposed in a deterministic environment. However, our model can be made more pragmatic and attractive if it is supposed in fuzzy, interval, or in stochastic environment for some parameter uncertainties or some other environmental characteristics. This conception is left for further research trend. As a part of future work, to make the system more realistic, we can include impreciseness in the parameters of the model to enhance our model.

Appendix

Let us define a Lyapunov function as

$$\kappa = V_1\left(X_p - X_p^* - X_p^* \ln \frac{X_p}{X_p^*}\right) + V_2\left(X_q - X_q^* - X_p^* \ln \frac{X_q}{X_q^*}\right) + V_3\left(X_r - X_r^* - X_r^* \ln \frac{X_r}{X_r^*}\right), \tag{A.1}$$

for the positive values of the constants V_1, V_2, and V_3 which will be specified soon after.

At this point, $\kappa(X_p, X_q, X_r) \geq 0$ since $\theta - 1 \geq \ln \theta$ for $\theta > 0$ and $\kappa(X_p^*, X_q^*, X_r^*) = 0$. Differentiating κ with regard to t alongside, the solutions of model (17) provide

$$\begin{aligned}
\frac{d\kappa}{dt} &= V_1 \frac{X_p - X_p^*}{X_p}\frac{dX_p}{dt} + V_2 \frac{X_q - X_q^*}{X_q}\frac{dX_q}{dt} + V_3 \frac{X_r - X_r^*}{X_r}\frac{dX_r}{dt} \\
&= V_1\left[\zeta_1\left(1 - X_p\xi_1^{-1}\right) - \zeta_2 X_r\left(\mu + \mu_1 X_p\right)^{-1} - \zeta_3 X_q + \zeta_4 X_q X_r\right] \\
&\quad + V_2\left[\varkappa_1\left(1 - X_q\xi_2^{-1}\right) - \varkappa_2 X_r\left(\rho + \rho_1 X_q\right)^{-1} - \varkappa_3 X_p + \varkappa_4 X_p X_r\right] \\
&\quad + V_3\left[-\phi_1 - \phi_2 X_r + \phi_3 X_p\left(\mu + \mu_1 X_p\right)^{-1} + \phi_4 X_q\left(\rho + \rho_1 X_q\right)^{-1}\right] \\
&= V_1\left[-\zeta_1\xi_1^{-1}\left(X_p - X_p^*\right) - \zeta_2\left\{X_r\left(\mu + \mu_1 X_p\right)^{-1} - X_r^*\left(\mu + \mu_1 X_p^*\right)^{-1}\right\} - \zeta_3\left(X_q - X_q^*\right) + \zeta_4\left(X_q X_r - X_q^* X_r^*\right)\right]\left(X_p - X_p^*\right) \\
&\quad + V_2\left[-\varkappa_1\xi_2\left(X_q - X_q^*\right) - \varkappa_2\left\{X_r\left(\rho + \rho_1 X_q\right)^{-1} - X_r^*\left(\rho + \rho_1 X_q^*\right)^{-1}\right\} - \varkappa_3\left(X_p - X_p^*\right) + \varkappa_4\left(X_p X_r - X_p^* X_r^*\right)\right]\left(X_q - X_q^*\right) \\
&\quad + V_3\left[-\phi_2\left(X_r - X_r^*\right) + \phi_3\left\{X_p\left(\mu + \mu_1 X_p\right)^{-1} - X_p^*\left(\mu + \mu_1 X_p^*\right)^{-1}\right\} + \phi_4\left\{X_q\left(\rho + \rho_1 X_q\right)^{-1} - X_q^*\left(\rho + \rho_1 X_q^*\right)^{-1}\right\}\right]\left(X_r - X_r^*\right) \\
&= -V_1\zeta_1\xi_1^{-1}\left(X_p - X_p^*\right)^{-2} - V_2\varkappa_1\xi_2^{-1}\left(X_q - X_q^*\right)^{-2} - V_3\phi_2\left(X_r - X_r^*\right)^{-2} \\
&\quad - \mu\left(V_1\zeta_2 - V_3\phi_3\right)\left(X_p - X_p^*\right)\left(X_r - X_r^*\right)\left(\mu + \mu_1 X_p\right)^{-1}\left(\mu + \mu_1 X_p^*\right)^{-1} \\
&\quad - \rho\left(V_2\varkappa_2 - V_3\phi_4\right)\left(X_q - X_q^*\right)\left(X_r - X_r^*\right)\left(\rho + \rho_1 X_q\right)^{-1}\left(\rho + \rho_1 X_q^*\right)^{-1} \\
&\quad - \mu_1 V_1\zeta_2\left(X_p^* X_r - X_p X_r^*\right)\left(\mu + \mu_1 X_p\right)^{-1}\left(\mu + \mu_1 X_p^*\right)^{-1} - \rho_1 V_2\varkappa_2\left(X_q^* X_r - X_q X_r^*\right)\left(\rho + \rho_1 X_q\right)^{-1}\left(\rho + \rho_1 X_q^*\right)^{-1} \\
&\quad - \left(V_1\zeta_3 + V_2\varkappa_3\right)\left(X_p - X_p^*\right)\left(X_q - X_q^*\right) + V_2\varkappa_4\left(X_p X_r - X_p^* X_r^*\right)\left(X_q - X_q^*\right).
\end{aligned} \tag{A.2}$$

By putting $V_1 = \phi_3$, $V_2 = \zeta_2\phi_4\varkappa_2^{-1}$, and $V_3 = \zeta_2$, subsequently making simpler $d\kappa/dt$, we acquired that

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= -\phi_3\zeta_1\xi_1^{-1}\left(X_p - X_p^*\right)^2 - \zeta_2\phi_4\varkappa_1\varkappa_2^{-1}\xi_2^{-1}\left(X_q - X_q^*\right)^2 - \zeta_2\phi_2\left(X_r - X_r^*\right)^2 \\ &\quad - \mu_1\phi_3\zeta_2\left(X_p^*X_r - X_pX_r^*\right)\left(\mu + \mu_1X_p\right)^{-1}\left(\mu + \mu_1X_p^*\right)^{-1} - \rho_1\zeta_2\phi_4\varkappa_2\left(X_q^*X_r - X_qX_r^*\right)\left(\rho + \rho_1X_q\right)^{-1}\left(\rho + \rho_1X_q^*\right)^{-1} \\ &\quad - \left(\phi_3\zeta_3 + \zeta_2\varkappa_3\phi_4\varkappa_2^{-1}\right)\left(X_p - X_p^*\right)\left(X_q - X_q^*\right) + \phi_3\zeta_4\left(X_qX_r - X_q^*X_r^*\right)\left(X_p - X_p^*\right) \\ &\quad + \zeta_2\varkappa_4\phi_4\varkappa_2^{-1}\left(X_pX_r - X_p^*X_r^*\right)\left(X_q - X_q^*\right).\end{aligned} \tag{A.3}$$

Clearly, $\mathrm{d}\kappa/\mathrm{d}t = 0$ at $\Gamma_7\left(X_p^*, X_q^*, X_r^*\right)$.
Now, $\mathrm{d}\kappa/\mathrm{d}t < 0$ if

$$X_p^* < X_p < \frac{X_p^*X_r}{X_r^*}, X_q^* < X_q < \frac{X_q^*X_r}{X_r^*} \text{ and } X_p^*X_q < X_pX_q^* \tag{A.4}$$

or

$$\frac{X_p^*X_r}{X_r^*} < X_p < X_p^*, \frac{X_q^*X_r}{X_r^*} < X_q < X_q^* \text{ and } X_pX_q^* < X_p^*X_q. \tag{A.5}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Kot, *Elements of Mathematical Ecology*, Cambridge University Press, Cambridge, UK, 2001.

[2] J. D. Murray, *Mathematical Biology*, Springer, New York, NY, USA, 1993.

[3] M. F. Elettreby, "Two prey one-predator model," *Chaos, Solitons & Fractals*, vol. 39, no. 5, pp. 2018–2027, 2009.

[4] Z. Hou, "On permanence of Lotka–Volterra systems with delays and variable intrinsic growth rates-Volterra systems with delays and variable intrinsic growth rates," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 2, pp. 960–975, 2013.

[5] D. Pal, G. Mahaptra, and G. P. Samanta, "Optimal harvesting of prey-predator system with interval biological parameters: a bioeconomic model," *Mathematical Biosciences*, vol. 241, no. 2, pp. 181–187, 2013.

[6] S. Ahmad, "On the nonautonomous Volterra-Lotka competition equations," *Proceedings of the American Mathematical Society*, vol. 117, no. 1, pp. 199–204, 1993.

[7] R. M. May, "Simple mathematical models with very complicated dynamics," *Nature*, vol. 261, no. 5560, pp. 459–467, 1976.

[8] J. M. Cushing, "Periodic lotka–volterra competition equations," *Journal of Mathematical Biology*, vol. 24, no. 4, pp. 381–403, 1986.

[9] A. A. Berryman, "The orgins and evolution of predator-prey theory," *Ecology*, vol. 73, no. 5, pp. 1530–1535, 1992.

[10] D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, "A model for tropic interaction," *Ecology*, vol. 56, no. 4, pp. 881–892, 1975.

[11] T. K. Kar, "Stability analysis of a prey–predator model incorporating a prey refuge," *Communications in Nonlinear Science and Numerical Simulation*, vol. 10, no. 6, pp. 681–691, 2005.

[12] C. S. Holling, "Some characteristics of simple types of predation and parasitism," *The Canadian Entomologist*, vol. 91, no. 7, pp. 385–398, 1959.

[13] M. A. Aziz-Alaoui and M. Daher Okiye, "Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes," *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1069–1075, 2003.

[14] Y. Kuznetsov and S. Rinaldi, "Remarks on food chain dynamics," *Mathematical Biosciences*, vol. 134, pp. 1–33, 1996.

[15] B. Li and Y. Kuang, "Simple food chain in a chemostat with distinct removal rates," *Journal of Mathematical Analysis and Applications*, vol. 242, no. 1, pp. 75–92, 2000.

[16] P. D. N. Srinivasu, B. S. R. V. Prasad, and M. Venkatesulu, "Biological control through provision of additional food to predators: a theoretical study," *Theoretical Population Biology*, vol. 72, no. 1, pp. 111–120, 2007.

[17] M. Bandyopadhyay, R. Bhattacharyya, and B. Mukhopadhyay, "Dynamics of an autotroph herbivore ecosystem with nutrient recycling," *Ecological Modelling*, vol. 176, no. 1-2, pp. 201–209, 2004.

[18] L. H. Erbe, H. Freedman, and V. H. Rao, "Three-species food chain models with mutual interference and time delays," *Mathematical Biosciences*, vol. 80, no. 1, pp. 57–80, 1986.

[19] H. I. Freedman and P. Waltman, "Persistence in a model of three competitive populations," *Mathematical Biosciences*, vol. 73, no. 1, pp. 89–101, 1985.

[20] N. Huda Gazi, S. Rahaman Khan, and C. Gopal Chakrabarti, "Integration of mussel in fish-farm: mathematical model and analysis," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 1, pp. 74–86, 2009.

[21] A. Maiti, A. K. Pal, and G. P. Samanta, "Effect of time delay on a food chain model," *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 189–203, 2008.

[22] A. Maiti, A. K. Pal, and G. P. Samanta, "Usefulness of biocontrol of pests in tea: a mathematical model," *Mathematical Modelling of Natural Phenomena*, vol. 3, no. 4, pp. 96–113, 2008.

[23] D. Pal, S. K. Mahato, and M. Mukherjee, *Food Chain Models with Interval Uncertainty*, Bluerose Publishers Pvt Ltd, India, 2022.

[24] D. Pal and G. S. Mahapatra, "A bioeconomic modeling of two-prey and one-predator fishery model with optimal harvesting policy through hybridization approach," *Applied Mathematics and Computation*, vol. 242, pp. 748–763, 2014.

[25] S. Gakkhar and B. Singh, "The dynamics of a food web consisting of two preys and a harvesting predator," *Chaos, Solitons & Fractals*, vol. 34, no. 4, pp. 1346–1356, 2007.

[26] D. Pal, G. S. Mahapatra, and G. P. Samanta, "Bifurcation analysis of predator–prey model with time delay and harvesting efforts using interval parameter," *Int. J. Dynam. Control*, vol. 3, pp. 199–209, 2015.

[27] J. Zhang, "Bifurcation analysis of a modified Holling–Tanner predator–prey model with time delay," *Applied Mathematical Modelling*, vol. 36, no. 3, pp. 1219–1231, 2012.
[28] A. K. Misra and B. Dubey, "A ratio-dependent predator-prey model with delay and harvesting," *Journal of Biological Systems*, vol. 18, no. 02, pp. 437–453, 2010.
[29] M. Bandyopadhyay and S. Banerjee, "A stage-structured prey–predator model with discrete time delay," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1385–1398, 2006.
[30] H. I. Freedman and V. Sree Hari Rao, "The trade-of between mutual interference and time lags in predator-prey systems," *Bulletin of Mathematical Biology*, vol. 45, pp. 991–1003, 1983.
[31] P. A. Naik, Z. Eskandari, Z. Avazzadeh, and J. Zu, "Multiple bifurcations of a discrete-time prey–predator model with mixed functional response," *International Journal of Bifurcation and Chaos*, vol. 32, no. 04, 2022.
[32] P. A. Naik, Z. Eskandari, and H. E. Shahraki, "Flip and generalized flip bifurcations of a two-dimensional discrete-time chemical model," *Mathematical Modelling and Numerical Simulation with Applications*, vol. 1, no. 2, pp. 95–101, 2021.
[33] P. A. Naik, Z. Eskandari, M. Yavuz, and J. Zu, "Complex dynamics of a discrete-time Bazykin–Berezovskaya prey-predator model with a strong Allee effect," *Journal of Computational and Applied Mathematics*, vol. 413, 2022.
[34] J. P. Tripathi, S. Abbas, and M. Thakur, "Stability analysis of two prey one predator model," *AIP Conference Proceedings*, vol. 1479, p. 905, 2012.
[35] J. P. Tripathi, S. Abbas, and M. Thakur, "Local and global stability analysis of a two prey one predator model with help," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 9, pp. 3284–3297, 2014.
[36] T. K. Kar and A. Batabyal, "Persistence and stability of a two prey one predator system," *International Journal of Engineering, Science and Technology*, vol. 2, pp. 174–190, 2010.
[37] K. Shiva Reddy, M. N. Srinivas, and C. V. Pavan Kumar, "Dynamics of a two prey and one predator system with time interruption and random fluctuations," *Pacific Science Review A: Natural Science and Engineering*, vol. 18, no. 2, pp. 150–156, 2016.
[38] B. Xie and F. Xu, "Stability analysis for a time-delayed nonlinear predator–prey model," *Advances in Difference Equations*, vol. 2018, no. 1, p. 122, 2018.
[39] Z. Zhang, R. K. Upadhyay, R. Agrawal, and J. Datta, "The gestation delay: a factor causing complex dynamics in gause-type competition models," *Complexity*, vol. 2018, Article ID 1589310, 21 pages, 2018.
[40] H. Arifah and K. P. Krisnawan, "A model of predator-prey differential equation with time delay," *Journal of Physics: Conference Series*, vol. 1320, no. 1, 2019.
[41] D. Savitri, A. suryanto, W. M. Kusumawinahyu, and A. Abadi, "Dynamics behaviour of two predators and one prey interaction with competition between predators," *IOP Conference Series: Materials Science and Engineering*, vol. 546, no. 5, 2019.
[42] D. Savitri, A. suryanto, W. M. Kusumawinahyu, and A. Abadi, "Dynamics of two preys – one predator system with competition between preys," *Journal of Physics: Conference Series*, vol. 1562, no. 1, 2020.
[43] S. Mondal and G. P. Samanta, "Dynamical behaviour of a two-prey and one-predator system with help and time delay," *Energ. Ecol. Environ*, vol. 5, no. 1, pp. 12–33, 2020.
[44] R. K. Naji and S. J. Majeed, "The dynamical analysis of a delayed prey-predator model with a refuge-stage structure prey population," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 15, no. 1, pp. 135–159, 2020.
[45] M. Ikbal, "Dynamics of predator-prey model interaction with intraspecific competition," *Journal of Physics: Conference Series*, vol. 1940, no. 1, 2021.
[46] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, NY, USA, 1977.
[47] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, Oxford University Press, New York, NY, USA, 2004.