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Research Article

Characterizing Topologically Dense Injective Acts and Their Monoid Connections

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In this paper, we explore the concept of topologically dense injectivity of monoid acts. It is shown that topologically dense injective acts constitute a class strictly larger than the class of ordinary injective ones. We determine a number of acts satisfying topologically dense injectivity. Specifically, any strongly divisible as well as strongly torsion free S-act over a monoid S is topologically dense injective if and only if S is a left reversible monoid. Furthermore, we establish a counterpart of the Skornjakov criterion and also identify a class of acts satisfying the Baer criterion for topologically dense injectivity. Lastly, some homological classifications for monoids by means of this type of injectivity of monoid acts are also provided.

1. Introduction and Preliminaries

Mathematical models of important notions in theoretical computer science and physics, such as automata and dynamical systems, can be represented by acts over semigroups or monoids. In the literature, various categorical properties of acts have been studied, including injectivity. The study of injective acts began with Berthiaume [1], who established that every act possesses an injective envelope. Since then, many authors have continued the work on such classes of acts, similar to the injectivity of modules over rings. Several generalizations of injective acts with respect to subclasses of monomorphisms other than weak injectivity can be found in many papers. For example, quasi-injective acts were considered in [2, 3]. Giuli [4] studied injectivity with respect to sequentially dense monomorphisms of acts over the monoid $(\mathbb{N}^{\infty}, \min)$, and these notions were later generalized to acts over any arbitrary semigroup in [5]. Zhang et al. [6, 7] classified monoids by C-injectivity and CC-injectivity, which are injectivities relative to all inclusions whose domains, and both domains and codomains, respectively, are cyclic. Shahbaz [8] studied $\mathscr{M}$-injectivity in the category of acts, where $\mathscr{M}$ is an arbitrary subclass of monomorphisms. Recently, Sedaghatjoo and Naghipoor [9] investigated classes of acts that are injective with respect to all embeddings with indecomposable domains or codomains. Vital injectivity for modules first appeared in [10, 11], and McMorris [12] investigated vital injectivity of acts over a monoid S with zero. This is injectivity with respect to all embeddings of vital right ideals into S, where right ideals I of S have the property that, for each non-zero $s \in S$, there exists a cancellable element $c \in S$ for which $sc \in I$.

In this paper, we extend the notion of vital right ideal to a new concept called topologically dense right ideal and more generally, topologically dense subact, and explore injectivity of acts with respect to all embeddings of topologically dense right ideals and topologically dense subacts (relative to the set of all subacts of an act which forms a topology on that act).

We prove that the category of acts over a left reversible monoid or a monoid containing a left zero element has enough topologically dense injectives. We show that the class of topologically dense injective acts is strictly larger than that of usual injective ones and identify a condition under which they coincide. Using some algebraic concepts, we find a number of acts satisfying topological dense injectivity. Particularly, any strongly divisible as well as strongly torsion free S-act is topologically dense injective if and only if S is a left reversible monoid.

Skornjakov [13] presented a criterion for injectivity of acts with a fixed element, which states that it is enough to consider injectivity with respect to all inclusions into cyclic acts. We provide a counterpart of the Skornjakov criterion for topological dense injectivity of all acts (with or without a fixed element).

The Baer criterion states that weak injectivity is equivalent to injectivity. In [14, 15], some classes of acts satisfying this criterion were found. Motivated by these studies, we find a class of acts for which the Baer criterion holds; that is, topologically dense injectivity and weak topologically dense injectivity are the same.

We also investigate the behavior of (weak) topologically dense injectivity of acts with respect to products, coproducts, and direct sums. Finally, we explore some kinds of weak topologically dense injectivity and topologically self-dense injectivity and present some homological classifications for monoids.

First we give some preliminaries needed in the sequel. Let S be a monoid. By a (*right*) *S-act* or *act over S*, we mean a set A together with a map $A \times S \longrightarrow A$, $(a, s) \longmapsto as$, such that for all $a \in A, s, t \in S$, $(as)t = a(st)$ and $a1 = a$. A subset B of A is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. An element $\theta \in A$ for which $\theta s = \theta$ for all $s \in S$ is said to be a *fixed element* of A. Clearly, S is an S-act with the operation as the action. Let A and B be two S-acts. A mapping $f\colon A \longrightarrow B$ is called a *homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. The category of all S-acts as well as all homomorphisms between them is denoted by **Act**-S. In this category, monomorphisms are exactly one-to-one homomorphisms. A subset I of a monoid S is called a *right ideal* of S if $xs \in I$ for any $x \in I$ and $s \in S$. A *congruence* on an S-act A is an equivalence relation ρ on A for which $a\rho a'$ implies that $(as)\rho(a's)$ for $a, a' \in A$ and $s \in S$. An S-act A is called *decomposable* if there exist proper subacts B and C of A such that $A = B \cup C$ and $B \cap C = \varnothing$. Otherwise, A is called *indecomposable*. An element $s \in S$ is called *left zero*, if $st = s$ for all $t \in S$. The notion of a *right zero* element is defined similarly. Also $s \in S$ is called zero if it is *left zero* as well as *right zero*. Note that the *zero* element, if exists, is unique. Throughout, S stands for a monoid unless otherwise stated. For undefined terms and notations about S-acts, we refer to [16].

2. Topologically Dense Injectivity in Act-S

In this section, the notion of topologically dense injective act is introduced. We characterize some classes of acts satisfying such kind of injectivity. Moreover, Skornjakov and Baer criteria are studied for topologically dense injectivity of acts as well.

For proceeding, first note that the set of all subacts of an S-act A including $\varnothing$ and A forms a topology on A, which has been studied in [17]. Regarding this topology, the closure of an open set (subact) B, denoted as $\overline{B}$, is the set of all elements $a \in A$ for which the intersection of B and every open set containing a is non-empty, that is, $\overline{B} = \{a \in A \mid aS \cap B \neq \varnothing\}$. So B is *topologically dense*, or briefly *dense*, in A if $\overline{B} = A$, i.e., if for every $a \in A$, there exists $s \in S$ such that $as \in B$. In this case, A is said to be a *dense extension* of B. Also B is closed in A if $\overline{B} = B$. So considering S as an S-act, a *dense right ideal* is a right ideal I of S which is a dense subact of S, that is, for every $s \in S$, there exists $x \in S$ such that $sx \in I$. For any S-acts A and B, a homomorphism $f\colon A \longrightarrow B$ is said to be a *dense homomorphism* if Im (f) is a dense subact of B. A dense homomorphism $f\colon A \longrightarrow B$ which is a monomorphism is called a *dense monomorphism*. In this case, we say that A is *densely embedded* into B.

It is easily checked that B is a dense subact of an S-act A if and only if $B \cap C \neq \varnothing$ for each non-empty subact C of A. In particular, the intersection of a right ideal and a dense right ideal of a monoid S is non-empty. Furthermore, if S is a commutative monoid or contains zero, then every right ideal of S is dense.

Recall from [12] that a right ideal I of a monoid S (with zero) is "vital" if for every (non-zero) $s \in S$, there exists a cancellable element $c \in S$ such that $sc \in I$, and an S-act A is "vital injective" if it is injective relative to all vital right ideals into S. This and a view of dense subacts motivate us to generalize these notions in the category **Act**-S, as follows.

Definition 1. Let A be an S-act. Then A is said to be *topologically dense injective*, or simply *densely injective*, if it is injective with respect to all dense monomorphisms, that is, for any dense monomorphism $f\colon B \longrightarrow C$ and a homomorphism $g\colon B \longrightarrow A$ there exists a homomorphism $h\colon C \longrightarrow A$ such that $hf = g$. Also A is called *weakly densely injective* if it is injective relative to all dense right ideals into S.

Clearly, an S-act A is densely injective if and only if any homomorphism $g\colon B \longrightarrow A$ from a dense subact B of an S-act C can be extended to C. So we may consider dense injectivity with respect to dense embeddings (inclusions) instead of dense monomorphisms.

Remark 2. Let A be a dense subact of an S-act B. It is clear that any fixed element of B (if exists) is also a fixed element of A. So if an S-act A is dense in an injective extension, then A contains a fixed element since each injective act has a fixed element.

Clearly, any injective act is densely injective. The following example shows that these two notions are actually different. It also demonstrates that, in contrast to the case of injectivity of acts, a densely injective act does not necessarily contain a fixed element. Furthermore, not all S-acts are (weakly) densely injective.

Example 1

(i) Let S be a group. Then any S-act A contains no proper dense subact since if B is a dense subact of A, for every $a \in A$, there exists $s \in S$ such that $as \in B$ and so $a = ass^{-1} \in B$, which means that $A = B$. This implies that each S-act A is densely injective. Indeed, if B is a dense subact of C and $f\colon B \longrightarrow A$ is a homomorphism, then $B = C$ and hence $g = f\colon C \longrightarrow A$ extends f. So if A is an S-act with no fixed element, in particular let A be a non-trivial group S as an act over itself, then it is densely injective but not injective.

(ii) Let $S = (\mathbb{Z}, \cdot)$. Then S is not a densely injective S-act. To see this, consider the S-act $\mathbb{Q}$ with usual multiplication as the action. Clearly, $\mathbb{Z}$ is a dense subact of $\mathbb{Q}$. Now it is easy to see that the identity mapping $id_{\mathbb{Z}}\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ is not extended to $\mathbb{Q}$.

(iii) If S as an S^1-act is weakly densely injective, then S contains a left identity. Indeed, considering the dense embedding $S \longrightarrow S^1$, there is a retraction $f\colon S^1 \longrightarrow S$ which implies $f(1)$ is a left identity of S.

(iv) Consider the monoid $(\mathbb{N}^{\infty}, \min)$. Using (ii), $\mathbb{N}$ is not a weakly densely injective $\mathbb{N}^{\infty}$-act.

It is well known that the category **Act**-S has enough injectives (with respect to all monomorphisms). In fact, for any S-act A, the cofree S-act $A^S = \{f\colon S \longrightarrow A \mid f \text{ is a map}\}$ with the action $(f \cdot s)(t) = f(st)$ for all $f \in A^S$ and $s, t \in S$, is injective and A is embedded into A^S (see [16], Theorem 3.1.5 and Corollary 3.1.6).

In what follows, our aim is to investigate whether **Act**-S has enough densely injectives where S is a commutative monoid. In fact, we construct a densely injective dense extension for any S-act. To this aim, let us give some preliminaries.

Let S be a commutative monoid and A be an S-act. Set

$$\mathscr{A} \coloneqq \left\{ f \in A^S \mid \exists s \in S, \forall t \in S, f(st) = f(s)t \right\}. \tag{1}$$

We show that $\mathscr{A}$ is a subact of the cofree act A^S. Let $f \in \mathscr{A}$ and $s \in S$. Then there exists $s' \in S$ such that $f(s't) = f(s')t$ for all $t \in S$. Using the commutativity, for all $t \in S$ we get

$$\begin{aligned}(f \cdot s)(s't) = f(ss't) = f(s'st) = f(s')(st) = \\ (f(s')s)t = f(s's)t = f(ss')t = (f \cdot s)(s')t,\end{aligned} \tag{2}$$

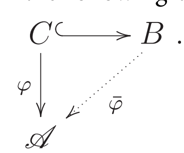
which means that $f \cdot s \in \mathscr{A}$. Now we have the following.

Theorem 3. *For a commutative monoid S, the S-act $\mathscr{A}$ is densely injective.*

Proof. Let B be an S-act, C be a dense subact of B, and $\varphi\colon C \longrightarrow \mathscr{A}$ be a homomorphism. Fix an element $a_0 \in A$. For any $b \in B$ define a mapping $\overline{\varphi}\colon B \longrightarrow \mathscr{A}$ by

$$\overline{\varphi}(b)(t) = \begin{cases} \varphi(bt)(1), & bt \in C, \\ a_0, & \text{otherwise}, \end{cases} \tag{3}$$

for any $b \in B, t \in S$ in the following diagram:



We show that $\overline{\varphi}(b) \in \mathscr{A}$. Since C is dense in B, $bs \in C$ for some $s \in S$. This implies that $\varphi(bs) \in \mathscr{A}$ and so there exists $s' \in S$ for which $\varphi(bs)(s't) = \varphi(bs)(s')t$ for any $t \in S$. Take $s''\colon = ss'$. Then for any $t \in S$, noting $bs'', bs''t \in C$ and being φ a homomorphism, we have

$$\begin{aligned}\overline{\varphi}(b)(s''t) = \varphi(bs''t)(1) = \varphi(bss't)(1) = (\varphi(bs) \cdot (s't))(1) = \varphi(bs)(s't) = \\ \varphi(bs)(s')t = (\varphi(bs) \cdot s')(1)t = \varphi(bss')(1)t = \varphi(bs'')(1)t = \overline{\varphi}(b)(s'')t.\end{aligned} \tag{4}$$

Now it is easily seen that $\overline{\varphi}$ is a homomorphism which extends φ, as required. □

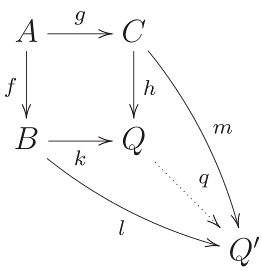
Corollary 4. *Let S be a commutative monoid and A an S-act. Then the S-act $\mathscr{A}$ is a densely injective dense extension of A.*

Proof. Let A be an S-act. Using Theorem 3, the S-act $\mathscr{A}$ is densely injective. It suffices to prove that A is densely embedded into $\mathscr{A}$. Define $\lambda\colon A \longrightarrow \mathscr{A}$ by $\lambda(a)\colon = \lambda_a\colon S \longrightarrow A, \lambda_a(s) = as$, for any $a \in A, s \in S$. Note that $\lambda(a) \in \mathscr{A}$ for any $a \in A$. Clearly, λ is a monomorphism. It remains to show that $\mathrm{Im}(\lambda)$ is a dense subact of $\mathscr{A}$. Let $f \in \mathscr{A}$. Then there exists $s \in S$ such that $f(st) = f(s)t$ for any $t \in S$. This implies that $(f \cdot s)(t) = f(st) = f(s)t = \lambda_{f(s)}(t)$ and so $f \cdot s = \lambda_{f(s)} \in \mathrm{Im}(\lambda)$ which completes the proof.

Here we recall the notion of pushout in a category. Let $f\colon A \longrightarrow B$ and $g\colon A \longrightarrow C$ be two morphisms of a category $\mathscr{C}$. The pair $((h,k), Q)$ with $h\colon C \longrightarrow Q, k\colon B \longrightarrow Q$ is called a *pushout* of the pair (f, g) if

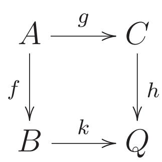
(i) $kf = hg$

(ii) For any pair $((l,m), Q')$ with $l\colon B \longrightarrow Q', m\colon C \longrightarrow Q'$ and $lf = mg$, there exists a unique morphism $q\colon Q \longrightarrow Q'$ such that $qk = l$ and $qh = m$, i.e., the following diagram is commutative:



□

Proposition 5. *In* **Act**-*S, pushouts transfer dense monomorphisms, that is, for a pushout diagram*



if g is a dense monomorphism, then so is k.

Proof. Recall from [16] that $Q = (B \sqcup C)/\theta$, θ is the congruence relation on $B \sqcup C$ generated by all pairs $H = \{(u_B f(a), u_C g(a))\colon a \in A\}$, $h = \gamma u_C\colon C \longrightarrow Q$, $k = \gamma u_B\colon B \longrightarrow Q$, $\gamma\colon B \sqcup C \longrightarrow Q$ is the natural epimorphism, and $u_B\colon B \longrightarrow B \sqcup C, u_C\colon C \longrightarrow B \sqcup C$ are coproduct injections. We show that k is a dense monomorphism. By [17], k is a monomorphism. So it suffices to show that k is dense. Let $[x]_\theta \in Q$. Then $x = u_B(b)$ for some $b \in B$, or $x = u_C(c)$ for some $c \in C$. In the former case, we have $[x]_\theta = k(b) \in \mathrm{Im}(k)$. In the latter case, using that g is dense, there exist $s \in S$ and $a \in A$ with $cs = g(a)$ and hence $[x]_\theta s = [u_C(c)]_\theta s = h(c)s = h(cs) = hg(a) = kf(a) \in \mathrm{Im}(k)$.

By a *dense retract* of an S-act A, we mean a dense subact B of A together with a homomorphism from A to B which maps B identically. Also A is called *densely absolute retract* if A is a dense retract of each of its dense extensions. Clearly, a dense retract of any densely injective S-act is densely injective.

In light of Proposition 5 and [18], Lemma 3.5(i), the following result is obtained. □

Theorem 6. *Let A be an S-act. Then the following assertions are equivalent:*

(i) A is densely injective.

(ii) A is densely absolute retract.

Recall that an extension B of an S-act A is essential if any homomorphism $f\colon B \longrightarrow C$ is a monomorphism whenever so is $f|_A$. Every minimal injective extension of an S-act A is said to be an injective envelope of A which is isomorphic to any injective essential extension of A. Moreover, for every S-act A there exists an injective envelope which is unique up to isomorphism and we denote it by $E(A)$. The reader is refereed to [1] for more details on these basic concepts. By a densely injective envelope of an S-act A, we mean a densely injective essential dense extension. The category Act-S is said to have enough densely injective envelopes if each S-act admits a densely injective envelope.

A monoid S is called *left reversible* if any two right ideals of S have a non-empty intersection. In particular, every commutative monoid is left reversible.

As we know, any injective S-act has at least one fixed element. One the other hand, for an S-act A, $E(A)\backslash A$ does not have two fixed elements by [19], Proposition 1. So if A has no fixed element, then $E(A)$ has only one fixed element. In the following, this unique element is denoted as 0.

Theorem 7. *Let A be an S-act. Then the following assertions hold:*

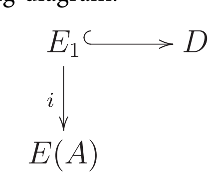
(i) If A has a fixed element, then $E(A)$ is a densely injective envelope of A.

(ii) Let S be a left reversible monoid. If A has no fixed element, then $E_1 = E(A)\backslash\{0\}$ is a subact of $E(A)$ which is a densely injective envelope of A.

Proof

(i) It is clear that $E(A)$ is densely injective. So it suffices to show that A is dense in $E(A)$. Using [19], Corollary 2, $E(A)\backslash A$ has no fixed element and then A is dense in $E(A)$ by [19], Proposition 1.

(ii) Let $b \in E_1$. Using [19], Proposition 1, $I_b = \{s \in S \mid bs \in A\}$ is a non-empty right ideal of S. If the right ideal $J_b = \{s \in S \mid bs = 0\}$ is non-empty, then $I_b \cap J_b \neq \varnothing$ by left reversibility, which contradicts the assumption. So $J_b = \varnothing$, which means that E_1 is a subact of $E(A)$. Moreover, it follows from [16], Lemma III.1.16, and [19], Proposition 1, that E_1 is an essential dense extension of A. Using Theorem 6, it suffices to show that E_1 is densely absolute retract. To this end, let D be a dense extension of E_1. Consider the following diagram:



in which i is the inclusion map. Since $E(A)$ is injective, there exists a homomorphism $f\colon D \longrightarrow E(A)$ that commutes the diagram. For any $d \in D$, if $f(d) = 0$, then $dS \cap E_1 = \varnothing$ which contradicts the fact that D is a dense extension of E_1. Thus $f(D) \subseteq E_1$ and so E_1 is densely absolute retract. Hence, $E(A)$ is a densely injective envelope of A. □

Corollary 8. *If S has a left zero element or S is a left reversible monoid, then the category* **Act**-*S has enough densely injective envelopes.*

Remark 9. The class of all dense extensions of S-acts are clearly composition closed, that is, if A is a dense subact of B and B is a dense subact of C, then A is dense in C. Then, in view of [18], Theorem 3.8 (v), any densely injective envelope of an S-act is a minimal densely injective dense extension.

In view of Remark 9 and Theorem 7 (i), we get the following.

Corollary 10. *Let A be an S-act with a fixed element. Then A is injective if and only if it is densely injective.*

A well-known criterion for injectivity of acts with a fixed element is the Skornjakov criterion stating that it suffices to verify the injectivity relative to all inclusions into cyclic acts, in which the fixed element plays an important role (see [13]). As for dense injectivity, we present an analogous criterion which needs no fixed element.

Theorem 11 (Skornjakov criterion for dense injectivity). *An S-act is densely injective if and only if it is injective relative to all dense embeddings into cyclic S-acts.*

Proof. We prove the non-trivial assertion. Let A be an S-act satisfying the assumption. Consider an S-act C, a dense subact B of C, and a homomorphism $f\colon B \longrightarrow A$. We have to show that there exists a homomorphism $\overline{f}\colon C \longrightarrow A$ which extends f. Let

$\Sigma\colon = (X, g) \mid B$ be a dense subact of $X \subseteq C$, $g\colon X \longrightarrow A$, and $g|_B = f\}$.

Σ is non-empty since $(B, f) \in \Sigma$. Consider a partial order relation on Σ as follows: $(X_1, g_1) \leq (X_2, g_2) \Leftrightarrow X_1 \subseteq X_2$ and $g_2|_{X_1} = g_1$.

For any chain $\{(X_i, g_i)_i\}_{i\in I}$ in Σ, the pair $(\underset{i\in I}{\cup} X_i, \overline{g})$ where $\overline{g}(x_i) = g_i(x_i)$ for $x_i \in X_i$ is an upper bound. By Zorn's lemma there exists a maximal element $(D, \widehat{g})$ in Σ. We shall show that $D = C$. Then, of course, $\overline{f} = \widehat{g}$ extends f.

Suppose that $D \neq C$. Then there exists $c \in C\backslash D$. Since B is dense in C, there exists $s \in S$ such that $cs \in B \subseteq D$ and so $D \cap cS \neq \varnothing$. Set $H\colon = D \cap cS$ and $h\colon = \widehat{g}|_H$. We claim that H is a dense subact of cS. Take any $cs \in cS \subseteq C$. Using the fact that B is dense in C, we get $(cs)t \in B$ for some $t \in S$ and hence $(cs)t = c(st) \in D \cap cS = H$. It follows from hypothesis that there exists a homomorphism $k\colon cS \longrightarrow A$ such that $k|_H = h$. Set $E\colon = D \cup cS$. Define $l\colon E \longrightarrow A$ by

$$l(x) = \begin{cases} \widehat{g}(x), & x \in D, \\ k(x), & x \in cS, \end{cases} \tag{5}$$

for every $x \in E$. Since $\widehat{g}|_H = h = k|_H$, l is well-defined and clearly a homomorphism. Also $l|_B = \widehat{g}|_B = f$. Moreover, since B is dense in C and D, it is dense in E and $D \subset E \subseteq C$ which contradicts the maximality of $(D, \widehat{g})$. □

Recall from [9] that an S-act A is said to be *indecomposable codomain injective* or *InC-injective* for short, if it is injective with respect to all embeddings into indecomposable acts. By [9], Corollary 2.8, an S-act is InC-injective if and only if it is injective relative to all embeddings into cyclic acts. Then, using Theorem 11, we get the following.

Corollary 12. *Any InC-injective S-act is densely injective.*

The following result follows from Corollary 12 and [9], Proposition 2.13.

Proposition 13. *If any densely injective S-act is injective, then S is not a left reversible monoid or S contains a left zero.*

Lemma 14. *The following assertions are equivalent for a monoid S:*

(i) S is left reversible.
(ii) Any right ideal of S is indecomposable.
(iii) Any right ideal of S is dense.
(iv) All subacts of indecomposable S-acts are indecomposable.
(v) Any two right ideals of S whose union is dense have a non-empty intersection.
(vi) Any dense right ideal of S is indecomposable.

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (i), (iv) ⇒ (ii) and (i) ⇒ (v) ⇔ (vi) are obvious.

(i) ⇔ (iv) Follows from [9], Proposition 2.2.

(vi) ⇒ (i) Suppose that there exist right ideals I_0 and J of S such that $I_0 \cap J = \varnothing$. Set $\Sigma_J = \{I \mid I$ is a right ideal of $S, I \cap J = \varnothing\}$ which is non-empty. Consider a partial order relation on Σ_J as follows:

$$I_1 \leq I_2 \Leftrightarrow I_1 \subseteq I_2. \tag{6}$$

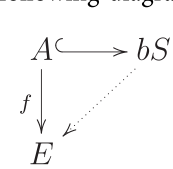
Let $\{I_i\}_{i\in I}$ be a chain in Σ_J. Clearly, $\cup_{i\in I} I_i$ is an upper bound. By Zorn's lemma there exists a maximal element I_1 in Σ. We claim that $I_1 \cup J$ is a dense right ideal of S; otherwise, there exists $s \in S$ such that $sS \cap (I_1 \cup J) = \varnothing$. It is clear that $(sS \cup I_1) \cap J = \varnothing$, so $I_1 \subset sS \cup I_1 \in \Sigma$, which is a contradiction. Thus $I_1 \cup J$ is dense in S and $I_1 \cap J = \varnothing$, which contradicts the assumption. □

Theorem 15. *Let S be a left reversible monoid. Then*

(i) Any dense injective S-act is InC-injective.
(ii) Any dense injective S-act is injective if and only if S has a left zero element.

Proof

(i) Consider the following diagram:



in which E is a dense injective S-act and A is a nonempty subact of a cyclic S-act bS. Consider the nonempty right ideal $I_b = \{s \in S \mid bs \in A\}$ of S. By Lemma 14, I_b is dense in S and so A is dense in bS. Indeed, for any $bt \in bS$, since I_b is dense in S, there exists $s \in S$ such that $ts \in I_b$ and thus $bts \in A$. Now since E is dense injective, there exists a homomorphism $g\colon bS \longrightarrow E$ such that $g|_A = f$.

(ii) If S has a left zero element, then by Corollary 10, any dense injective S-act is injective. For the converse, since S is left reversible, S has a left zero element by Proposition 13.

Let A be an S-act. The S-act $A \cup \{0\}$ with a fixed element 0 adjoined to A is denoted by A^0. □

Proposition 16. *Let S be a left reversible monoid and A be a densely injective S-act. Then $A \leq E(A) \leq A^0$.*

Proof. If A has a fixed element, then by Corollary 10, $A = E(A)$. Now let A have no fixed element. By Theorem 7 (ii), $E(A)\backslash\{0\}$ is a dense injective envelope of A which implies $E(A)\backslash\{0\} = A$.

In what follows, a class of densely injective acts is obtained. To this end, let us list some preliminaries.

The notions of torsion free and divisible S-acts are known and defined by using the right and left cancellable elements of S, respectively (see [16]). In [2], torsion freeness and divisibility are considered in a much stronger sense (without imposing the cancellability properties on elements of S) which we call here strong torsion freeness (see also [20]) and strong divisibility defined as follows.

Let A be an S-act. Then A is called strongly torsion free if for any $a, b \in A$ and for any $s \in S$, the equality $as = bs$ implies $a = b$. Also we say that A is strongly divisible if $As = A$ for each $s \in S$, that is, for any $a \in A$, there exists $b \in A$ such that $a = bs$. □

Lemma 17. *Let S be a left reversible monoid and A be an S-act. Then $E(A)$ is strongly torsion free if and only if so is A.*

Proof. Suppose that A is torsion free. If $E(A)$ is not torsion free, then there exist $b_1, b_2 \in E(A), s \in S$ such that $b_1 s = b_2 s$ but $b_1 \neq b_2$. Define a relation ρ on $E(A)$ by

$$x\rho y \Leftrightarrow xs = ys \text{ for some } s \in S. \tag{7}$$

We show that ρ is a congruence on $E(A)$. The reflexivity and symmetry are clear. For transitivity, let $x\rho y, y\rho z$ for $x, y, z \in E(A)$. Then there exist $s, s' \in S$ such that $xs = ys$ and $ys' = zs'$. Since S is left reversible, there exist $t, t' \in S$ such that $st = s't'$ and so $xst = yst = ys't' = zs't' = zst$, which means that $x\rho z$. Let $x\rho y$ and $t \in S$; then there exists $s \in S$ such that $xs = ys$. Left reversibility of S gives that there exist $s', t' \in S$ with $tt' = ss'$ so that $xtt' = xss' = yss' = ytt'$. Thus $xt\rho yt$, as desired. Now, since $b_1 s = b_2 s$ and $b_1 \neq b_2$, $\rho \neq \Delta_{E(A)}$. Using [16], Lemma 3.1.15, $\rho|_A \neq \Delta_A$ where $\rho|_A = \rho \cap (A \times A)$. This implies the existence of $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $s \in S$ such that $a_1 s = a_2 s$, which contradicts being strong torsion free of A. The converse is clear. □

Lemma 18. *Let A be a strongly divisible as well as strongly torsion free S-act. Then A is closed in each of its strongly torsion free extension.*

Proof. Let C be a strongly torsion free extension of A and $c \in \overline{A}$. Then there exists $s \in S$ such that $cs \in A = As$ which implies $cs = as$ for some $a \in A$. Since C is strongly torsion free, $c = a \in A$ and hence $\overline{A} = A$. □

Theorem 19. *Let S be a left reversible monoid. Then any strongly divisible as well as strongly torsion free S-act A is densely injective.*

Proof. By Lemma 17, $E(A)$ is a strongly torsion free S-act and by Corollary 8, A has a densely injective envelope $E_d(A)$. Since $E_d(A)$ is an essential extension of A, there is a monomorphism $h\colon E_d(A) \longrightarrow E(A)$ which implies that $E_d(A)$ is a strongly torsion free S-act. Now we are done using Lemma 18. □

A densely injective act over a left reversible monoid is not necessarily strongly divisible nor strongly torsion free. For this, consider the monoid $S = (\mathbb{N}, \max)$ which is a densely injective S-act (see Example 4 (i)) but not strongly divisible nor strongly torsion free. In the next section, we discuss the converse of Theorem 19 (see Proposition 39).

In view of Corollary 10 and Theorem 19, a class of injective acts is characterized in the following.

Corollary 20. *Any strongly divisible and strongly torsion free act with a fixed element over a left reversible monoid is injective.*

Theorem 21. *For an S-act A, any strongly torsion free dense extension B of A is essential.*

Proof. Suppose that $f\colon B \longrightarrow C$ is a homomorphism such that $f|_A$ is a monomorphism. Let $f(b) = f(b')$ for $b, b' \in B$. Since A is a dense subact of B, there exist $s, s' \in S$ such that $bs, b's' \in A$ and so $I_b = \{s \in S \mid bs \in A\}$ and $I_{b'} = \{s \in S \mid b's \in A\}$ are non-empty. Since A is a dense subact of B, I_b and $I_{b'}$ are dense right ideals of S and so $I_b \cap I_{b'} \neq \varnothing$. So there exists $t \in I_b \cap I_{b'}$ which means that $bt, b't \in A$. Then $f(bt) = f(b)t = f(b')t = f(b't)$ and so $bt = b't$. Now since B is strongly torsion free, $b = b'$ and hence f is a monomorphism. □

The next result presents a criterion for an injective extension of an S-act to be an injective envelope.

Corollary 22. *If B is an injective strongly torsion free dense extension of an S-act A, then B is an injective envelope of A.*

Example 2

(i) Consider $\mathbb{Z}$ and $\mathbb{Q}$ as $(\mathbb{N}, \cdot)$-acts with usual multiplication as the actions. Then $\mathbb{Q}$ is a dense extension of $\mathbb{Z}$. Moreover, $\mathbb{Q}$ is strongly torsion free and strongly divisible. Then, using Corollary 20, $\mathbb{Q}$ is injective. Now it follows from Corollary 22 that $\mathbb{Q}$ is an injective envelope of $\mathbb{Z}$.

(ii) Consider $\mathbb{N}$ and $\mathbb{Z}$ as $(\mathbb{N}_0, +)$-acts with usual addition as the actions. Then $\mathbb{Z}$ is a dense extension of $\mathbb{N}$. Moreover, $\mathbb{Z}$ is strongly torsion free and strongly divisible. Then Theorem 19 implies that $\mathbb{Z}$ is densely injective but not injective because it has no fixed element.

(iii) By Theorem 21, $\mathbb{Z}$ is an essential extension of $\mathbb{N}$. Now using part (ii) and Proposition 16, since $(\mathbb{N}_0, +)$ is left reversible, we conclude that $\mathbb{Z} \cup \{\infty\}$ is an injective envelope of S-acts $\mathbb{N}$ and $\mathbb{Z}$.

The condition that weak injectivity coincides to injectivity is known as the Baer criterion for injectivity. However, although this condition holds for injectivity of modules over a ring with unit, it fails for injectivity of acts over an arbitrary monoid (see [16]). The next result gives a class of acts satisfying this criterion. As we shall see in the last section, Baer criterion also fails for dense injectivity of acts (see Example 3).

Theorem 23. *Let A be a strongly torsion free S-act. Then A is densely injective if and only if it is weakly densely injective.*

Proof. It is clear that each densely injective S-act is weakly densely injective. For the converse, let A be weakly densely injective. Assume that B is a dense subact of a cyclic S-act cS and $f: B \longrightarrow A$ is a homomorphism. We show that there exists a homomorphism $\overline{f}: cS \longrightarrow A$ which extends f. Let $s \in S$. Then there exists $t \in S$ such that $(cs)t \in B$ and hence $st \in I_c = \{s \in S \mid cs \in A\}$ which means that I_c is dense in S. Consider a homomorphism $g: I_c \longrightarrow A$ given by $g(s) = f(cs)$. So there is a homomorphism $h: S \longrightarrow A$ which extends g. Define $\overline{f}: cS \longrightarrow A$ by $\overline{f}(cs) = h(s)$ for any $s \in S$. Let $cs_1 = cs_2$; then there exists $t \in S$ such that $cs_1 t = cs_2 t \in B$. So

$$\begin{aligned} h(s_1)t &= h(s_1 t) = g(s_1 t) = f(cs_1 t) = f(cs_2 t) \\ &= g(s_2 t) = h(s_2 t) = h(s_2)t, \end{aligned} \tag{8}$$

and hence $\overline{f}(cs_1) = h(s_1) = h(s_2) = \overline{f}(cs_2)$, which means that $\overline{f}$ is well-defined. Now it is not difficult to check that $\overline{f}$ is a homomorphism which extends f. □

In view of Lemma 14 and Theorem 23, we have the following.

Corollary 24. *Let S be a left reversible monoid and A be a strongly torsion free S-act. Then A is densely injective if and only if it is weakly injective.*

Theorem 25. *Let S be a left reversible monoid. Then an S-act is injective if and only if it is densely injective as well as injective relative to all closed subacts.*

Proof. Let B be a subact of an S-act C and $f: B \longrightarrow A$ be a homomorphism. Set $D: = \{x \in C \mid xs \in B \text{ for some } s \in S\}$. It is clear that B is a dense subact of D and D is a closed subact of C. Now since A is dense injective, there exists a homomorphism $g: D \longrightarrow A$ such that $g|_B = f$. Moreover, since A is closed injective, there exists a homomorphism $h: C \longrightarrow A$ such that $h|_D = g$. Hence, $h|_B = (h|_D)|_B = g|_B = f$, which means that A is injective. The converse holds trivially. □

3. Products, Coproducts, and Direct Sums of (Weakly) Densely Injective S-Acts

This section is devoted to study the behavior of (weak) dense injectivity of acts with respect to products, coproducts, and direct sums. The product of a family of S-acts is their Cartesian product with the componentwise action, and the coproduct is their disjoint union with natural action. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. For a family $\{A_i \mid i \in I\}$ of S-acts with a unique fixed element 0, the direct sum $\oplus_{i\in I} A_i$ is defined to be the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

The following result shows that (weak) dense injectivity well-behaves under products as usual.

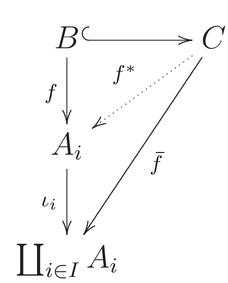
Proposition 26. *Let $\{A_i \mid i \in I\}$ be a family of S-acts. Then the product $\prod_{i\in I} A_i$ is (weakly) densely injective if each A_i is (weakly) densely injective. The converse also holds if each A_i has a fixed element.*

Proof See [8], Theorem 3.24. □

It is known that the usual injectivity is not transferred from a coproduct of acts to all of its components in general. For instance, taking a non-trivial group S, the S-act $S \sqcup S^0$ where $S^0 = S \cup \{0\}$ is injective, whereas S is not an injective S-act. In contrast to the case of injectivity, the next result shows that the dense injectivity is inherited from coproducts to their components.

Proposition 27. *Let $\{A_i \mid i \in I\}$ be a family of S-acts. If the coproduct $\coprod_{i\in I} A_i$ is (weakly) densely injective, then so is each A_i.*

Proof. Assume that $\coprod_{i\in I} A_i$ is densely injective. Let $i \in I$. We show that A_i is densely injective. Let B be a dense subact of C and consider the diagram



where f is a homomorphism and ι_i is the canonical injection. Since $\coprod_{i\in I} A_i$ is densely injective, there exists a homomorphism $\overline{f}: C \longrightarrow \coprod_{i\in I} A_i$ such that $\overline{f}|_B = \iota_i f = f$. We claim that $\mathrm{Im}\overline{f} \subseteq A_i$. Let there exist $x \in C$ and $j \in I, j \neq i$, such that $\overline{f}(x) \in A_j$. Since B is dense in C, $xs \in B$ for some $s \in S$ and so $\overline{f}(xs) = \iota_i f(xs) = f(xs) \in A_i$. On the other hand, $\overline{f}(xs) = \overline{f}(x)s \in A_j$. Then $\overline{f}(xs) \in A_i \cap A_j$ which is a contradiction. Now considering $f^*: = \overline{f}: C \longrightarrow A_i$, we get $f^*|_B = f$. The proof for weak dense injectivity is the same. □

Recall that all coproducts of injective S-acts are injective if and only if S is left reversible (see [16], Propositions 3.1.13 and 3.5.4). In the following, a counterpart of these results for (weak) dense injectivity of acts is presented.

Theorem 28. *The following statements are equivalent for any monoid S:*

(i) All coproducts of (weakly) densely injective S-acts are (weakly) densely injective.

(ii) $\Theta \sqcup \Theta$ is (weakly) densely injective.

(iii) For some S-act A, $A \sqcup A$ is (weakly) densely injective.

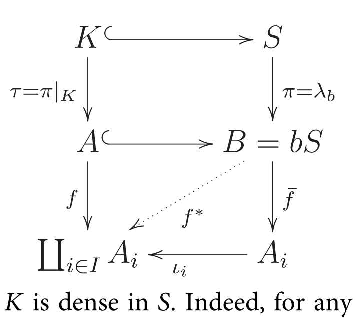
(iv) S is left reversible.

Proof. We just need to prove the assertion for dense injectivity.

The implication (i) ⇒ (ii) ⇒ (iii) is trivial.

(iii) ⇒ (iv) Using a same method to the proof of Proposition 2.12 (iii) ⇒ (iv) in [9] for the densely injective case, the result is obtained.

(iv) ⇒ (i) Let A_i be a densely injective S-act for each $i \in I$. We apply Theorem 11 to prove that $\coprod_{i\in I} A_i$ is densely injective. Suppose that A is a dense subact of a cyclic S-act $B = bS$ and $f: A \longrightarrow \coprod_{i\in I} A_i$ is a homomorphism. Moreover, consider the epimorphism $\pi: = \lambda_b: S \longrightarrow B$, the right ideal $K: = \pi^{-1}(A)$ of S, and $\tau: = \pi|_K: K \longrightarrow A$ in the following diagram:



Note that K is dense in S. Indeed, for any $s \in S, \pi(s) = bs \in B$ and since A is dense in B, there exists $t \in S$ such that $\pi(st) = \pi(s)t = bst \in A$ and so $st \in \pi^{-1}(A) = K$. We claim that there exists $i \in I$ for which $\mathrm{Im} f \subseteq A_i$. Otherwise, $\mathrm{Im} f \cap A_i$ and $\mathrm{Im} f \cap A_j$ are non-empty for some $i, j \in I$, $i \neq j$, which clearly gives that $\mathrm{Im} f$ is a decomposable subact of $\coprod_{i\in I} A_i$. Using [16], Lemma 1.5.36, this implies that A and hence K are decomposable which contradicts the left reversibility of S. This gives the existence of $i \in I$ such that $\mathrm{Im} f \subseteq A_i$. Since A_i is densely injective by the assumption, f can be extended to a homomorphism $\overline{f}: B \longrightarrow A_i$. Hence, taking $f^*: = \iota_i \overline{f}: B \longrightarrow \coprod_{i\in I} A_i$ we have $f^*|_A = f$, as required. □

Now, we are ready to prove the converse of Theorem 19.

Proposition 29. *For a monoid S, if any strongly divisible as well as strongly torsion free S-act is (weakly) densely injective, then S is left reversible.*

Proof. Consider the S-act $\Theta \sqcup \Theta$ which is clearly strongly divisible and strongly torsion free. It follows from the hypothesis that $\Theta \sqcup \Theta$ is (weakly) densely injective. Now, using Theorem 28, the assertion holds. □

Note that each S-act A with trivial action, i.e., $as = a$ for any $a \in A, s \in S$, is densely injective. As for the injectivity of acts with trivial actions, we have the following.

Proposition 30. *Let A be a non-singleton S-act with trivial action. Then the following statements are equivalent:*

(i) A is injective.

(ii) A is principally weakly injective.

(iii) S is left reversible.

Proof. (i) ⇒ (ii) Trivial.

(ii) ⇒ (iii) Let A be a principally weakly injective S-act. If S is not left reversible, there exist principal right ideals J and K of S such that $J \cap K = \varnothing$. Fix two distinct elements $a, b \in A$. Consider the homomorphism $f: J \sqcup K \longrightarrow A$ defined by $f(x) = \begin{cases} a & x \in J, \\ b & x \in K. \end{cases}$ It is easily checked that f can not be extended to S which contradicts the assumption.

(iii) ⇒ (i) Let S be a left reversible monoid. Since A is isomorphic to the coproduct of singleton S-acts, it is injective by [16], Proposition 3.1.13. □

The following result follows clearly from [8], Theorem 3.30.

Theorem 31. *Let $\{A_i \mid i \in I\}$ be a family of S-acts with a unique fixed element 0 such that the direct sum $\oplus_{i\in I} A_i$ is (weakly) densely injective. Then each A_i is (weakly) densely injective.*

A monoid S is called *(densely) Noetherian* if every (dense) right ideal of S is finitely generated. It is easy to check that a monoid S is (densely) Noetherian if and only if it satisfies the ascending chain condition on its (dense) right ideals, that is, for every ascending chain

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots \quad (9)$$

of (dense) right ideals of S, there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \cdots$.

Lemma 32. *A monoid S is Noetherian if and only if it is densely Noetherian.*

Proof. It is clear that any Noetherian monoid is densely Noetherian. Conversely, suppose that S is not Noetherian, so there exists a right ideal J of S which is not finitely generated. Set $\Sigma = I \mid I$ is a non-finitely generated right ideal of $S \neq \varnothing$. Consider a partial order relation on Σ as follows: $I_1 \leq I_2 \Leftrightarrow I_1 \subseteq I_2$.

Let $\{I_i\}_{i\in I}$ be a chain in Σ. Clearly, $\cup_{i\in I} I_i$ is an upper bound. By Zorn's lemma there exists a maximal element I_0 in Σ. We claim that I_0 is a dense right ideal of S; otherwise, there exists $s \in S$ such that $sS \cap I_0 = \varnothing$. Thus $sS \cup I_0 \notin \Sigma$ whence $sS \cup I_0$ and then I_0 is finitely generated, which is a contradiction. Thus I_0 is a non-finitely generated dense right ideal of S, which means that S is not densely Noetherian. □

Theorem 33. *For a monoid S with zero, the following conditions are equivalent:*

(i) Each direct sum of densely injective S-acts is densely injective.

(ii) Each direct sum of weakly densely injective S-acts is weakly densely injective.

(iii) S is Noetherian.

Proof. (i) $\Leftrightarrow$ (iii) Follows from Corollary 10 and [3], Theorem 1.

(ii) $\Leftrightarrow$ (iii) Follows from [8], Theorem 3.34 and Lemma 32. □

Theorem 34. *Let S have a left zero element. Then each direct sum of densely injective S-acts is densely injective if and only if each direct sum of densely injective S-acts is a retract of their direct product.*

Proof. $(\Rightarrow)$ Since S has a left zero element, $\oplus A_i$ is a dense subact of $\prod A_i$. So we are done using Theorem 6.

$(\Leftarrow)$ By Proposition 26, each direct product of densely injective S-acts is densely injective and clearly every retract of a densely injective S-act is densely injective. □

4. Classifying Monoids by (Principal, *fg*-) Weak Dense and Self-Dense Injectivities

In this section, we study some usual types of weak dense injectivity and self-dense injectivity of acts. By means of these notions, some homological classification results for monoids are also obtained.

Definition 35. An S-act A is called *principally (fg-) weakly densely injective* if it is injective relative to all principal (finitely generated) dense right ideals into S.

Remark 36

(i) Let A be an S-act. Similarly to the case of weak injectivity of acts, one has A is (*fg*-) weakly densely injective if and only if for any (finitely generated) dense right ideal I of S and any homomorphism $f\colon I \longrightarrow A$ there exists $a \in A$ such that $f(x) = ax$ for any $x \in I$.

(ii) As we mentioned in Lemma 14, if S is left reversible, then every right ideal of S is dense. Then, in this case, (principal, fg-) weak dense injectivity and (principal, *fg*-) weak injectivity coincide. Therefore, in view of [16], Examples 3.4.6, 3.5.6, the notions of principal weak dense injectivity, *fg*-weak dense injectivity and weak dense injectivity are actually different.

Let A be an S-act. Then any $a \in A$ is called a *dense element* if for every $a' \in A$, there exist $s, s' \in S$ such that $as = a's'$ (i.e., $aS \cap a'S \neq \varnothing$). Clearly, $a \in A$ is dense if and only if aS is a dense subact of A. So every element of A is dense element if and only if every subact of A is dense. An element $s \in S$ is called *regular* if there exists $x \in S$ such that $s = sxs$. If all dense elements of S are regular, then S is called a *densely regular monoid*. The monoids $(\mathbb{N} \cup \{\infty\}, \min)$ and $(\mathbb{N}, \max)$ are examples of densely regular monoid.

Using [9], Proposition 2.1, the next result is immediate.

Proposition 37. *Let S be a left reversible monoid. Then an S-act A is indecomposable if and only if all elements of A are dense.*

Proposition 38. *Let A be an S-act. Then the following statements are equivalent:*

(i) A is principally weakly densely injective.

(ii) For any dense principal right ideal sS of S and any homomorphism $f\colon sS \longrightarrow A$ *there exists* $a \in A$ *such that* $f(x) = ax$ *for any* $x \in sS$.

(iii) For a dense element $s \in S$ *and any* $a \in A$ *with* $\ker \lambda_s \subseteq \ker \lambda_a$, $a = bs$ *for some* $b \in A$.

Proof. The proof is similar to that of [16], Proposition 3.3.2. □

Corollary 39. *The following assertions hold for any monoid S:*

(i) Let ρ *be a right congruence on S. The factor act* S/ρ *is principally weakly densely injective if and only if for any dense element* $s \in S$ *and any* $t \in S$ *for which* $sx = sy, x, y \in S$, *implies* $(tx)\rho(ty)$, *there exists* $u \in S$ *such that* $t\rho(us)$.

(ii) A right ideal $zS, z \in S$, *of S is principally weakly densely injective if and only if for any dense element* $s \in S$ *and any* $t \in S$ *for which* $sx = sy, x, y \in S$, *implies* $ztx = zty$, *there exists* $u \in S$ *such that* $zt = zus$. *In particular, if zS is a principally weakly densely injective dense right ideal of S, then z is a regular element.*

An S-act A is said to be *densely divisible* if $As = A$ for any left cancellable dense element $s \in S$, that is, for any $a \in A$ there exists $b \in A$ such that $a = bs$.

Corollary 40. *Every principally weakly densely injective act is densely divisible.*

The converse of Corollary 40 is not generally true (see [16], Example 3.3.11).

The next three theorems are proved similarly to the well-known homological classification results for monoids by different kinds of weak injectivity of acts which can be found, for example, in [16].

Theorem 41. *The following conditions are equivalent:*

(i) All S-acts are principally weakly densely injective.

(ii) All dense right ideals of S are principally weakly densely injective.

(iii) All finitely generated dense right ideals of S are principally weakly densely injective.

(iv) All principal dense right ideals of S are principally weakly densely injective.

(v) S is densely regular.

Theorem 42. *The following conditions are equivalent:*

(i) All S-acts are fg-weakly densely injective.

(ii) All dense right ideals of S are fg-weakly densely injective.

(iii) All finitely generated dense right ideals of S are fg-weakly densely injective.

(iv) S is a densely regular monoid whose dense finitely generated right ideals are principal.

Theorem 43. *The following conditions are equivalent:*

(i) All S-acts are weakly densely injective.

(ii) All dense right ideals of S are weakly densely injective.

(iii) All dense right ideals of S have an idempotent generator.

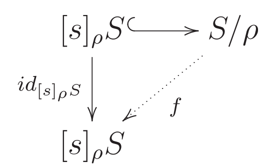
(iv) S is a densely regular principal dense right ideal monoid.

The following example shows that the Baer criterion fails for dense injectivity of acts as injectivity (see [7], Example 12).

Example 3. Let S be a monoid with the multiplication table:

.	s	t	u	v	1
s	s	s	u	u	s
t	s	t	u	v	t
u	s	s	u	u	u
v	s	t	u	v	v
1	s	t	u	v	1

It is easily checked that S is a densely regular principal dense right ideal monoid. Then all S-acts are weakly densely injective by Theorem 43. Consider the monocyclic right congruence $\rho = \rho(t, v)$ on S. Clearly, $(s, u) \notin \rho$ and $[s]_\rho S$ is a dense subact of the S-act S/ρ. We show that the S-act $[s]_\rho S = \{[s]_\rho, [u]_\rho\}$ is not densely injective. Consider the following diagram:



Suppose that there exists a homomorphism $f\colon S/\rho \longrightarrow [s]_\rho S$ extending $id_{[s]_\rho S}$. If $f([1]_\rho) = [s]_\rho$, then $f([t]_\rho) = f([1]_\rho t) = [s]_\rho t = [s]_\rho$ and $f([v]_\rho) = f([1]_\rho v) = [s]_\rho v = [u]_\rho$ which is a contradiction. Similarly, the case $f([1]_\rho) = [u]_\rho$ yields also a contradiction.

A monoid S is said to be *self-densely injective* if S is densely injective as an S-act.

In the following, we study self-dense injectivity property for monoids.

Let K be a dense right ideal of S and $q \in S$. For $s \in S$, put $K_s = \{u \in S \mid su \in K\}$. Then K_s is a (non-empty) dense right ideal of S. Define a relation $\rho(K, q)$ on S by

$$s\rho(K,q)t \Leftrightarrow K_s = K_t \text{ and } qsu = qtu \text{ for all } u \in K_s. \quad (10)$$

Then $\rho(K, q)$ is a right congruence on S.

Using Theorem 11, a same argument to [16], Theorem 4.5.3, gives the following result.

Theorem 44. *A monoid S is self-densely injective if and only if for any dense right ideal K of S and any homomorphism $f\colon K \longrightarrow S$ there exists $q \in S$ such that $f(a) = qa$ for all $a \in K$ and $s\rho(K,q)t, s, t \in S$, implies $qs = qt$.*

Analogous to [16], Theorems 4.5.10, 4.5.11, 4.5.12, we have the next three results.

Theorem 45. *The following conditions are equivalent:*

(i) All principal dense right ideals of S are densely injective.

(ii) S is a densely regular self-densely injective monoid.

Theorem 46. *The following conditions are equivalent:*

(i) All finitely generated dense right ideals of S are densely injective.

(ii) S is a densely regular self-densely injective monoid whose dense finitely generated right ideals are principal.

Theorem 47. *The following conditions are equivalent:*

(i) All dense right ideals of S are densely injective.

(ii) S is a densely regular self-densely injective principal dense right ideal monoid.

Recall from [16] that an idempotent $e \in S$ is called *right special* if for any right congruence ρ on S there exists $k \in eS$ such that $(ke)\rho e$ and $u\rho v, u, v \in S$, implies $(ku)\rho(kv)$.

Using Theorem 11, the next result for dense injectivity is similar to the injectivity case. The proof is just an adaptation of the proof of [16], Theorem 4.5.13.

Theorem 48. *All S-acts are densely injective if and only if S is a densely regular principal dense right ideal monoid all idempotents of which are special.*

Finally, the following example shows that self-dense injectivity does not imply self-injectivity.

Example 4

(i) Consider the monoid $S = (\mathbb{N}, \max)$. Then it follows from Theorem 48 that S is a self-densely injective monoid. Since S has no zero, it is not self-injective.

(ii) Each non-trivial group S is a self-densely injective but not self-injective act over itself (see Example 1 (i)).

5. Conclusion

The examination of injectivity concerning various classes of monomorphisms in a category holds significant importance across multiple mathematical domains. Numerous authors have explored this concept within diverse categories, each pertaining to distinct classes of monomorphisms. This paper delves into the exploration of topological dense injectivity within monoid acts. We establish that the category of acts over a left reversible monoid or a monoid that includes a left zero element possesses enough topologically dense injectives. It is revealed that topologically dense injective acts form a class strictly larger than the class of ordinary injective acts and we identify a condition under which they coincide. We pinpoint several acts that meet the criteria for topological dense injectivity. Specifically, a strongly divisible as well as strongly torsion free S-act over a monoid S is topologically dense injective if and only if S is a left reversible monoid. Moreover, we give a counterpart of the Skornjakov criterion and identify a class of acts that adhere to the Baer criterion for topological dense injectivity. Finally, we present various homological classifications for monoids based on this form of injectivity in monoid acts.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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