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Research Article

Computing the (l, k)-Clique Metric Dimension of Graphs via (Edge) Corona Products and Integer Linear Programming Model

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Let G be a graph with n vertices and $C_G = \{X\colon X \text{ is an } l\text{-clique of } G\}$. A vertex $v \in V(G)$ is said to resolve a pair of cliques $\{X, Y\}$ in G if $d_G(v, X) \neq d_G(v, Y)$ where d_G is the distance function of G. For a pair of cliques $\{X, Y\}$, the resolving neighbourhood of X and Y, denoted by $R_G\{X, Y\}$, is the collection of all vertices which resolve the pair $\{X, Y\}$. A subset S of $V(G)$ is called an (l, k)-clique metric generator for G if $|R_G\{X, Y\} \cap S| \geq k$ for each pair of distinct l-cliques X and Y of G. The (l, k)-clique metric dimension of G, denoted by $l - \mathrm{cdim}_k(G)$, is defined as $\min\{|S|\colon S \text{ is an } (l, k)\text{-clique metric generator of } G\}$. In this paper, the (l, k)-clique metric dimension of corona and edge corona of two graphs are computed. In addition, an integer linear programming model is presented for the (l, k)-clique metric basis for a given graph G and its l-cliques.

1. Introduction

Throughout this paper all graphs are assumed to be finite, simple, connected, and undirected. For a positive integer number n, we use the notation $[n]$ instead of $\{1, \ldots, n\}$. A clique is a collection of vertices of a graph in which every two distinct vertices are adjacent. An l-clique is a clique with l vertices. For an l-clique X of a graph G, we will also use the notation $N[X]$ to denote $\{v \in V(G)\colon v \text{ is adjacent to a vertex of } X\} \cup X$. For a vertex v and an l-clique X of a graph G, notation $d_G(v, X)$ denotes $\min\{d_G(x, v)\colon x \in X\}$ where $d_G(x, v)$ is as usual the number of edges on a shortest x, v-path.

Let G be a graph with n vertices and $C_G = \{X\colon X \text{ is an } l\text{-clique of } G\}$. For a pair of cliques $\{X, Y\}$, the resolving neighbourhood of X and Y, denoted by $R_G\{X, Y\}$, is the collection of all vertices which resolve the pair $\{X, Y\}$. A subset S of $V(G)$ is called an (l, k)-clique metric generator ((l, k)-CMG for short) for G if $|R_G\{X, Y\} \cap S| \geq k$ for each pair of distinct l-cliques X and Y of G. A (l, k)-clique metric generator of minimum cardinality is called an (l, k)-clique metric basis of G. The cardinality of an (l, k)-clique metric basis of G, denoted by $l - \mathrm{cdim}_k(G)$, is said to be (l, k)-clique metric dimension ((l, k)-CMD for short) of G.

Here, (l, k)-CMD is considered as a generalization of the concept of a k-metric dimension presented in [1]. Indeed, $(1, k)$-CMD is known as k-metric dimension and is denoted by $\dim_k(G)$ in [1]. In addition, (1,1)-CMD is known as metric dimension which is the first version of this type of invariants (see [2] for more details). After that, other versions of metric dimension such as edge metric dimension and mixed metric dimension were also defined (see [3–6] for more information about these topics). In what follows, we will also use notation $\mathrm{edim}_k(G)$ instead of $2 - \mathrm{cdim}_k(G)$.

In the next section, we need an extension of the concept k-metric dimensional of graph defined in [7] as follows.

A graph is (l, k)-clique metric dimensional if k is the largest integer such that there exists a (l, k)-clique metric generator for G.

Consider graph G shown in Figure 1. We want to compute $2 - \mathrm{cdim}_2(G)$. Then first, we find $R_G\{e, e'\}$ for each pair of distinct 2-cliques (edges) e and e' of G. $R_G\{e_1, e_2\} = \{u_1, u_3\}$, $R_G\{e_1, e_3\} = \{u_1, u_2, u_3, u_4\}$, $R_G\{e_1, e_4\} = \{u_2, u_4\}$, $R_G\{e_2, e_3\} = \{u_2, u_4\}$, $R_G\{e_2, e_4\} = \{u_1, u_2, u_3, u_4\}$, $R_G\{e_2, e_5\} = \{u_3, u_4\}$, $R_G\{e_3, e_4\} = \{u_1, u_3\}$, $R_G\{e_3, e_5\} = \{u_2, u_3\}$, $R_G\{e_4, e_5\} = \{u_1, u_2\}$. According to $R_G\{e_1, e_2\} = \{u_1, u_3\}$, $R_G\{e_1, e_4\} = \{u_2, u_4\}$, $R_G\{e_2, e_5\} = \{u_3, u_4\}$, and $R_G\{e_3, e_4\} = \{u_1, u_3\}$, we can conclude that vertices u_1, u_2, u_3, u_4 must be the members of each (2,2)-clique metric generator of G. Therefore, $S = \{u_1, u_2, u_3, u_4\}$ is a (2,2)-clique metric basis of G and so $2 - \mathrm{cdim}_2(G) = 4$.

As another example, we compute $2 - \mathrm{cdim}_3$ of graph G depicted in Figure 1. This graph has two 3-cliques $X = u_1u_2u_4$ and $Y = u_2u_3u_4$. Thus, $R_G\{X, Y\} = \{u_1, u_3\}$. Then $S = \{u_1, u_3\}$ is a (3,2)-clique metric basis of G and so $3 - \mathrm{cdim}_2(G) = 2$.

Lots of work have been done in k-metric generator sets of graphs. We recommend [1] for more details on this topic. Estrada-Moreno et al. studied k-metric dimension of corona product graphs in [8]. In this paper, we give $l - \mathrm{cdim}_k(G \circ H)$ and $l - \mathrm{cdim}_k(G \diamond H)$ in terms of the global forcing numbers of H, the order, and size of G. We also present an integer linear programming model for the (l, k)-clique metric basis for a given graph G and its l-cliques.

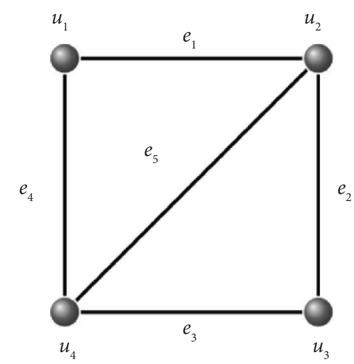

Figure 1: Graph G.

2. Main Results

To state our main results, we need to introduce the concept of k-global forcing set for l-cliques as an extension of the idea of global forcing sets for l-cliques of a graph which was presented in [9].

Let k and l be two positive integer numbers. A k-global forcing set for l-cliques of a graph G is a subset S of $V(G)$ with this property that $|(X \cap S)\Delta(Y \cap S)| \geq k$ for any two distinct l-cliques X and Y of G. A k-global forcing set for l-cliques of G with minimum cardinality is called a minimum k-global forcing set for l-cliques of G, and its cardinality, denoted by $\phi^k_{\mathrm{glc}}(G)$, is called the k-global forcing number for l-cliques of G.

For finding a global forcing set for l-cliques of G, an ILP model was presented in [9]. We extend this model to achieve the following ILP for finding a k-global forcing set for l-cliques of G.

Let G be a graph with $\{v_1, \ldots, v_n\}$ and let $C_G = \{X_1, \ldots, X_t\}$ be the set of all l-cliques of G. Suppose that $A_G = [a_{\mathrm{ij}}]$ is a $t \times n$ matrix, where $a_{\mathrm{ij}} = 1$ if $v_j \in X_i$, and $a_{\mathrm{ij}} = 0$ otherwise. The goal is to minimize $F(w_1, \ldots, w_n) = \sum_{i=1}^{n} w_i$ subject to the constraints:

(i) $|a_{i1} - a_{j1}|w_1 + |a_{i2} - a_{j2}|w_2 + \cdots + |a_{in} - a_{jn}|w_n \geq k$, $1 \leq i < j \leq t$.

(ii) $w_i \in \{0, 1\}, i \in [n]$ with $w_i = 1$ if vertex $v_1 \in S$ and $w_i = 0$ otherwise.

It is not difficult to see that if $\{w'_1, \ldots, w'_n\}$ is a set of values for which F attains its minimum, then $S = \{v_i\colon w'_i = 1$ for $i \in [n]\}$ is a minimum k-global forcing set for l-cliques of G. In addition, $F(w'_1, \ldots, w'_n) = \phi^k_{\mathrm{glc}}(G)$.

2.1. (l, k)-CMD of Corona Product. Let G and H be two graphs with $V(G) = \{g_1, \ldots, g_n\}$. The corona product $G \circ H$ is obtained from one copy of G and n copies of H by joining with an edge each vertex of the ith copy of H, $i \in [n]$, to g_i, see [10]. In this subsection, H_i, $i \in [n]$, denotes the ith copy of H in $G \circ H$.

Theorem 1. *Let G be a graph with $n > 1$ vertices and H be a graph with more than one $(l-1)$-clique. Then*

$$l - \mathrm{cdim}_k(G \circ H) = n \cdot \phi^k_{g(l-1)c}(H). \tag{1}$$

Proof. Let S_i be a k-global forcing set for $(l-1)$-cliques of H_i that $|S_i| = \phi^k_{g(l-1)c}(H)$. Set $S = \cup_{i=1}^{n} S_i$. Obviously, $|S| = n \cdot \phi^k_{g(l-1)c}(H)$. We claim that S is an (l, k)-CMG of $G \circ H$. To prove our claim, we investigate the following cases for l-cliques X and Y in $G \circ H$.

Case 1. X and Y are two distinct $(l-1)$-cliques of H_i for an $i \in [n]$. Then $|R_{G\circ H}\{X, Y\} \cap S_i| \geq k$ and consequently $|R_{G\circ H}\{X, Y\} \cap S| \geq k$.

Case 2. X and Y are two distinct l-cliques of H_i for an $i \in [n]$. Then there exist $(l-1)$-cliques X' and Y' in H_i such that $X' \subseteq X$ and $Y' \subseteq Y$. Thus, $|R_{G\circ H}\{X', Y'\} \cap S_i| \geq k$. This concludes that $|R_{G\circ H}\{X, Y\} \cap S_i| \geq k$ and so $|R_{G\circ H}\{X, Y\} \cap S| \geq k$.

Case 3. X and Y do not satisfy in Case 1 and Case 2. In this case, it is not difficult to check that there exists H_i, $i \in [n]$, such that $V(H_i) \subseteq R_{G\circ H}\{X, Y\}$. Hence, $|R_{G\circ H}\{X, Y\} \cap S_i| \geq k$ and so $|R_{G\circ H}\{X, Y\} \cap S| \geq k$.

According to cases 1–3, S is an (l, k)-CMG for $G \circ H$ which implies that $l - \mathrm{cdim}_k(G \circ H) \leq n \cdot \phi^k_{g(l-1)c}(H)$. Then it is sufficient to prove that $l - \mathrm{cdim}_k(G \circ H) \geq n \cdot \phi^k_{g(l-1)c}(H)$.

Let S' be an (l, k)-clique metric basis of $G \circ H$. Thus, it is enough to prove that $S' \cap V(H_i)$ is a k-global forcing set for $(l-1)$-cliques of H_i. Let X and Y be two distinct $(l-1)$-cliques of H_i. Since S' is an (l, k)-clique metric basis of $G \circ H$, then there exist at least k vertices $w_1, \ldots, w_k \in S'$ such that $d_{G\circ H}(X', w_i) \neq d_{G\circ H}(Y', w_i)$ for every $i \in [n]$ where $X' = X \cup \{g_i\}$ and $Y' = Y \cup \{g_i\}$. On the other hand, clearly

$d_{G\circ H}(X, v) = d_{G\circ H}(Y, v)$ for each $v \in V(G\circ H)V(H_i)$. Thus, we deduce that $\{w_1, \ldots, w_k\} \subseteq X\Delta Y$. Thus, $\{w_1, \ldots, w_k\} \subseteq S'$ is a k-global forcing set for $(l-1)$-cliques X and Y. Therefore, $l-\mathrm{cdim}_k\ (G\circ H) \geq |\cup_{i=1}^{n}(V(H_i)) \cap S'| \geq \sum_{i=1}^{n}\phi^k_{g(l-1)c}(H_i) = n\cdot\phi^k_{g(l-1)c}(H)$. □

Consider graph $\Gamma = P_2 \circ P_3$ shown in Figure 2. In this figure, 3-cliques are as $X_1 = \{g_1, h_{1_1}, h_{2_1}\}$, $X_2 = \{g_1, h_{2_1}, h_{3_1}\}$, $X_3 = \{g_2, h_{1_2}, h_{2_2}\}$, $X_4 = \{g_2, h_{2_2}, h_{3_2}\}$. Hence, $R_{P_2\circ P_3}\{X_1, X_2\} = R_{P_2\circ P_3}\{X_1 - \{g_1\}, X_2 - \{g_1\}\} = (X_1 - \{g_1\})\Delta(X_2 - \{g_1\}) = \{h_{1_1}, h_{3_1}\}$. By a similar argument, we have $R_{P_2\circ P_3}\{X_3, X_4\} = \{h_{1_2}, h_{3_2}\}$, $R_{P_2\circ P_3}\{X_1, X_3\} \supseteq \{h_{1_1}, h_{2_1}, h_{3_1}\}$, $R_{P_2\circ P_3}\{X_1, X_4\} \supseteq \{h_{1_1}, h_{2_1}, h_{3_1}\}$, $R_{P_2\circ P_3}\{X_2, X_3\} \supseteq \{h_{1_1}, h_{2_1}, h_{3_1}\}$, and $R_{P_2\circ P_3}\{X_2, X_4\} \supseteq \{h_{1_1}, h_{2_1}, h_{3_1}\}$. Since $\min_{i,j\in[4]}\{|R\{X_i, X_j\}| = 2$, then $k \leq 2$. Now, we are ready to compute $3-\mathrm{cdim}_2(P_2\circ P_3)$ by the previous theorem. Clearly, $\phi^2_{g2c}(P_3) = 2$. Therefore, $3-\mathrm{cdim}_2(\Gamma) = 3-\mathrm{cdim}_2(P_2\circ P_3) = 2\cdot\phi^2_{g2c}(P_3) = 2\cdot 2 = 4$.

Theorem 2. *Let G be a graph and H with $|E(H)| > 1$. Then $G\circ H$ is (2,2)-clique metric dimensional and*

$$\mathrm{edim}_2(G\circ H) = |V(G)||V(H)|. \quad (2)$$

Proof. First, we show that $G\circ H$ is (2,2)-clique metric dimensional. In other words, we prove that $G\circ H$ has no $(2,k)$-clique metric generator if $k > 2$. For this aim, we show that $k = \min_{e,e'\in E(G)}\{|R_{G\circ H}\{e, e\}|\} = 2$ for any $e, e' \in E(G\circ H)$. Suppose $e = g_i h_{j_i}$ and $e' = g_i h_{t_i}$. Thus, $R_{G\circ H}\{e, e'\} = \{h_{j_i}, h_{t_i}\}$ and so $k = \min_{e,e'\in E(G)}\{|R_{G\circ H}\{e, e'\}|\} = 2$.

Now, according to Theorem 1, we have $\mathrm{edim}_2(G\circ H) = n\cdot\phi^2_{g1c}(H)$. On the other hand, it is not difficult to check that $\phi^2_{g1c}(H) = |V(H)|$. Therefore, $\mathrm{edim}_2(G\circ H) = |V(G)\,\|\,V(H)|$. □

2.2. (l,k)-CMD of Edge Corona Product. Let G be a graph of size m and H be a graph. The edge corona product $G\diamondsuit H$ of graphs G and H is obtained from one copy of G and m copies of H by joining with an edge each vertex of the i^{th} copy of H, $i \in [m]$, to vertices of the i^{th} edge of G, cf. [11]. If $e \in E(G)$, then the copy of H in $G\circ H$ corresponding to the i^{th} edge of G will be denoted with H_i.

Theorem 3. *Let $l \geq 3$ be a positive integer number and H be a graph with more than one $(l-2)$-clique. If G is a graph of size m without any pendant vertices such that $N(X')\Delta N(Y') \neq \varnothing$ for every two 2-cliques X' and Y' in G and $|X''\Delta Y''| = 2$ for every two l-cliques X'' and Y'' in G, then*

$$l-\mathrm{cdim}_k(G\diamondsuit H) = m\cdot\phi^k_{g(l-2)c}(H). \quad (3)$$

Proof. Suppose that S_i is a k-global forcing set for $(l-2)$-cliques of H_i that $|S_i| = \phi^k_{g(l-2)c}(H)$ and set $S = \cup_{i=1}^{m}S_i$. To achieve $l-\mathrm{cdim}_k(G\diamondsuit H) \leq m\cdot\phi^k_{g(l-2)c}(H)$, we prove that S is a (l,k)-CMG of $G\diamondsuit H$. To do this, we investigate below cases for two distinct l-cliques X and Y of $G\diamondsuit H$.

Case 1. X and Y are l-cliques of H_i for an $i \in [n]$. Then there exist $(l-2)$-cliques X' and Y' in H_i such that $X' \subseteq X$ and $Y' \subseteq Y$. Thus, $|R_{G\diamondsuit H}\{X', Y'\} \cap S_i| \geq k$. Hence, $|R_{G\diamondsuit H}\{X, Y\} \cap S_i| \geq k$ and so $|R_{G\diamondsuit H}\{X, Y\} \cap S| \geq k$. Let $e_i = gg' \in E(G)$. A similar argument shows that $|R_{G\diamondsuit H}\{X, Y\} \cap S| \geq k$ where $X - \{g\}$ and $Y - \{g\}$ are $(l-1)$-cliques of H_i, or $X - \{g, g'\}$ and $Y - \{g, g'\}$ are $(l-2)$-cliques of H_i, for an $i \in [n]$.

Case 2. $X - \{g\}$ and $Y - \{g\}$ are $(l-1)$-cliques in H_i and H_j where $i \neq j$ and $e_i = gg', e_j = gg'' \in E(G)$. Clearly there exist two $(l-2)$-cliques X' and X'' in H_i such that $X', X'' \subseteq X - \{g\}$. Then $|R_{G\diamondsuit H}\{X', X''\} \cap S_i| \geq k$. Hence, $|R_{G\diamondsuit H}\{X, Y\} \cap S_i| \geq k$ and so $|R_{G\diamondsuit H}\{X, Y\} \cap S| \geq k$.

Case 3. X and Y do not satisfy in Case 1 and Case 2. In this case, one can check that there exists H_i such that $V(H_i) \subseteq R_{G\diamondsuit H}\{X, Y\}$. Then $|R_{G\diamondsuit H}\{X, Y\} \cap S_i| \geq k$ which concludes $|R_{G\diamondsuit H}\{X, Y\} \cap S| \geq k$.

Therefore, S is a (l,k)-CMG for $G\diamondsuit H$.

Now we prove $l-\mathrm{cdim}_k(G\diamondsuit H) \geq m\cdot\phi^k_{g(l-2)c}(H)$. Let S' be (l,k)-clique metric basis of $G\diamondsuit H$. It is enough to prove $S' \cap V(H_i)$ is a k-global forcing set for $(l-2)$-cliques of H_i, for $i \in [m]$. Suppose that X and Y are two distinct $(l-2)$-cliques of H_i, for $i \in [m]$. Let $e_i = gg' \in E(G)$. Since S' is a (l,k)-clique metric basis of $G\circ H$, then there exist at least k vertices $w_1, \ldots, w_k \in S'$ such that $d_{G\diamondsuit H}(X', w_i) \neq d_{G\diamondsuit H}(Y', w_i)$ for every $i \in [k]$ where $X' = X \cup \{g, g'\}$ and $Y' = Y \cup \{g, g'\}$. On the other hand, clearly $d_{G\circ H}(X, v) = d_{G\circ H}(Y, v)$ for each $v \in V(G\circ H)V(H_i)$. Thus, we conclude that $\{w_1, \ldots, w_k\} \subseteq X\Delta Y$. Hence, $\{w_1, \ldots, w_k\} \subseteq S'$ is a k-global forcing set for $(l-2)$-cliques X and Y in H. Therefore, $l-\mathrm{cdim}_k(G\diamondsuit H) \geq |\cup_{i=1}^{m}(V(H_i)) \cap S'| \geq \sum_{i=1}^{m}\phi^k_{g(l-2)c}(H_i) = m\cdot\phi^k_{g(l-2)c}(H)$. □

Consider $\Gamma = C_4 \diamondsuit P_2$ shown in Figure 3. $X_1 = \{g_1, g_2, h_{1_1}\}$, $X_2 = \{g_1, g_2, h_{2_1}\}$, $X_3 = \{g_2, g_3, h_{1_2}\}$, $X_4 = \{g_2, g_3, h_{2_2}\}$, $X_5 = \{g_3, g_4, h_{1_3}\}$, $X_6 = \{g_3, g_4, h_{2_3}\}$, $X_7 = \{g_1, g_4, h_{1_4}\}$, $X_8 = \{g_1, g_4, h_{2_4}\}$, $X_9 = \{g_1, h_{1_1}, h_{2_1}\}$, $X_{10} = \{g_2, h_{1_1}, h_{2_1}\}$, $X_{11} = \{g_2, h_{1_2}, h_{2_2}\}$, $X_{12} = \{g_3, h_{1_2}, h_{2_2}\}$, $X_{13} = \{g_3, h_{1_3}, h_{2_3}\}$, $X_{14} = \{g_4, h_{1_3}, h_{2_3}\}$, $X_{15} = \{g_4, h_{1_4}, h_{2_4}\}$, $X_{16} = \{g_1, h_{1_4}, h_{2_4}\}$ are 3-cliques of Γ. Then $R_{C_4\diamondsuit P_2}\{X_1, X_2\} = R_{C_4\diamondsuit P_2}\{X_1 - \{g_1, g_2\}, X_2 - \{g_1, g_2\}\} = \{h_{1_1}, h_{2_1}\}$, $R_{C_4\diamondsuit P_2}\{X_3, X_4\} = R_{C_4\diamondsuit P_2}\{X_3 - \{g_2, g_3\}, X_4 - \{g_2, g_3\}\} = \{h_{1_2}, h_{2_2}\}$. In addition,

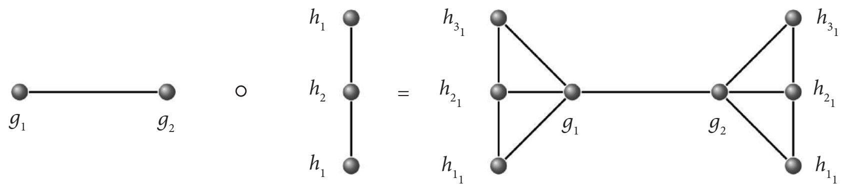

Figure 2: The corona product of P_2 and P_3.

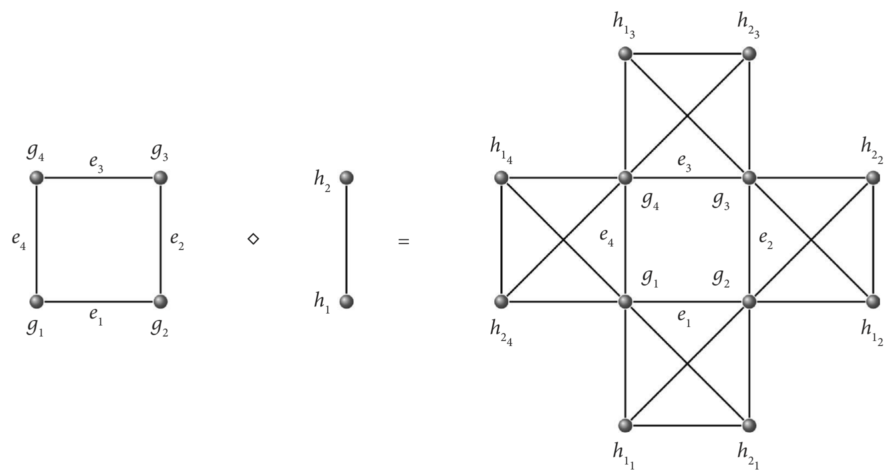

Figure 3: The edge corona product of C_4 and P_2.

by similar argument, we have $R_{C_4\diamond P_2}\{X_5, X_6\} = \{h_{1_3}, h_{2_3}\}$, $R_{C_4\diamond P_2}\{X_7, X_8\} = \{h_{1_4}, h_{2_4}\}$, $R_{C_4\diamond P_2}\{X_1, X_3\} \supseteq \{h_{1_4}, h_{2_4}\}$, $R_{C_4\diamond P_2}\{X_1, X_4\} \supseteq \{h_{1_4}, h_{2_4}\}$, $R_{C_4\diamond P_2}\{X_1, X_5\} \supseteq \{h_{1_1}, h_{2_1}\}$, $R_{C_4\diamond P_2}\{X_1, X_6\} \supseteq \{h_{1_1}, h_{2_1}\}$, $R_{C_4\diamond P_2}\{X_1, X_7\} \supseteq \{h_{1_2}, h_{2_2}\}$, $R_{C_4\diamond P_2}\{X_1, X_8\} \supseteq \{h_{1_2}, h_{2_2}\}$, $R_{C_4\diamond P_2}\{X_1, X_9\} \supseteq \{h_{1_2}, h_{2_2}\}$, $R_{C_4\diamond P_2}\{X_1, X_{10}\} \supseteq \{h_{1_4}, h_{2_4}\}$, $R_{C_4\diamond P_2}\{X_1, X_{11}\} \supseteq \{h_{1_4}, h_{2_4}\}$, $R_{C_4\diamond P_2}\{X_1, X_{12}\} \supseteq \{h_{1_4}, h_{2_4}\}$, $R_{C_4\diamond P_2}\{X_1, X_{13}\} \supseteq \{h_{1_1}, h_{2_1}\}$, $R_{C_4\diamond P_2}\{X_1, X_{14}\} \supseteq \{h_{1_1}, h_{2_1}\}$, $R_{C_4\diamond P_2}\{X_1, X_{15}\} \supseteq \{h_{1_2}, h_{2_2}\}$, $R_{C_4\diamond P_2}\{X_1, X_{16}\} \supseteq \{h_{1_2}, h_{2_2}\}$. By a similar method, one can obtain other $R_{C_4\diamond P_2}\{X_i, X_j\}$. Since $\min_{1\le i<j\le 16}\{\,|R\{X_i, X_j\}\,| = 2$, then $k \le 2$. Now, we are ready to compute $3-\mathrm{cdim}_2(\Gamma)$ by the previous theorem. Clearly, $\phi^2_{g1c}(P_3) = 2$. Therefore, $3-\mathrm{cdim}_2(\Gamma) = 3-\mathrm{cdim}_2(P_2 \diamond P_3) = m\cdot\phi^2_{g1c}(P_3) = 4\cdot 2 = 8$.

Theorem 4. *If H is a nontrivial graph G is a graph of order n with this property that $N(X')\Delta N(Y') \ne \varnothing$ for every two 2-cliques X' and Y' in G. Then $G\diamond H$ is a (2,2)-clique metric dimensional and*

$$\mathrm{edim}_2(G\diamond H) = |E(G)\|V(H)|. \quad (4)$$

Proof. In order to show that $G\diamond H$ is a (2,2)-clique metric dimensional, we need to prove $k = \min_{e,e'\in E(G)}\{|R_{G\diamond H}\{e, e\}|\} = 2$ for any $e, e' \in E(G\diamond H)$. Suppose $e = g_i h_{j_i}$ and $e' = g_i h_{t_i}$. Thus, $R_{G\diamond H}\{e, e'\} = \{h_{j_i}, h_{t_i}\}$ and so $k = \min_{e,e'\in E(G)}\{|R_{G\diamond H}\{e, e'\}|\} = 2$.

Now, let S be a (2,2)-clique metric basis of $G\diamond H$. Assume, to the contrary, that there exists $x \in S\cap V(H_i)$ for an $i \in [n]$. Then $R\{\mathrm{gx}, \mathrm{gy}\} < 2$ where g is an end point of e_i, which is a contradiction. This concludes that $\mathrm{edim}_2(G\diamond H) \ge |E(G)\|V(H)|$. On the other hand, obviously $\cup_{i=1}^n V(H_i)$ is a (2,2)-clique metric generator of $G\diamond H$ and so $\mathrm{edim}_2(G\diamond H) \le |E(G)\|V(H)|$. Therefore, $\mathrm{edim}_2(G\diamond H) = |E(G)\|V(H)|$. □

2.3. Integer Linear Programming Model. In [9], Afkhami et al. gave an integer linear programming model (ILPM) to deal with the l-clique metric dimension. Motivated by this work, we here present an ILPM for the (l,k)-clique metric basis for a given graph G and its l-cliques. Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Suppose that $C_G = \{X_1, \ldots, X_t\}$ is the set of all l-cliques of G. In addition, suppose that $D_G = [d_{ij}]$ is a $t\times n$ matrix such that $i\in[t]$ and $j\in[n]$. For $x_i \in \{0,1\}$, $i\in[n]$, define $F(x_1,\ldots,x_n) = \sum_{i=1}^n x_i$. The goal is to minimize F subject to the constraints

$$\left|d_{i1}-d_{j1}\right|x_1 + \left|d_{i2}-d_{j2}\right|x_2 + \cdots + \left|d_{\mathrm{in}}-d_{\mathrm{jn}}\right|x_n \ge k, \quad 1\le i<j\le t. \quad (5)$$

Clearly, if $x_1', \ldots, x_n'$ is a set of values for which F is attained, then $S = \{v_i\colon x_i' = 1\}$ is a (l,k)-clique metric basis for G.

3. Application of $(2, k)$-Clique Metric Generator in Self-Driving Car Navigation

A self-driving car needs to determine its position on the city's streets uniquely. In other words, each street of the city needs code which uniquely determines its location. Therefore, if we consider the city as a graph G that edges of G are corresponding to the city's streets, then an edge metric generator of G would be the codes of streets. We note that a self-driving car calculates its location by measuring the distance to a set of landmarks placed in certain vertices. In this case, if there are two positions which are only distinguished by a single landmark and communication with this landmark is lost, then the self-diving car cannot find its position. To fix this problem, we have to improve the accuracy of the detection or the robustness of the system. To do this, we should have a family of detectors, say k detectors, such that every pair of edges is distinguished by them.

4. Concluding Remarks

(l, k)-Clique metric dimension of a graph is a parameter that is difficult to compute and that frequently arises in applications. In the present work, we have studied its behavior under corona and edge corona products. It would be of interest to investigate this invariant under other products of graphs such as Cartesian product, lexicographic product, and strong product. We have also presented an integer linear programming model for finding (l, k)-clique metric dimension of a graph. Then, another interesting thing would be to apply heuristic methods like greedy algorithms, local search algorithms, or metaheuristic algorithms (e.g., simulated annealing and genetic algorithms) for finding near-optimal solutions efficiently.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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