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Research Article

Algorithmic Complexity and Bounds for Domination Subdivision Numbers of Graphs

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Let $G = (V, E)$ be a simple graph. A subset $D \subseteq V$ is a dominating set if every vertex not in D is adjacent to a vertex in D. The domination number of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. The domination subdivision number $\mathrm{sd}_\gamma(G)$ of G is the minimum number of edges that must be subdivided (each edge can be subdivided at most once) in order to increase the domination number. In 2000, Haynes et al. showed that $\mathrm{sd}_\gamma(G) \leq d_G(v) + d_G(v) - 1$ for any edge $uv \in E(G)$ with $d_G(u) \geq 2$ and $d_G(v) \geq 2$ where G is a connected graph with order no less than 3. In this paper, we improve the above bound to $\mathrm{sd}_\gamma(G) \leq d_G(u) + d_G(v) - |N_G(u) \cap N_G(v)| - 1$, and furthermore, we show the decision problem for determining whether $\mathrm{sd}_\gamma(G) = 1$ is NP-hard. Moreover, we show some bounds or exact values for domination subdivision numbers of some graphs.

1. Introduction

For terminology and notation on the graph theory not given here, the reader is referred to Xu [1]. Let $G = (V, E)$ be a finite, undirected, and simple graph, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set of G. For a vertex $x \in V(G)$, let $N_G(x) = \{y\colon xy \in E(G)\}$ be the *open set of neighbors* of x and $N_G[x] = N_G(x) \cup \{x\}$ be the *closed set of neighbors* of x. The cardinality of $V(G)$ is called the *order* of G. The degree of vertex $x \in V(G)$ is the cardinality of $N_G(x)$. The maximum degree and minimum degree of G are denoted $\Delta(G)$ and $\delta(G)$, respectively. If $\Delta(G) = \delta(G) = k$ for graph G, then G is called a *k-regular graph*. For any edge $e \in E(G)$, we denote G_e as a new graph by subdividing the edge e in G. For any edge $e = uv \in E(G)$, we may view e as a two vertex set $\{u, v\}$.

A subset $D \subseteq V$ is a *dominating set* of G if every vertex in $V - D$ has at least one neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G. A dominating set D is called a *minimum dominating set* of G if $|D| = \gamma(G)$. Domination is an important and classic notion that has become one of the most widely researched topics in graph theory and is also used to study the property of networks frequently. A thorough study of domination appears in the books [2, 3] by Haynes, Hedetniemi, and Slater. Among various problems related to the domination number, some focus on graph alterations and their effects on the domination number. As for different applications, there are also many variated dominations, such as Italian domination [4], 2-rainbow domination [5], research on Zagreb indices by domination [6].

The *domination subdivision number* of a graph G, denoted by $\mathrm{sd}_\gamma(G)$, equals the minimum number of edges that must be subdivided in order to obtain a graph G' for which $\gamma(G') > \gamma(G)$. Since the domination number of graph K_2 does not change when its only edge is subdivided, we must assume here that the graph G is of order no less than 3. Domination subdivision number of graph has been widely studied, see [7–9] for examples.

For a graph parameter, knowing whether or not there exists a polynomial-time algorithm to compute its exact value is the essential problem. If the decision problem corresponding to the computation of this parameter is NP-hard or NP-complete, then polynomial-time algorithms for this parameter do not exist unless $\mathrm{NP} = P$. The problem of determining the domination number has been proven NP-complete for chordal bipartite graphs [10]. There are many other results on complexity for variations of domination; these results can be found in the two books [3, 11] and the survey [12].

Many famous networks are bipartite graphs, such as hypercube graphs, partial cube, grid graphs, and median graphs. If we know the decision problem for the domination subdivision problem is NP-hard, then the studies on the domination subdivision number are more meaningful. So we should be concerned about the algorithmic complexity of the domination subdivision problem in bipartite graphs. In this paper, we will show that the decision problem for the domination subdivision number is NP-hard even for bipartite graphs. In other words, there are no polynomial-time algorithms to compute these parameters unless $P = \mathrm{NP}$.

2. Preliminary Results

In the book [13], Garey and Johnson provide three steps to prove a decision problem to be NP-hard. We applied these three steps to prove that our decision problem is NP-hard. Our proof involved a polynomial transformation from the well-known NP-complete problem, the 3-satisfiability problem. In this section, we will recall some terms related to the 3-satisfiability problem.

(i) U is a set of Boolean variables.

(ii) A truth assignment for U is a mapping $t: U \longrightarrow \{T, F\}$. If $t(u) = T$, then u is considered "true" under t; if $t(u) = F$, then u is considered "false" under t.

(iii) u and $\overline{u}$ are literals over U when u is a variable in U. The literal u (resp. $\overline{u}$) is true under t if and only if the variable u is true (resp. false) under t.

(iv) A clause over U is a set of literals over U. It represents the disjunction of these literals and it is satisfied by a truth assignment t if and only if at least one of its elements is true under t.

(v) A collection $\mathscr{C}$ of clauses over U is satisfiable if and only if there exists a truth assignment t for U that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment t is called a satisfying truth assignment for $\mathscr{C}$.

The 3-satisfiability problem is defined as finding a satisfying truth assignment for a collection $\mathscr{C}$ of clauses over U.

3-satisfiability problem (3SAT):

Instance: *A collection*$\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$*of clauses over a finite set*U*of variables such that*$|C_j| = 3$*for*$j = 1, 2, \ldots, m$.

Question: *Is there a truth assignment for*U*that satisfies all the clauses in* $\mathscr{C}$*?*

Theorem 1 (Theorem 3.1 in [13]). *The 3-satisfiability problem is NP-complete.*

A dominating set D is called an *efficient dominating set* of graph G if $|N_G[v] \cap D| = 1$ for every vertex $v \in V(G)$. An efficient dominating set of a graph G is always a minimum dominating set [14, 15].

Lemma 2 (Berge [16]). *For any graph* G,

$$\gamma(G) \geq \frac{|V(G)|}{\Delta(G) + 1}. \tag{1}$$

Lemma 3 (Huang and Xu [17]). *Let* G *be a* k*-regular graph. Then,*

$$\gamma(G) \geq \frac{|V(G)|}{(k+1)}, \tag{2}$$

with equality if and only if G *has an efficient dominating set. In addition, if* G *has an efficient dominating set, then every efficient dominating set must be a minimum dominating set, and vice versa.*

3. Bounds

Let G be a simple graph. Let $X \subseteq V(G)$ and $x \in X$. The private neighborhood of x with respect to X is defined as the set

$$\begin{aligned} \mathrm{PN}(x, X, G) &= N_G[x] \backslash N_G[X - x] \\ &= \{u \in V(G) | N_G[u] \cap X = \{x\}\}. \end{aligned} \tag{3}$$

For any edge, $e = uv \in E(G)$ and $D \subseteq V(G)$, if $e \cap D = \{u\}$, then we denote $\overline{e \cap D} = \{v\}$.

Theorem 4. *Let* $e = uv$ *be an edge in* G*. Subdivide* e *by a new vertex* w*. Then,* $\gamma(G_e) > \gamma(G)$ *iff* $\gamma(G - E_G(u) \triangle E_G(v)) > \gamma(G)$, $|e \cap D| \leq 1$, *and* $\overline{e \cap D} \in PN(e \cap D, G)$ *if* $|e \cap D| = 1$ *for any minimum dominating set* D *of* G.

Proof. Assume $\gamma(G_e) > \gamma(G)$. Suppose to the contrary that $\gamma(G - E_G(u) \triangle E_G(v)) = \gamma(G)$. Let D be a minimum dominating set of $G - E_G(u) \triangle E_G(v)$. Then, $|D| = \gamma(G)$ and $D \cap \{u, v\} = 1$ (if $D \cap \{u, v\} = 2$, then D is also a dominating set of G_e contradicts $\gamma(G_e) > \gamma(G)$). Let $D' = (D \backslash \{u, v\}) \cup \{w\}$. Then, D' is a dominating set of G_e with cardinality $\gamma(G)$, which is a contradiction with $\gamma(G_e) > \gamma(G)$. Hence,

$$\gamma(G - E_G(u) \triangle E_G(v)) > \gamma(G). \tag{4}$$

Let D be a minimum dominating set of G. If $|e \cap D| = 2$ or $|e \cap D| = 1$ and $\overline{e \cap D} \notin PN(e \cap D, D, G)$, then D is also a minimum dominating set D of G_e. So $|e \cap D| \leq 1$ and $\overline{e \cap D} \in PN(e \cap D, D, G)$ if $|e \cap D| = 1$ for any minimum dominating set D of G.

Assume $\gamma(G - E_G(u) \triangle E_G(v)) > \gamma(G)$, $|e \cap D| \leq 1$, and $\overline{e \cap D} \in PN(e \cap D, D, G)$ if $|e \cap D| = 1$ for any minimum dominating set D of G. Suppose to the contrary that $\gamma(G_e) = \gamma(G)$. Let D_e be a minimum dominating set of G_e. Then, $1 \leq |D_e \cap \{u, v, w\}| \leq 2$. If $D_e \cap \{u, v, w\} = 2$, then we can assume without loss of generality that $D_e \cap \{u, v, w\} = \{u, v\}$, and hence, D_e is also a minimum dominating set of G, a contradiction with $|e \cap D| \leq 1$ for any minimum dominating set D of G. Thus, $|D_e \cap \{u, v, w\}| = 1$. If $\{w\} = D_e \cap$

$\{u, v, w\}$, then $D_e - w + u$ is a minimum dominating set of $G - E_G(u) \triangle E_G(v)$, which is a contradiction with $\gamma(G - E_G(u) \triangle E_G(v)) > \gamma(G)$. If $D_e \cap \{u, v, w\} = \{u\}$ or $\{v\}$, then D_e is a minimum dominating set of G and assume without loss of generality $D_e \cap \{u, v\} = \{u\}$. Note that $N_{G_e}(v) \cap D_e \neq \emptyset$ and D_e is also a minimum dominating set of G since $\gamma(G_e) = \gamma(G)$. So $N_{G_e}(v) \cap (D_e - u) \neq \emptyset$, a contradiction with $v \in PN(u, D_e, G)$. □

Theorem 5. *For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v, where $d_G(u) \geq 2$ and $d_G(v) \geq 2$,*

$$\mathrm{sd}_\gamma(G) \leq d_G(u) + d_G(v) - |N_G(u) \cap N_G(v)| - 1. \quad (5)$$

Proof. Let

$$N_G(u) = \{u_1, u_2, \ldots, u_a, v\}, \quad (6)$$

and let

$$N_G(v) = \{v_1, v_2, \ldots, v_b, u, u_{a-c+1}, \ldots, u_a\}, \quad (7)$$

where $c = |N_G(u) \cap N_G(v)|$ and $b = d_G(v) - 1 - |N_G(u) \cap N_G(v)|$. Let

$$S = E_G(u) \cup \{vw\colon w \in N_G(v) \backslash N_G[u]\}, \quad (8)$$

and let G' be the graph that results from subdividing all edges in S. We will show $\gamma(G') > \gamma(G)$. Let the subdivided vertex of uv be w', and the subdivided vertex of uu_i is u_i' and the subdivided vertex of vv_j is v_j' for each $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$. Let D' be a minimum dominating set of G'. Clearly, $1 \leq |D' \cap \{u, w', v\}| \leq 2$ and we can assume $D' \cap \{u, w', v\} = \{u, v\}$ if $|D' \cap \{u, w', v\}| = 2$. Let $A' = D' \cap \{u_1', u_2', \ldots, u_a'\}$ and $A = \{u_i\colon u_i' \in A', 1 \leq i \leq a\}$, $B' = D' \cap \{v_1', v_2', \ldots, v_b'\}$ and $B = \{v_j\colon v_j' \in B', 1 \leq j \leq b\}$.

Assume $D' \cap \{u, w', v\} = \{u, v\}$. Then, $(D' \backslash (A' \cup B')) \cup A \cup B - u$ is a dominating set of G with cardinality $|D'| - 1$. Thus, $\gamma(G') = |D'| \geq \gamma(G) + 1$. Next, assume $|D' \cap \{u, w', v\}| = 1$. We consider the following three cases.

Case 1: $D' \cap \{u, w', v\} = w'$.

Then, $(D' \backslash (A' \cup B')) \cup A \cup B - w'$ is a dominating set of G with cardinality $|D'| - 1$. Thus, $\gamma(G') = |D'| \geq \gamma(G) + 1$.

Case 2: $D' \cap \{u, w', v\} = u$.

If $B' \neq \emptyset$, then $(D' \backslash A') \cup A - u - B' + v$ is a dominating set of G with cardinality no more than $|D'| - 1$. Thus, $\gamma(G') = |D'| > \gamma(G)$.

Suppose $B' = \emptyset$. Then, v should be dominated by some vertex in $N_G(u) \cap N_G(v)$. Therefore, $(D' \backslash (A' \cup B')) \cup A \cup B - u$ is a dominating set of G with cardinality $|D'| - 1$. Thus, $\gamma(G') = |D'| \geq \gamma(G) + 1$.

Case 3: $D' \cap \{u, w', v\} = v$.

Then, u should be dominated by some vertex in A' which implies $A' \neq \emptyset$. Then, $(D' \backslash B') \cup B - v - A' + u$ is a dominating set of G with cardinality no more than $|D'| - 1$. Thus, $\gamma(G') = |D'| > \gamma(G)$.

Note that $|S| = d_G(u) + d_G(v) - |N_G(u) \cap N_G(v)| - 1$. So

$$\mathrm{sd}_\gamma(G) \leq d_G(u) + d_G(v) - |N_G(u) \cap N_G(v)| - 1. \quad (9)$$

□

Corollary 6 (Haynes et al. [18]). *For any connected graph G and edge uv, where $d_G(u) \geq 2$ and $d_G(v) \geq 2$,*

$$\mathrm{sd}_\gamma(G) \leq d_G(v) + d_G(v) - 1. \quad (10)$$

Proposition 7. *Let $e = uv$ be an edge in G and w be the inserted vertex in e. If w belongs to every γ-set of G_e and $\gamma(G - E_G(u) \triangle E_G(v)) > \gamma(G)$, then $\gamma(G_e) > \gamma(G)$.*

Proof. Since w belongs to every γ-set D_e of G_e, $u, v \notin D_e$. Then, $D = D_e - w + u$ is a dominating set of $G - E_G(u) \triangle E_G(v)$. Since $|D_e| = |D| \geq \gamma(G - E_G(u) \triangle E_G(v)) > \gamma(G)$, $\gamma(G_e) = |D_e| > \gamma(G)$. □

Proposition 8. *Let G be a $k \geq 2$-regular graph and it has an efficient dominating set. Then, $sd_\gamma(G) = 1$.*

Proof. By Lemma 3, $\gamma(G) = |V(G)|/(k + 1)$. Let G_e be a graph by subdividing any edge e of G. Since G is k-regular and $\Delta(G_e) = k$ where $k \geq 2$, $\Delta(G_e) = k$. By Lemma 2,

$$\gamma(G_e) \geq \frac{|V(G_e)|}{k+1} = \frac{|V(G)| + 1}{k+1} > \frac{|V(G)|}{k+1}. \quad (11)$$

Hence, $\gamma(G_e) > \gamma(G)$ which implies that $\mathrm{sd}_\gamma(G) = 1$.

An efficient dominating set is also known as perfect codes in coding theory. There are many classical graphs that have efficient dominating sets, such as cycle C_n where $n \equiv 0 \pmod 3$, star graph, and pancake graph [19], some Circulant graphs, and Harary graphs [17], some Möbius ladders [20]. The domination subdivision numbers of these graphs are 1. □

Proposition 9. *Let G be a graph and let u be a support vertex that has at least two leaves. Then, $sd_\gamma(G) = 1$.*

Proof. Let $e = uv \in E(G)$ where v is a leaf and let w be another leaf corresponding to u. Let G_e be the graph from subdividing the edge e and let D_e be a minimum dominating set of G_e. Note that $|D_e \cap N_{G_e}[v]| \geq 1$ and $|D_e \cap N_{G_e}[w]| \geq 1$. Since $|D_e \cap N_{G_e}[w]| \geq 1$, we can without loss of generality assume $u \in D_e$. Then, $D_e \backslash N_{G_e}[v]$ is a dominating set of G with cardinality at most $|D_e| - 1$. Therefore,

$$\gamma(G_e) = |D_e| > \gamma(G), \quad (12)$$

which implies that $\mathrm{sd}_\gamma(G) = 1$. □

Proposition 10. *Let G be a graph and u, v be two adjacent support vertices. Then, $sd_\gamma(G) \le 3$.*

Proof. Let $e = uv \in E(G)$ where u and v are both support vertices. Let p and q be two leaves corresponding to u and v, respectively. Let G' be the graph from subdividing three edges uv, up, vq of G, where the inserted vertices are w, s and t, respectively. Let D be a minimum dominating set of G'. To dominate w, p, q in G', we need $\{u, v, w\} \cap D \ne \emptyset$, $\{p, s\} \cap D \ne \emptyset$, $\{q, t\} \cap D \ne \emptyset$. Note that

$$(D \backslash \{u, v, w, p, q, s, t\}) \cup \{u, v\}, \tag{13}$$

is a dominating set of G. We have

$$\gamma\left(G'\right) = |D| \ge |(D \backslash \{u, v, w, p, q, s, t\}) \cup \{u, v\}| > \gamma(G), \tag{14}$$

which implies that $\mathrm{sd}_\gamma(G) \le 3$. □

Theorem 11. *Let G be a nonempty graph. Then, $sd_\gamma(G \circ K_1) = 3$.*

Proof. Denote $V(G) = \{u_1, u_2, \ldots, u_n\}$ and the corresponding vertex of u_i is v_i for $1 \le i \le n$. Clearly, $\gamma(G \circ K_1) = |V(G)|$. Let G' be a graph by subdividing any two edges of $G \circ K_1$. If the two edges both belong to $E(G)$, then $V(G)$ is also a dominating set of G'. If the two edges are both pendent edges $u_i v_i$ and $u_j v_j$ for some i, j, then $V(G) - u_i - u_j + u_i' + u_j'$ is also a dominating set of G' where u_i' and u_j' are the inserted vertices. If one of the two edges is in $E(G)$ and the other is a pendent edge $u_i v_i$ for some i, then $V(G) - u_i + u_i'$ is also a dominating set of G' where u_i' is the inserted vertices of $u_i v_i$. Thus $\mathrm{sd}_\gamma(G \circ K_1) \ge 3$.

Let $u_i u_j$ be an edge in $E(G)$. We subdivide the three edges $u_i u_j$, $u_i v_i$, and $u_j v_j$. Let the inserted vertices are w', u_i' and u_j' and G' be the resulting graph. To dominate all the pendent vertices, we need at least n vertices except u_i, u_j, w' in G'. To dominate w', we need at least one vertex in $\{u_i, u_j, w'\}$. Therefore, $\gamma(G') \ge n + 1$ and hence, $\mathrm{sd}_\gamma(G \circ K_1) = 3$. □

4. NP-Hardness of Domination Subdivision Number

In this section, we show the algorithmic complexity of the problem for determining the domination subdivision number of a graph. We first state the problem as the following decision problem.

Domination subdivision problem:

Instance: *A* nonempty *graph G*.

Question: *Is* $\mathrm{sd}_\gamma(G) = 1$*?*

For proving the algorithmic complexity of domination subdivision problem is NP-hard, we follow the method introduced in [21] which is to prove the algorithmic complexity of bondage problem is NP-hard. The steps are similar but the constructed graph and details are different.

Theorem 12. *The domination subdivision problem is NP-hard for bipartite graphs.*

Proof. We start the proof by using 3SAT problem which is a well-known NP-complete problem by Theorem 1. Let Boolean variables set $U = \{u_1, u_2, \ldots, u_n\}$ and clauses set $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary 3SAT instance where $|C_j| = 3$ for each $j \in [m]$. To reduce the above instance of 3SAT to an instance of domination subdivision problem, we will construct a graph G from the above instance, and then prove $\mathscr{C}$ is satisfiable if and only if $\mathrm{sd}_\gamma(G) = 1$.

For each variable $u_i \in U$, we create a graph H_i with vertex set $V(H_i) = \{u_i, v_i, \overline{u}_i, p_i, q_i\}$ and edge set $E(H_i) = \{u_i v_i, v_i \overline{u}_i, u_i p_i, \overline{u}_i p_i, p_i q_i\}$. We then create a single vertex c_j for each $C_j = \{x_j, y_j, z_j\} \in \mathscr{C}$ and add three edges $c_j x_j, c_j y_j, c_j z_j$. Finally, we add a path $P = s_1 s_2 s_3$ with length 2, and join s_1 and s_3 to vertex c_j for every $j \in [m]$. Figure 1 shows an example of constructed G where $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, \overline{u}_2, u_3\}, C_2 = \{\overline{u}_1, u_2, u_4\}$, $C_3 = \{\overline{u}_2, u_3, u_4\}$.

Note that G contains $5n + m + 3$ vertices and $5n + 5m + 2$ edges; this construction can be done in polynomial time. To demonstrate that this is truly a transformation, it is necessary to establish that $\mathrm{sd}_\gamma(G) = 1$ if and only if when there exists a truth assignment for U which satisfies all the clauses in $\mathscr{C}$.

Assume D be a minimum dominating set of G. Note that $|D \cap V(P)| \ge 1$, $|D \cap N_G[v_i]| \ge 1$ and $|D \cap N_G[q_i]| \ge 1$ for every $i \in [n]$. Hence,

$$\gamma(G) = |D| \ge \sum_{i=1}^{n} |D \cap V(H_i)| + |D \cap V(P)| \ge 2n + 1. \tag{15}$$

Suppose that $\gamma(G) = 2n + 1$. Then, $|D \cap V(P)| = 1$, $|D \cap V(H_i)| = 2$ for every $i \in [n]$, and $c_j \notin D$ for all $j \in [m]$. Because q_i should be dominated by D, $|D \cap \{u_i, \overline{u}_i\}| \le 1$ for every $i \in [n]$. Since all vertices in $V(P)$ can only be dominated by $D \cap V(P)$ and $|D \cap V(P)| = 1$, $D \cap V(P) = \{s_2\}$.

For any edge $e \in E(G)$, we claim that $\gamma(G_e) \le 2n + 2$. If $e = p_k q_k$ for some $k \in [n]$ and the inserted vertex is w, then

$$\{w, v_k\} \cup \left(\cup_{i \in [n] \backslash \{k\}} \{p_i, v_i\} \right) \cup \{s_1, s_2\}, \tag{16}$$

is a dominating set of G_e with cardinality $2n + 2$. If $e \in \left\{ (\cup_{i \in [n]} E(H_i)) \cup E_G(s_1) \cup E_G(s_3) \right\} \backslash (\cup_{i \in [n]} \{p_i q_i\})$, then

$$\left(\cup_{i \in [n]} \{p_i, v_i\} \right) \cup \{s_1, s_3\}, \tag{17}$$

is a dominating set of G_e with cardinality $2n + 2$. If $e = c_l u$ for some $l \in [m]$ and $u \in U$ or $u \in \overline{U}$ (assume without loss of generality $u \in V(H_k)$), then

$$\{u, p_k\} \cup \left(\cup_{i \in [n] \backslash \{k\}} \{p_i, v_i\} \right) \cup \{s_1, s_2\}, \tag{18}$$

is a dominating set of G_e with cardinality $2n + 2$.

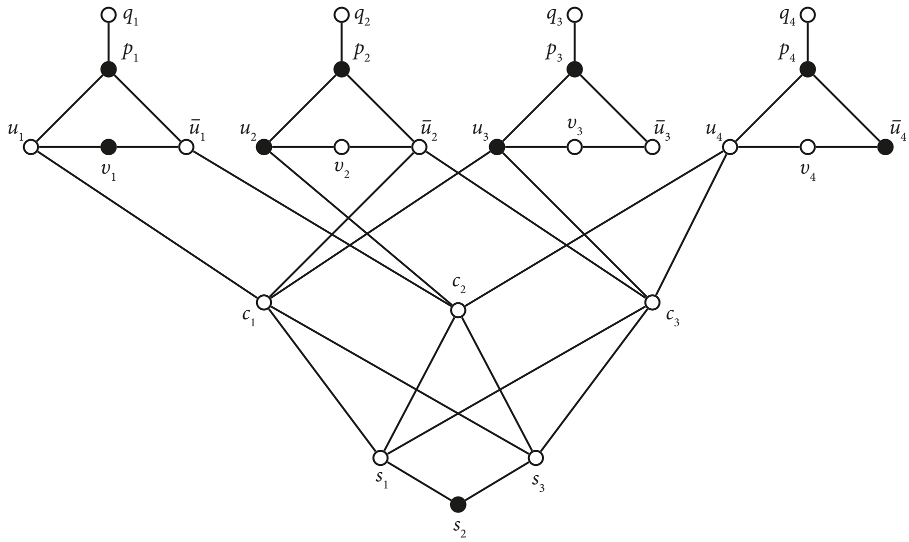

Figure 1: The constructed graph G with $\gamma(G) = 9$. The set of bold vertices is a minimum dominating set of G.

We then claim that $\gamma(G) = 2n + 1$ if and only if $\mathscr{C}$ is satisfiable. Assume $\gamma(G) = 2n + 1$. Let D be a minimum dominating set of G. Define a function $t: U \longrightarrow \{T, F\}$ by

$$t(u_i) = \begin{cases} T, & \text{if } u_i \in D, \\ F, & \text{otherwise,} \end{cases} \quad i = 1, 2, \ldots, n. \tag{19}$$

By the above discussions, $|D \cap \{u_i, \overline{u}_i\}| \leq 1$ for every $i \in [n]$. Hence, the definition of t is well-defined. Recall that $D \cap V(P) = \{s_2\}$. For any clause $C_j \in \mathscr{C}$ where $j \in [m]$, there exists some integer i with $i \in [n]$ such that c_j should be dominated by $u_i \in D$ or $\overline{u}_i \in D$, without loss of generality we say $u_i \in D$. This implies $t(u_i) = T$ by (19), which deduces that the clause C_j is satisfied. Therefore, $\mathscr{C}$ is satisfiable. Conversely, assume that $\mathscr{C}$ is satisfiable by t, where $t: U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. We create a subset $D' \subseteq V(G)$ as follows. Put u_i (resp. $\overline{u}_i$) to D' when u_i is true (resp. flase) under t. Because t is a satisfying truth assignment for $\mathscr{C}$, at least one literal u in C_j is true under t for each $j \in [m]$ which implies $c_j \in N_G[D']$. Hence $D' \cup \{s_2\}$ is a dominating set of G with cardinality $2n + 1$. By (15), $\gamma(G) \geq 2n + 1$, and then $\gamma(G) = 2n + 1$.

Finally, we claim $\gamma(G) = 2n + 1$ if and only if $\mathrm{sd}_\gamma(G) = 1$. Suppose $\gamma(G) = 2n + 1$. We subdivide $e = s_1 s_2$ and let the inserted vertex be s. By contradiction, assume $\gamma(G) = \gamma(G_e)$ and D be a minimum dominating set of G_e. By the similar discussions as above, $|D \cap V(H_i)| \geq 2$ for every $i \in [n]$. Since $s_1 s s_2 s_3$ is a path in G_e, at least 2 vertices of $\{c_1, c_2, \ldots, c_m, s_1, s_2, s_3\}$ should be in D. So $\gamma(G_e) \geq 2n + 2 > \gamma(G)$, and then $\mathrm{sd}_\gamma(G) = 1$. Suppose $\mathrm{sd}_\gamma(G) = 1$. Let e' be an edge with $\gamma(G) < \gamma(G_{e'})$. By the above discussions, $\gamma(G_{e'}) \leq 2n + 2$. Hence, $2n + 1 \leq \gamma(G) < \gamma(G_{e'}) \leq 2n + 2$. Therefore, $\gamma(G) = 2n + 1$.

By the above discussions, we see that $\mathrm{sd}_\gamma(G) = 1$ if and only if there exists a truth assignment t for U which satisfies all the clauses of $\mathscr{C}$. We complete the proof. □

5. Domination Subdivision Number for Some Cartesian Product Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. The *Cartesian product* of G_1 and G_2 is an undirected graph, denoted by $G_1 \square G_2$, where $V(G_1 \square G_2) = V_1 \times V_2$, two distinct vertices $x_1 x_2$ and $y_1 y_2$, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, are linked by an edge in $G_1 \square G_2$ if and only if either $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_2 = y_2$ and $x_1 y_1 \in E(G_1)$. Throughout this paper, the notation P_n and C_n denote a path with vertex set $[n] = \{1, 2, \ldots, n\}$. For integers $m \geq 2$ and $n \geq 3$, the Cartesian product of G_1 with order m and G_2 with order n is $G_1 \square G_2$ that has vertex set

$$\left\{v_{ij}: i \in [m], j \in [n]\right\}. \tag{20}$$

Let $(G_1)_v = V(G_1) \times \{v\}$ and $(G_2)_u = \{u\} \times V(G_2)$ where $v \in V(G_2)$ and $u \in V(G_1)$ are called the layers of G_1 and G_2, respectively. Figure 2 is the Cartesian product of P_2 and C_{10}.

If we want to compute the domination subdivision number of a graph, we need to know the exact value of its domination number. We start with the following classical results.

Theorem 13 (Jacobson and Kinch [22]). *For $n \geq 2$,*

$$\gamma(P_2 \square P_n) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{21}$$

Theorem 14 (Nandi et al. [23]). *For $n \geq 3$,*

$$\gamma(P_2 \square C_n) = \begin{cases} \left\lceil \dfrac{n+1}{2} \right\rceil, & \text{when } n \text{ is not a multiple of 4,} \\ \dfrac{n}{2}, & \text{when } n \text{ is a multiple of 4.} \end{cases} \tag{22}$$

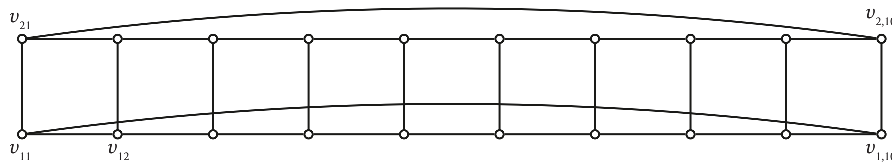

Figure 2: Graph $P_2\square C_{10}$.

Lemma 15. *For $n\geq 3$ and $n\equiv 0\,(\mathrm{mod}\,4)$,*

$$\mathrm{sd}_\gamma(P_2\square C_n)=1. \quad (23)$$

Proof. Let $n\geq 3$ and $n\equiv 0\,(\mathrm{mod}\,4)$. By Theorem 14, $\gamma(P_2\square C_n)=n/2$. Note that $\gamma(P_2\square C_n)$ is a 3-regular graph. Since

$$\gamma(P_2\square C_n)=\frac{n}{2} = \frac{|V(G)|}{4}, \quad (24)$$

$P_2\square C_n$ has an efficient dominating set by Lemma 3. By Proposition 8, $\mathrm{sd}_\gamma(P_2\square C_n)=1$. □

Lemma 16. *For $n\geq 3$ and $n\equiv 1\,(\mathrm{mod}\,4)$,*

$$\mathrm{sd}_\gamma(P_2\square C_n)=1. \quad (25)$$

Proof. Let $n\geq 3$ and $n\equiv 1\,(\mathrm{mod}\,4)$. Let $e=v_{12}v_{22}$ and let D be a minimum dominating set of $(P_2\square C_n)_e$ where the inserted vertex in e is u. Suppose $|(P_2)_2\cap D|\geq 2$. Then $D\backslash(P_2)_2$ can dominate the vertices in $P_2\square C_n-(P_2)_1\cup(P_2)_2\cup(P_2)_3$. Note that $P_2\square C_n-(P_2)_1\cup(P_2)_2\cup(P_2)_3\cong P_2\square P_{n-3}$. By Theorem 13,

$$\left|D\backslash(P_2)_2\right|\geq\gamma(P_2\square P_{n-3})=\left\lceil\frac{n-3+1}{2}\right\rceil = \frac{n-1}{2}. \quad (26)$$

Hence,

$$|D|=\left|D\backslash(P_2)_2\right|+\left|(P_2)_2\cap D\right|\geq\frac{n-1}{2}+2 = \frac{n+3}{2}>\frac{n+1}{2}. \quad (27)$$

In the following, assume $|(P_2)_2\cap D|=1$.

Suppose $(P_2)_2\cap D=\{u\}$. Then, $D\backslash(P_2)_2$ can dominate the vertices in $P_2\square C_n-(P_2)_2$. Note that $P_2\square C_n-(P_2)_2\cong P_2\square P_{n-1}$. By Theorem 14,

$$\left|D\backslash(P_2)_2\right|\geq\gamma(P_2\square P_{n-1})=\left\lceil\frac{n-1+1}{2}\right\rceil = \frac{n+1}{2}. \quad (28)$$

Hence,

$$\left|D\right|=\left|D\backslash(P_2)_2\right|+\left|(P_2)_2\cap D\right|\geq\frac{n+1}{2}+1 = \frac{n+3}{2}>\frac{n+1}{2}. \quad (29)$$

Suppose finally $|(P_2)_2\cap D|=1$, $u\notin D$ and suppose without loss of generality $(P_2)_2\cap D=\{v_{12}\}$. Then, v_{21} or v_{23} (say v_{21}) must be in D to dominate v_{22}. Note that any vertex of D from $(C_n)_i$ dominates three vertices of $(C_n)_i$ including itself and any vertex of D from $(C_n)_{2-i}$ dominates one vertex of $(C_n)_i$ in $(P_2\square C_n)_e$. Since v_{21} and v_{12} dominate a common vertex v_{11} in $(P_2\square C_n)_e$,

$$3\left|D\cap(C_n)_1\right|+\left|D\cap(C_n)_2\right|-1\geq\left|(C_n)_1\right|=n. \quad (30)$$

Since v_{12} cannot dominate v_{22} in $(P_2\square C_n)_e$,

$$3\left|D\cap(C_n)_2\right|+\left|D\cap(C_n)_1\right|-1\geq\left|(C_n)_1\right|=n. \quad (31)$$

By summing (30) and (31), we have

$$|D|=\left|D\cap(C_n)_1\right|+\left|D\cap(C_n)_2\right|\geq\left|(C_n)_1\right|\geq\frac{n+1}{2}. \quad (32)$$

If the equality holds in (32), then the equalities hold in (30) and (31) which implies $|D\cap(C_n)_1|=n+1/4$ contradicts $|D\cap(C_n)_1|$ is an integer. Hence

$$|D|>\frac{n+1}{2}. \quad (33)$$

By Theorem 14, $\gamma(P_2\square C_n)=n+1/2$. So $\gamma((P_2\square C_n)_e)>\gamma(P_2\square C_n)$ which implies that $\mathrm{sd}_\gamma(P_2\square C_n)=1$. □

Lemma 17. *For $n\geq 3$ and $n\equiv 3\,(\mathrm{mod}\,4)$,*

$$\mathrm{sd}_\gamma(P_2\square C_n)=2. \quad (34)$$

Proof. Let $n\geq 3$ and $n\equiv 3\,(\mathrm{mod}\,4)$. By Theorem 14, $\gamma(P_2\square C_n)=n+1/2$. Let $e=v_{11}v_{21}$ or $e=v_{11}v_{12}$. Then

$$D=\left\{v_{11}\right\}\cup\left\{v_{2j},v_{1k}\colon\ j\equiv 2\,(\mathrm{mod}\,4),k\equiv 0\,(\mathrm{mod}\,4)\right\}, \quad (35)$$

with cardinality $n+1/2$ is a dominating set of $(P_2\square C_n)_e$. Since there are only two types of edge in $P_2\square C_n$,

$$\gamma((P_2\square C_n)_e)=|D| = \gamma(P_2\square C_n), \quad (36)$$

for any edge e in $P_2\square C_n$. Hence, $\mathrm{sd}_\gamma(P_2\square C_n)>1$.

Let G be the graph that results from subdividing two edges $v_{11}v_{21}$ and $v_{13}v_{23}$, by adding subdivision vertices u and w. Let D be a minimum dominating set of G. Suppose $|(P_2)_1 \cap D| \geq 2$. Then, $D\backslash (P_2)_1$ can dominate the vertices in $P_2 \square C_n - (P_2)_1 \cup (P_2)_2 \cup (P_2)_n$. Note that $P_2 \square C_n - (P_2)_1 \cup (P_2)_2 \cup (P_2)_n \cong P_2 \square P_{n-3}$. By Theorem 13,

$$\begin{aligned} \left|D\backslash (P_2)_1\right| &\geq \gamma(P_2 \square P_{n-3}) = \left\lceil \frac{n-3+1}{2} \right\rceil \\ &= \frac{n-1}{2}. \end{aligned} \tag{37}$$

Hence,

$$\begin{aligned} |D| &= \left|D\backslash (P_2)_1\right| + \left|(P_2)_1 \cap D\right| \geq \frac{n-1}{2} + 2 \\ &= \frac{n+3}{2} > \frac{n+1}{2}. \end{aligned} \tag{38}$$

Suppose $|(P_2)_3 \cap D| \geq 2$. Then, $D\backslash (P_2)_1$ can dominate the vertices in $P_2 \square C_n - (P_2)_2 \cup (P_2)_3 \cup (P_2)_4$. Note that $P_2 \square C_n - (P_2)_2 \cup (P_2)_3 \cup (P_2)_4 \cong P_2 \square P_{n-3}$. By Theorem 13,

$$\begin{aligned} \left|D\backslash (P_2)_3\right| &\geq \gamma(P_2 \square P_{n-3}) = \left\lceil \frac{n-3+1}{2} \right\rceil \\ &= \frac{n-1}{2}. \end{aligned} \tag{39}$$

Hence,

$$\begin{aligned} |D| &= \left|D\backslash (P_2)_3\right| + \left|(P_2)_3 \cap D\right| \geq \frac{n-1}{2} + 2 \\ &= \frac{n+3}{2} > \frac{n+1}{2}. \end{aligned} \tag{40}$$

In the following, assume $|(P_2)_1 \cap D| = |(P_2)_3 \cap D| = 1$.

Suppose $(P_2)_1 \cap D = \{u\}$. Then $D\backslash (P_2)_1$ can dominate the vertices in $P_2 \square C_n - (P_2)_1$. Note that $P_2 \square C_n - (P_2)_1 \cong P_2 \square P_{n-1}$. By Theorem 13,

$$\begin{aligned} \left|D\backslash (P_2)_1\right| &\geq \gamma(P_2 \square P_{n-1}) = \left\lceil \frac{n-1+1}{2} \right\rceil \\ &= \frac{n+1}{2}. \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} |D| &= \left|D\backslash (P_2)_1\right| + \left|(P_2)_1 \cap D\right| \geq \frac{n+1}{2} + 1 \\ &= \frac{n+3}{2} > \frac{n+1}{2}. \end{aligned} \tag{42}$$

Suppose $(P_2)_3 \cap D = \{w\}$. Then, $D\backslash (P_2)_3$ can dominate the vertices in $P_2 \square C_n - (P_2)_3$. Note that $P_2 \square C_n - (P_2)_3 \cong P_2 \square P_{n-1}$. By Theorem 13,

$$\begin{aligned} \left|D\backslash (P_2)_3\right| &\geq \gamma(P_2 \square P_{n-1}) = \left\lceil \frac{n-1+1}{2} \right\rceil \\ &= \frac{n+1}{2}. \end{aligned} \tag{43}$$

Hence,

$$\begin{aligned} |D| &= \left|D\backslash (P_2)_3\right| + \left|(P_2)_3 \cap D\right| \geq \frac{n+1}{2} + 1 \\ &= \frac{n+3}{2} > \frac{n+1}{2}. \end{aligned} \tag{44}$$

Suppose finally $|(P_2)_1 \cap D| = |(P_2)_3 \cap D| = 1$, $u, w \notin D$ and suppose without loss of generality $(P_2)_1 \cap D = \{v_{11}\}$. Then v_{22} or v_{2n} must be in D to dominate v_{21}. Assume v_{13} in D to dominate the vertex w and v_{23} not in D. Since v_{22} should be dominated by D, $v_{22} \in D$ or $v_{12} \in D$ (say $v_{22} \in D$ since it is more efficient). Note that any vertex of D from $(C_n)_i$ dominates three vertices of $(C_n)_i$ including itself and any vertex of D from $(C_n)_{2-i}$ dominates at most one vertex of $(C_n)_i$ in G. Since v_{11}, v_{22}, and v_{13} dominate a common vertex v_{12} in G,

$$3\left|D \cap (C_n)_1\right| + \left|D \cap (C_n)_2\right| - 2 \geq \left|(C_n)_1\right| = n. \tag{45}$$

Since v_{11} and v_{13} cannot dominate vertices in $(C_n)_2$ corresponding to G,

$$3\left|D \cap (C_n)_2\right| + \left|D \cap (C_n)_1\right| - 2 \geq \left|(C_n)_1\right| = n. \tag{46}$$

By summing (45) and (46), we have

$$|D| = \left|D \cap (C_n)_1\right| + \left|D \cap (C_n)_2\right| \geq \frac{n+2}{2} > \frac{n+1}{2}. \tag{47}$$

Assume v_{23} in D dominates the vertex w and v_{13} not in D. If $v_{22} \in D$, then $(D\backslash\{v_{22}\}) \cup \{v_{21}\}$ is also a minimum dominating set of G, and this case has been solved in the second paragraph. If $v_{12} \in D$, then $(D\backslash\{v_{12}\}) \cup \{v_{13}\}$ is also a minimum dominating set of G, and this case also has been solved in the second paragraph. The remaining case is $v_{12} \notin D$ and $v_{22} \notin D$. Since v_{21} and v_{13} should be dominated by D in G, v_{2n} and v_{14} must be in D. Note that any vertex of D from $(C_n)_i$ dominates three vertices of $(C_n)_i$ including itself and any vertex of D from $(C_n)_{2-i}$ dominates at most one vertex of $(C_n)_i$ in G. Since v_{11} and v_{2n} dominate a common vertex v_{1n} in G, and v_{23} cannot dominate v_{13}, we have

$$3\left|D \cap (C_n)_1\right| + \left|D \cap (C_n)_2\right| - 2 \geq \left|(C_n)_1\right| = n. \tag{48}$$

Since v_{11} cannot dominate vertices in $(C_n)_2$ corresponding to G and v_{14} dominates v_{24} which is also dominated by v_{23} in G,

$$3\left|D \cap (C_n)_2\right| + \left|D \cap (C_n)_1\right| - 2 \geq \left|(C_n)_1\right| = n. \tag{49}$$

By summing (48) and (49), we have

$$|D| = \left|D \cap (C_n)_1\right| + \left|D \cap (C_n)_2\right| \geq \frac{n+2}{2} > \frac{n+1}{2}. \tag{50}$$

By Theorem 14, $\gamma(P_2 \square C_n) = n + 1/2$. By the above discussions, $\gamma(G) = |D| > n + 1/2 = \gamma(P_2 \square C_n)$ which implies that $\mathrm{sd}_\gamma(P_2 \square C_n) = 1$. □

Lemma 18. *For $n \geq 3$ and $n \equiv 2 \pmod 4$,*

$$\mathrm{sd}_\gamma(P_2 \square C_n) = 3. \tag{51}$$

Proof. By Theorem 14, $\gamma(P_2\square C_n) = n + 2/2$. We first show $\mathrm{sd}_\gamma(P_2\square C_n) \geq 3$. Assume $e_1 = v_{11}v_{21}$ and $e_2 = v_{1j}v_{2j}$ where $2 \leq j \leq n/2$. Let G be the graph that results from subdividing the two edges e_1 and e_2. If $j \equiv 2, 3 \pmod 4$, then

$$D = \{v_{11}\} \cup \{v_{2r}, v_{1(r+2)}: r \equiv 3 \pmod 4, 3 \leq r < j\} \cup \{v_{2j}\} \cup \{v_{1s}, v_{2(s+2)}: s \equiv 0 \pmod 4, j < s < n\}, \tag{52}$$

with cardinality $n + 2/2$ is a dominating set of G. If $j \equiv 0, 1 \pmod 4$, then

$$D = \{v_{1r}, v_{2(r+2)}: r \equiv 1 \pmod 4, 1 \leq r < j\} \cup \{v_{1j}\} \cup \{v_{2s}, v_{1(s+2)}: s \equiv 2 \pmod 4, j < s < n\} \cup \{v_{2n}\}, \tag{53}$$

with cardinality $n + 2/2$ is a dominating set of G. By the above discussion, we can easily check that the above constructed set D of G is also a dominating set of G when $e_2 = v_{1j}v_{1(j+1)}$ and $j \equiv 1, 3 \pmod 4$. If $e_2 = v_{1j}v_{1(j+1)}$ and w be the subdivision vertex where $j \equiv 2 \pmod 4$, then

$$D = \{v_{21}\} \cup \{v_{1r}, v_{2(r+2)}: r \equiv 3 \pmod 4, 3 \leq r < j\} \cup \{w\} \cup \{v_{2s}, v_{1(s+2)}: s \equiv 0 \pmod 4, j < s < n\}, \tag{54}$$

with cardinality $n + 2/2$ is a dominating set of G. If $e_2 = v_{1j}v_{1(j+1)}$ and w be the subdivision vertex where $j \equiv 0 \pmod 4$, then

$$D = \{v_{1r}, v_{2(r+2)}: r \equiv 1 \pmod 4, 1 \leq r < j\} \cup \{w\} \cup \{v_{1s}, v_{2(s+2)}: s \equiv 0 \pmod 4, j + 2 < s < n\} \cup \{v_{2(j+2)}\}, \tag{55}$$

with cardinality $n + 2/2$ is a dominating set of G.

Assume $e_1 = v_{11}v_{12}$ and $e_2 = v_{kj}v_{k(j+1)}$ where $2 \leq j \leq n/2$ and $k \in [2]$. Let G be the graph that results from subdividing the two edges e_1 and e_2. If $j \equiv 0 \pmod 4$, then

$$D = \{v_{11}\} \cup \{v_{2r}, v_{1(r+2)}: r \equiv 2 \pmod 4, 2 \leq r < j\} \cup \{v_{2(j+1)}\} \cup \{v_{1s}, v_{2(s+2)}: s \equiv 3 \pmod 4, j < s < n\}, \tag{56}$$

with cardinality $n + 2/2$ is a dominating set of G. If $j \equiv 2 \pmod 4$, then

$$D = \{v_{11}\} \cup \{v_{2r}, v_{1(r+2)}: r \equiv 2 \pmod 4, 2 \leq r < j\} \cup \{v_{2j}\} \cup \{v_{1s}, v_{2(s+2)}: s \equiv 3 \pmod 4, j < s < n\}, \tag{57}$$

with cardinality $n + 2/2$ is a dominating set of G. If $j \equiv 1 \pmod 4$ and w be the subdivision vertex for e_1, then

$$D = \{w\} \cup \{v_{2r}, v_{1(r+2)}: r \equiv 3 \pmod 4, 3 \leq r < j\} \cup \{v_{2(j+1)}\} \cup \{v_{1s}, v_{2(s+2)}: s \equiv 0 \pmod 4, j < s < n\}, \tag{58}$$

with cardinality $n + 2/2$ is a dominating set of G. If $j \equiv 3 \pmod 4$ and w be the subdivision vertex for e_1, then

$$D = \{w\} \cup \{v_{2r}, v_{1(r+2)}: r \equiv 3 \pmod 4, 3 \leq r < j\} \cup \{v_{2j}\} \cup \{v_{1s}, v_{2(s+2)}: s \equiv 0 \pmod 4, j < s < n\}, \tag{59}$$

with cardinality $n + 2/2$ is a dominating set of G. By symmetry and the above discussions, we see that $\gamma(G) = \gamma(P_2\square C_n)$ where G is the graph that results from subdividing any two edges of $P_2\square C_n$. Hence, $\mathrm{sd}_\gamma(P_2\square C_n) > 2$.

Let G' be the graph that results from subdividing three edges $v_{11}v_{12}$, $v_{22}v_{23}$, and $v_{13}v_{14}$, by adding subdivision vertices u, v, and w. Let D be a minimum dominating set of G'. In the following we prove $|D| > n + 2/2 = \gamma(P_2\square C_n)$ by Theorem 14. We can easily check $|D| > n + 2/2$ for $n = 6$. Assume $n \geq 10$ below. Suppose $|((P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4) \cap D| \geq 4$. Then, $D \backslash ((P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4)$ can dominate $G' - (P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4 \cup (P_2)_5 \cup (P_2)_n$ which is isomorphic to $P_2\square P_{n-6}$. Hence

$$\begin{aligned} |D| &= |D \backslash ((P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4)| + |((P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4) \cap D| \\ &\geq \gamma(P_2\square P_{n-6}) + 4 \\ &\geq \left\lceil \frac{n-6+1}{2} \right\rceil + 4 \\ &> \frac{n+2}{2}, \end{aligned} \tag{60}$$

the last but one inequality comes from Theorem 13. In the following, assume $|((P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4) \cap D| = 3$ (cannot be smaller than 3 since u, v, w should be dominated by 3 distinct vertices in D).

Assume $|\{u, v, w\} \cap D| \geq 2$ and we say $u, v \in D$ for example. For dominating the vertex w, either v_{13} or w in D. Note that any vertex of D from $(C_n)_i$ or the corresponding subdivision vertices dominates three vertices of $(C_n)_i$ including itself and any vertex of D from $(C_n)_{2-i}$ dominate at most one vertex of $(C_n)_i$ in G'. Since v cannot dominate vertices in $(C_n)_2$,

$$3\left|D \cap \left((C_n)_1 \cup \{u, w\}\right)\right| + \left|D \cap \left((C_n)_2 \cup \{v\}\right)\right| - 1 \geq \left|(C_n)_1 \cup \{u, w\}\right| = n + 2. \tag{61}$$

Since v_{13} and v have a common neighbor v_{23}, u, w have no neighbors in $(C_n)_2$.

$$3\left|D \cap \left((C_n)_2 \cup \{v\}\right)\right| + |D \cap \cup \{u, w\})| - 2 \geq \left|(C_n)_2 \cup \{v\}\right| = n + 1. \tag{62}$$

By summing (61) and (62), we have

$$|D| = \left|D \cap \left((C_n)_1 \cup \{u, w\}\right)\right| + \left|D \cap \left((C_n)_2 \cup \{v\}\right)\right| \geq \frac{n+3}{2} > \frac{n+2}{2}. \tag{63}$$

Assume $|\{u, v, w\} \cap D| = 1$. Suppose $u \in D$ and $v, w \notin D$. Note that v_{22} and v should be dominated by D, and also v_{13} and w should be dominated by D. Since $|((P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_4) \cap D| = 3$, v_{22} and v_{13} must be in D. Then $D \backslash ((P_2)_1 \cup (P_2)_2 \cup (P_2)_3)$ can dominate $G' - (P_2)_1 \cup (P_2)_2 \cup (P_2)_3 \cup (P_2)_n$ which is isomorphic to $P_2 \square P_{n-4}$. Hence

$$\begin{aligned} |D| &= \left|D \backslash \left((P_2)_1 \cup (P_2)_2 \cup (P_2)_3\right)\right| + \left|\left((P_2)_1 \cup (P_2)_2 \cup (P_2)_3\right) \cap D\right| \\ &\geq \gamma\left(P_2 \square P_{n-4}\right) + 3 \\ &\geq \left\lceil \frac{n-4+1}{2} \right\rceil + 3 \\ &> \frac{n+2}{2}, \end{aligned} \tag{64}$$

the last but one inequality comes from Theorem 13. Similar discussions for $v \in D$ and $u, w \notin D$.

By the above discussions, $\gamma(G') = |D| > n + 2/2 = \gamma(P_2 \square C_n)$ by Theorem 14, which implies that $\mathrm{sd}_\gamma(P_2 \square C_n) \leq 3$. We complete the proof of this Lemma.

Combing Lemmas 15, 16, 17, and 18, we show the exact value for domination subdivision number of $P_2 \square C_n$. □

Theorem 19. *For $n \geq 3$,*

$$\mathrm{sd}_\gamma\left(P_2 \square C_n\right) = \begin{cases} 1, & n \equiv 0,\ 1 \pmod 4, \\ 2, & n \equiv 3 \pmod 4, \\ 3, & n \equiv 2 \pmod 4. \end{cases} \tag{65}$$

6. Conclusions

In this paper, we prove the decision problem for determining whether $\mathrm{sd}_\gamma(G) = 1$ is NP-hard. This result is fundamental for a mathematical parameter. This result shows it is meaningful to study the bounds and other properties of the domination subdivision number of graphs. Since it is difficult to determine whether $\gamma(G_e) > \gamma(G)$ for any edge e in graph G, it is obvious that the domination subdivision problem is not a NP problem. So we only prove the decision problem for determining whether $\mathrm{sd}_\gamma(G) = 1$ is NP-hard but not NP-complete. Moreover, we show a better tight bound $\mathrm{sd}_\gamma(G) \leq d_G(u) + d_G(v) - |N_G(u) \cap N_G(v)| - 1$ for domination subdivision number of connected graphs.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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