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Research Article

A Convergent Legendre Spectral Collocation Method for the Variable-Order Fractional-Functional Optimal Control Problems

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In this paper, a numerical method is applied to approximate the solution of variable-order fractional-functional optimal control problems. The variable-order fractional derivative is described in the type III Caputo sense. The technique of approximating the optimal solution of the problem using Lagrange interpolating polynomials is employed by utilizing the shifted Legendre–Gauss–Lobatto collocation points. To obtain the coefficients of these interpolating polynomials, the problem is transformed into a nonlinear programming problem. The proposed method offers a significant advantage in that it does not require the approximation of singular integral. In addition, the matrix differentiation is calculated accurately and efficiently, overcoming the difficulties posed by variable-order fractional derivatives. The convergence of the proposed method is investigated, and to validate the effectiveness of our proposed method, some examples are presented. We achieved an excellent agreement between numerical and exact solutions for different variable orders, indicating our method's good performance.

1. Introduction

Nowadays, fractional calculus, a branch of mathematics that studies the properties of derivatives and integrals of non-integer order, is essential due to its increasing applications in the sciences and engineering. Particularly, mathematical modeling of phenomena based on fractional calculus has proven to exhibit more realistic behavior [1–4].

A mild solution for a control problem governed by fractional stochastic evolution inclusion using the Caputo derivative with nonlocal conditions was proved by Abuasbeh et al. [5] via the fixed-point theorem of convex multiple-valued maps.

In [6], Khan et al. address the issue of resilient base containment control for fractional-order multiagent systems (FOMASs) that have mixed time delays. Niazi and his colleagues [7] prove the existence of a solution for an initial value problem involving a hybrid fractional differential equation with delay.

In recent years, solving optimal control problems governed by a fractional dynamical system has become one of the most popular topics in control theory, and it has encouraged many researchers to come up with an efficient numerical approach to solving them. A new method for finding the approximate solution of fractional optimal control problems with the Caputo–Fabrizio (CF) fractional integro-differential equation is presented in [8]. This method is based on the Gegenbauer polynomials definition and utilizes the modified operational matrix of the CF-fractional derivative. In [9], Heidari and Razzaghi attempted to solve two classes of fractional optimal control problems (OCPs) with delay. Their approach involves using the Legendre–Gauss collocation method and extended Chebyshev cardinal wavelets to solve the Hamilton–Jacobi–Bellman equation numerically. Ghanbari and Razzaghi [10] have proposed a new numerical method to solve fractional optimal control problems (FOCPs). Their method is based on generalized fractional-order Chebyshev wavelets (GFOCW) and the incomplete beta function. A Fibonacci wavelet operational matrix and the Galerkin method were used by Sabermahani and Ordokhani [11] to solve fractional optimal control problems with equal and unequal constraints. In [12], authors use modified hat functions as basic functions to approximate control and state variables. The properties of these basis functions, the Caputo derivative and the Riemann–Liouville integral simplify the FOCP to nonlinear algebraic equations. For more study, see also the related works [13–15].

Recently, the variable-order fractional optimal control problems have attracted the attention of many researchers. This is because variable-order fractional calculus offers greater flexibility in selecting the most appropriate order for accurately describing real-world problems. Heydari et al. [16] used a method based on the Chebyshev cardinal functions and Lagrange multipliers to reduce the problem to a system of algebraic equations. An efficient method was presented by Heydari and Avazzadeh [17] through the Legendre wavelets and their operational matrix of variable-order fractional integration in the Riemann–Liouville sense. Hassani et al. [18] examined two-dimensional variable-order fractional optimal control problems. Using the transcendental Bernstein series and extending problem variables based on these series, they transformed the optimal control problem into a simple optimization problem. Bhrawy and Zaky in [19] proposed a method using shifted Chebyshev polynomials and an operational matrix to approximate the solution of a variable-order fractional functional Dirichlet boundary value problem.

The novelty of our work lies in the fact that there is no study on the numerical solution of variable-order fractional-functional optimal control (VOFFOC) problems, and we look at this issue for the first time. The purpose of this paper is to present a numerical scheme that will be able to approximate the VOFFOC problem efficiently. Our scheme is based on the Legendre spectral collocation methods and reduces the problem into a nonlinear programming (NLP) problem.

The structure of the paper is as follows. In Section 2, some necessary preliminaries are given. In Section 3, a new direct method is described to solve the VOFFOC problems. Section 4 discusses the convergence of the proposed method. In Section 5, some numerical examples are given to show the method's efficiency. Finally, the conclusions and suggestions are presented in Section 6.

2. Some Preliminaries

In this section, we present some basic definitions and mathematical preliminaries related to the fixed-order and variable-order fractional integrals and derivatives [20].

Definition 1. Consider the function $\chi(.)$ defined on the finite interval $[0,T]$. For fixed-order $\varepsilon>0$, the left and right RL fractional integrals are ${}_0I_t^{\varepsilon}\chi(t)$ and ${}_tI_T^{\varepsilon}\chi(t)$ and are defined by

$$\begin{aligned}{}_0I_t^{\varepsilon}\chi(t)&=\frac{1}{\Gamma(\varepsilon)}\int_0^t (t-\tau)^{\varepsilon-1}\chi(\tau)d\tau,\quad 0<t\leq T,\\ {}_tI_T^{\varepsilon}\chi(t)&=\frac{1}{\Gamma(\varepsilon)}\int_t^T (\tau-t)^{\varepsilon-1}\chi(\tau)d\tau,\quad 0\leq t<T.\end{aligned}\tag{1}$$

Definition 2. Let $\chi(.)$ be a function on the interval $[0,T]$. The left and right *RL* fractional derivatives of fixed-order $0<\varepsilon<1$ are denoted by ${}_0D_t^{\varepsilon}\chi(t)$ and ${}_tD_T^{\varepsilon}\chi(t)$, respectively, and defined by

$$\begin{aligned}{}_0D_t^{\varepsilon}\chi(t)&=\frac{1}{\Gamma(1-\varepsilon)}\frac{d}{dt}\int_0^t (t-\tau)^{-\varepsilon}\chi(\tau)d\tau,\quad t>0,\\ {}_tD_T^{\varepsilon}\chi(t)&=\frac{(-1)}{\Gamma(1-\varepsilon)}\frac{d}{dt}\int_t^T (\tau-t)^{-\varepsilon}\chi(\tau)d\tau,\quad t<T.\end{aligned}\tag{2}$$

Definition 3. Suppose that function $\chi(.)$ is defined on $[0,T]$. The left and right Caputo fractional derivatives of $\chi(.)$ of fixed-order $0<\varepsilon<1$ are denoted by ${}_0^CD_t^{\varepsilon}\chi(t)$ and ${}_t^CD_T^{\varepsilon}\chi(t)$, respectively, and defined by

$$\begin{aligned}{}_0^CD_t^{\varepsilon}\chi(t)&=\frac{1}{\Gamma(1-\varepsilon)}\int_0^t (t-\tau)^{-\varepsilon}\chi'(\tau)\,d\tau,\quad 0\leq t<T,\\ {}_t^CD_T^{\varepsilon}\chi(t)&=\frac{(-1)}{\Gamma(1-\varepsilon)}\int_t^T (\tau-t)^{-\varepsilon}\chi'(\tau)\,d\tau,\quad 0<t\leq T.\end{aligned}\tag{3}$$

We now present the basic concepts of variable-order fractional calculus and consider the fractional order in the derivative and integral to be a continuous function on $(0,T)$. First, we introduce the generalization of a fixed-order fractional integral called the variable-order Riemann–Liouville integral.

Definition 4. Assuming that the continuously differentiable function χ is defined on $(0,T)$. The left and right Riemann–Liouville fractional integrals of order $\varepsilon(.)$ are defined as follows:

$$\begin{aligned}{}_0^{RL}I_t^{\varepsilon(t)}\chi(t)&=\int_0^t\frac{1}{\Gamma(\varepsilon(t))}(t-\tau)^{\varepsilon(t)-1}\chi(\tau)d\tau,\quad t>0,\\ {}_t^{RL}I_T^{\varepsilon(t)}\chi(t)&=\int_t^T\frac{1}{\Gamma(\varepsilon(t))}(\tau-t)^{\varepsilon(t)-1}\chi(\tau)d\tau,\quad t<T.\end{aligned}\tag{4}$$

Definition 5. Assume that $\chi\colon [0,T]\longrightarrow\mathbb{R}$ is a continuously differentiable function and $\varepsilon\colon [0,T]\longrightarrow[0,1]$ is a given function.

(1) The type I left and right Caputo variable-order fractional derivatives (VOFDs) of $\chi(t)$ of order $\varepsilon(.)$, respectively, are defined by

$$
\begin{aligned}
{}_{0}^{C}D_t^{\varepsilon(t)}\chi(t) &= \frac{1}{\Gamma(1-\varepsilon(t))}\frac{d}{dt}\int_0^t (t-\tau)^{-\varepsilon(t)}[\chi(\tau)-\chi(0)]d\tau,\\
{}_{T}^{C}D_t^{\varepsilon(t)}\chi(t) &= \frac{-1}{\Gamma(1-\varepsilon(t))}\frac{d}{dt}\int_t^T (t-\tau)^{-\varepsilon(t)}[\chi(\tau)-\chi(T)]d\tau.
\end{aligned}
\tag{5}
$$

(2) The type II left and right Caputo VOFDs of $\chi(t)$ of order $\varepsilon(.)$, respectively, are given by

$$
\begin{aligned}
{}_{T}^{C}\mathscr{D}_t^{\varepsilon(t)}\chi(t) &= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\varepsilon(t))}\int_0^t (t-\tau)^{-\varepsilon(t)}[\chi(\tau)-\chi(0)]dt\right),\\
{}_{T}^{C}\mathscr{D}_t^{\varepsilon(t)}\chi(t) &= \frac{d}{dt}\left(\frac{-1}{\Gamma(1-\varepsilon(t))}\int_t^T (t-\tau)^{-\varepsilon(t)}[\chi(\tau)-\chi(T)]d\tau\right).
\end{aligned}
\tag{6}
$$

(3) The type III left and right Caputo derivatives of $\chi(t)$ of order $\varepsilon(t)$, respectively, are defined by

$$
\begin{aligned}
{}_{0}^{C}\mathbb{D}_t^{\varepsilon(t)}\chi(t) &= \frac{1}{\Gamma(1-\varepsilon(t))}\int_0^t (t-\tau)^{-\varepsilon(t)}\chi'(\tau)d\tau,\\
{}_{t}^{C}\mathbb{D}_T^{\varepsilon(t)}\chi(t) &= \frac{-1}{\Gamma(1-\varepsilon(t))}\int_t^T (t-\tau)^{-\varepsilon(t)}\chi'(\tau)d\tau.
\end{aligned}
\tag{7}
$$

Lemma 6 (see [20]). *Let $\chi(t) = (t-a)^{\beta}$ for $t \in [a,b]$ where $\beta > 0$. Then,*

$$
\begin{aligned}
{}_{a}I_t^{\varepsilon(t)}\chi(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\varepsilon(t)+1)}(t-a)^{\beta+\varepsilon(t)},\\
{}_{a}^{C}\mathbb{D}_t^{\varepsilon(t)}\chi(t) &= \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\varepsilon(t)+1)}(t-a)^{\beta-\varepsilon(t)}, & \beta \geq 1,\\ 0, & \beta < 1. \end{cases}
\end{aligned}
\tag{8}
$$

In this paper, we will focus on the type III Caputo VOFDs.

3. Implementation of the Method for VOFFOC Problem

In this paper, we consider the following VOFFOC problem

$$
\text{Minimize } L(Y(.),V(.)) = \int_0^T P(t,Y(t),V(t))dt, \tag{9}
$$

$$
\text{subject to } \begin{cases} {}_{0}^{C}\mathbb{D}_t^{\varepsilon(t)}Y(t) = G(t,Y(t),Y(\phi(t)),V(t)), & 0 < t \leq T,\\ Y(0) = Y_0, \end{cases} \tag{10}
$$

where $Y_0 \in \mathbb{R}$ is given, P: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}$ and G: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}$ and ϕ: $[0,T] \longrightarrow [0,T]$ are continuously differentiable functions, $Y(t)$ and $V(t)$ are the state and control variables, respectively, ε: $[0,T] \longrightarrow [0,1]$ is a continuous function, and ${}_{0}^{C}\mathbb{D}_t^{\varepsilon(t)}$ is the type III Caputo VOFD operator. We assume that there exists a smooth optimal solution $(Y^*(.),V^*(.))$ for the above VOFFOC problem. We are going to propose a convergent and efficient method to approximate the optimal solution. In implementing the method, we use the following Lagrange interpolating polynomials

$$h_j(t) = \prod_{\substack{i=0\\ i\neq j}}^{N} \frac{t-t_i}{t_j-t_i}, \quad j = 0, 1, \ldots, N, \tag{11}$$

where t_i for $i = 0, 1\ldots, N$ are the shifted Legendre–Gauss–Lobatto (SLGL) points. These points can be given by $t_i = T/2(\tau_i + 1)$, where τ_i are the roots of polynomials

$$q_{N+1}(\tau) = \left(1-\tau^2\right)J_N'(\tau), \ \tau \in [-1, 1], \tag{12}$$

where $J_N(t)$ is the Legendre polynomial of degree N defined by the following recurrence formula on interval $[-1, 1]$

$$\begin{cases} J_{j+1}(\tau) = \dfrac{2j+1}{j+1}(\tau)J_j(\tau) - \dfrac{j}{j+1}J_{j-1}(\tau), & j = 1, 2, \ldots \\ J_0(t) = 1, J_1(\tau) = \tau. \end{cases} \tag{13}$$

The Lagrange polynomials have the useful delta Kronecker property, i.e.,

$$h_j(t_k) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{14}$$

Now, we approximate the variables of the problem (9) and (10) in terms of the Lagrange interpolating polynomials as follows:

$$Y(t) \approx Y^N(t) = \sum_{j=0}^{N} \bar{y}_j h_j(t), V(t) \approx V^N(t) = \sum_{j=0}^{N} \bar{v}_j h_j(t), \tag{15}$$

where $\bar{y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N)$ and $\bar{v} = (\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_N)$ are unknown coefficients. With the interpolation property of the Lagrange interpolating polynomials, we obtain

$$Y(t_k) \approx \bar{y}_k, V(t_k) \approx \bar{v}_k, \quad k = 0, 1, \ldots, N. \tag{16}$$

Also, to approximate the VOFD of state variable $Y(.)$, we gain

$${}_0^C\mathbb{D}_t^{\varepsilon(t)}Y(t) \approx {}_0^C\mathbb{D}_t^{\varepsilon(t)}Y_N(t) = \sum_{j=0}^{N} \bar{y}_k \, {}_0^C\mathbb{D}^{\varepsilon(t)} h_j(t). \tag{17}$$

So, at the collocation points, we have

$${}_0^C\mathbb{D}_t^{\varepsilon(t)}Y(t)|_{t=t_k} \approx \sum_{j=0}^{N} \bar{y}_k F_{k,j+1}^{\varepsilon}, \quad k = 1, 2, .., N, \tag{18}$$

where for $k = 1, 2, \ldots, N$ and $j = 0, 1, \ldots, N$

$$F_{k,j+1}^{\varepsilon} = {}_0^C D_t^{\varepsilon(t)} h_j(t)|_{t=t_k} = \frac{1}{\Gamma(1-\varepsilon(t_k))}\int_0^{t_k} (t_k - z)^{-\varepsilon(t_k)} h_j'(z)dz. \tag{19}$$

Here, we define the type III Caputo VOF differentiation matrix as $F^{\varepsilon} = (F_{k,j+1}^{\varepsilon})_{N\times(N+1)}$. Note that the elements of matrix differentiation F^{ε} can not be directly and exactly calculated by relation (11). Some Jacobi–Gauss quadrature formulas can be used for approximating the singular integral in (19) and for obtaining the matrix differentiation. But, here, we are going to present an exact and efficient method to gain this matrix. In the presented method, we use Lemma 6 and the properties of interpolating polynomials, and we do not need to approximately calculate a singular integral.

First, we consider the following expansion for the Lagrange polynomials

$$h_j(t) = \sum_{p=0}^{N} \eta_{pj} t^p, \quad 0 \le t \le T, j = 0, 1, \ldots, N, \tag{20}$$

where η_{pj} are coefficients that we can calculate as follows. By evaluating (14) and (20), we obtain

$$\sum_{p=0}^{N} \eta_{pj} t_k^p = \delta_{jk}, \quad j = 0, 1, \ldots, N, k = 0, 1, \ldots, N. \tag{21}$$

Relation (21) can be represented as the following matrix form

$$\Lambda \eta_j = \delta_j, \ j = 0, 1, \ldots, N, \tag{22}$$

where

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & t_1 & t_1^2 & \ldots & t_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & t_N^2 & \ldots & t_N^N \end{bmatrix}, \eta_j = \begin{bmatrix} \eta_{0j} \\ \eta_{1j} \\ \vdots \\ \eta_{Nj} \end{bmatrix}, \delta_j = \begin{bmatrix} \delta_{0j} \\ \delta_{1j} \\ \vdots \\ \delta_{Nj} \end{bmatrix}. \tag{23}$$

Since Λ is an invertible matrix, we gain $\eta_j = (\Lambda^{-1})_{(j+1)}$ for $j = 0, 1, \ldots, N$ where $(\Lambda^{-1})_{j+1}$ is the $(j+1)$-th column of matrix Λ^{-1}. So,

$$\eta_{pj} = \left(\Lambda^{-1}\right)_{(p+1)(j+1)}, \quad j, p = 0, 1, \ldots, N. \tag{24}$$

Hence, by evaluating (20) and (24), we obtain

$$h_j(t) = \sum_{p=0}^{N} \left(\Lambda^{-1}\right)_{(p+1)(j+1)} t^p, \quad j = 0, 1, \ldots, N. \tag{25}$$

Now, using Lemma 6 and relation (25), the components of type III Caputo VOF differentiation matrix $F^{\varepsilon} = (F^{\varepsilon(t)}_{kj})_{N\times(N+1)}$ can be shown as

$$F^{\varepsilon}_{k,j+1} = {}^{C}_{0}D^{\varepsilon(t)}_{t}\, h_j(t)\,|_{t=t_k} = \sum_{p=1}^{N} \left(\Lambda^{-1}\right)_{(p+1)(j+1)} \frac{\Gamma(p+1)}{\Gamma(p+1-\varepsilon(t_k))}\, t_k^{p-\varepsilon(t_k)}, \tag{26}$$

for $k = 1, 2, \ldots, N$ and $j = 0, 1, \ldots, N$.

We will use the following theorem to approximate the objective functional (9).

Lemma 7. *Consider a polynomial $p(.)$ of degree at most $(2N-1)$, on the interval $[0,T]$. We have*

$$\int_0^T p(t)dt = \sum_{k=0}^{N} \omega_k\, p(t_k), \tag{27}$$

where $\{t_k\}_{k=0}^{N}$ are the SLGL points on $[0,T]$ and $\omega_k = \mathrm{T}/N(N+1)(J_N(t_k))^2$ for $k = 0, 1, \ldots, N$.

Proof. It can be gained by employing the transformation $x = 2t/T - 1$ in Theorem 3.29 of [21].

As a result of Lemma 7, for any continuous function $q(.)$ on $[0,\mathrm{T}]$, we obtain

$$\int_0^T q(t)dt = \lim_{N\longrightarrow\infty} \sum_{k=0}^{N} \omega_k\, q(t_k). \tag{28}$$

Here, $(t_k, \omega_k)_{k=0}^{N}$ is the sequence of SLGL nodes and weights.

Now, by relations (16), (18), and (28), we discretize the problem (9) and (10), as follows:

$$\text{Minimize}\quad L_N(\bar{y},\bar{v}) = \sum_{k=0}^{N} P\left(t_k, \bar{y}_k, \bar{v}_k\right)\omega_k, \tag{29}$$

$$\text{subject to}\begin{cases} \displaystyle\sum_{j=0}^{N} \bar{y}_k F^{\varepsilon}_{k,j+1} - G\left(t_k, \bar{y}_k, \sum_{j=0}^{N} \bar{y}_j h_j(\phi(t_k)), \bar{v}_k\right) = 0, \quad k = 1, 2, \ldots, N, \\ \bar{y}_0 - Y_0 = 0. \end{cases} \tag{30}$$

By obtaining the decision variables $\bar{y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N)$ and $\bar{v} = (\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_N)$ of solving the above NLP problem, we can find an approximate solution for the main system. □

4. Convergence Analysis of the Method

In this section, we analyze the convergence of the method. We first rewrite the problem (29) and (30) as the following equivalent form:

$$\text{Minimize}\quad L_N(\bar{y},\bar{v}) = \sum_{k=0}^{N} P\left(t_k, \bar{y}_k, \bar{v}_k\right)\omega_k, \tag{31}$$

$$\text{subject to}\begin{cases} {}^{C}_{0}\mathbb{D}^{\varepsilon(t_k)}_{t_k} Y_N(t_k) - G(t_k, Y_N(t_k), Y_N(\phi(t_k)), V_N(t_k)) = 0, \quad k = 1, \ldots, N, \\ Y_N(0) = Y_0, \end{cases} \tag{32}$$

where $Y_N(t)$ and $V_N(t)$ satisfy (15). We assume the problem (31) and (32) (or equivalently (29) and (30)) is feasible.

Assumption 8. We assume that the optimal solution of VOFFOC problem (1) and (2) has a Lagrange interpolating polynomial (based on the SLGL points), which uniformly converges to it.

Theorem 9. *Assume that $(\bar{y}^*_j, \bar{v}^*_j)_{j=0}^{N}$ is the optimal solution of (22) and (23) and define $\bar{Y}_N(t) = \sum_{j=0}^{N} \bar{y}^*_j h_j(t)$ and $\bar{V}_N(t) = \sum_{j=0}^{N} \bar{v}^*_j h_j(t)$ on $[0,T]$. Also, assume $(\bar{Y}_N(.), \bar{V}_N(.))_{N=N_0}^{\infty}$ uniformly converges to $(\bar{Y}(.), \bar{V}(.))$ such that $\bar{Y}(.)$ and $\bar{V}(.)$ are continuously differentiable and ${}^{C}_{0}\mathbb{D}^{\varepsilon(t)}_{t}\bar{Y}(.)$ is in $C(0,T)$. Then, $(\bar{Y}(.), \bar{V}(.))$ is an optimal solution for the main VOFFOC problem (1) and (2).*

Proof. We first show that $(\bar{Y}(.),\bar{V}(.))$ is a feasible solution for the VOFOC problem (1) and (2). Suppose that $t\in[0,T]$ is given. Since SLGR points $\{t_k\}_{k=0}^{N}$ with $N\longrightarrow\infty$ is dense on $[0,\ T]$, there exists a subsequence $\left\{t_{k_j}\right\}_{j=0}^{\infty}$ such that $\lim_{j\longrightarrow\infty}k_j=\infty$ and $\lim_{j\longrightarrow\infty}t_{k_j}=t$. By continuity of functions $G(.,.,.,.)$ and ${}_0^C\mathbb{D}_t^{\epsilon(t)}\bar{Y}(.)$, and (23), we obtain

$$
{}_0^C\mathbb{D}_t^{\epsilon(t)}\bar{Y}(t)-G(t,\bar{Y}(t),\bar{Y}(\phi(t)),\bar{V}(t))=\lim_{N\longrightarrow\infty}\lim_{j\longrightarrow\infty}\left({}_0^C\mathbb{D}_t^{\epsilon\left(t_{k_j}\right)}\bar{Y}_N\left(t_{k_j}\right)-G\left(t_{k_j},\bar{Y}_N\left(t_{k_j}\right),\bar{Y}_N\left(\phi\left(t_{k_j}\right)\right),\bar{V}_N\left(t_{k_j}\right)\right)\right)=0. \tag{33}
$$

Also, for the initial condition

$$
\bar{Y}(0)-Y_0=\lim_{N\longrightarrow\infty}\left(\bar{Y}_N(0)-Y_0\right)=0. \tag{34}
$$

Now, we want to show that $(\bar{Y}(.),\bar{V}(.))$ is an optimal solution for the VOFOC problem (9) and (10). By objective function (31), we obtain

$$
L_N\left(\bar{y}^*,\bar{\nu}^*\right)=\sum_{k=0}^{N}P\left(t_k,\bar{y}_k^*,\bar{\nu}_k^*\right)\omega_k. \tag{35}
$$

Also, by objective functional (1) and replacing continuous function $q(.)$ in relation (28) with $P(.,\bar{Y}(.),\bar{V}(.))$, we gain

$$
J(\bar{Y}(.),\bar{V}(.))=\int_0^T P(t,\bar{Y}(t),\bar{V}(t))dt=\lim_{N\longrightarrow\infty}\sum_{k=0}^{N}P\left(t_k,\bar{Y}(t_k),\bar{V}(t_k)\right)\omega_k. \tag{36}
$$

Moreover, since $\sum_{k=0}^{N}\omega_k=T$ and $(\bar{Y}_N(.),\bar{V}_N(.))$ is uniformly convergent to $(\bar{Y}(.),\bar{V}(.))$, we obtain

$$
\begin{aligned}
&\lim_{N\longrightarrow\infty}\left\|\sum_{k=0}^{N}\omega_k\left(P\left(t_k,\bar{Y}(t_k),\bar{V}(t_k)\right)-P\left(t_k,\sum_{j=0}^{N}\bar{y}_j^*h_j(t_k),\sum_{j=0}^{N}\bar{\nu}_j^*h_j(t_k)\right)\right)\right\|_\infty\\
&\le L_1\lim_{N\longrightarrow\infty}\sum_{k=0}^{N}\omega_k\left(\left\|\bar{Y}(t_k)-\sum_{j=0}^{N}\bar{y}_j^*h_j(t_k)\right\|_\infty+\left\|\bar{V}(t_k)-\sum_{j=0}^{N}\bar{\nu}_j^*h_j(t_k)\right\|_\infty\right)\\
&=L_1\lim_{N\longrightarrow\infty}\sum_{k=0}^{N}\omega_k\left(\left\|\bar{Y}(t_k)-\bar{Y}_N(t_k)\right\|_\infty+\left\|\bar{V}(t_k)-\bar{V}_N(t_k)\right\|_\infty\right)\\
&\le L_1T\lim_{N\longrightarrow\infty}\left(\left\|\bar{Y}(.)-\bar{Y}_N(.)\right\|_\infty+\left\|\bar{V}(.)-\bar{V}_N(.)\right\|_\infty\right)=0,
\end{aligned} \tag{37}
$$

where $L_1>0$ is the Lipschitz constant of continuously differentiable function $P(.,.,.)$. Thus, by (26)–(28), we gain

$$
\begin{aligned}
L(\bar{Y}(.),\bar{V}(t)) &= \int_0^T P(t,\bar{Y}(t),\bar{V}(t))dt \\
&= \lim_{N\longrightarrow\infty}\left(\sum_{k=0}^{N}\omega_k P\left(t_k,\sum_{j=0}^{N}\bar{y}_j^* h_j(t_k),\sum_{j=0}^{N}\bar{v}_j^* h_j(t_k)\right)\right. \\
&\quad \left.+\sum_{k=0}^{N}\omega_k\left[P\left(t_k,\bar{Y}(t_k),\bar{V}(t_k)\right)-P\left(t_k,\sum_{j=0}^{N}\bar{y}_j^* h_j(t_k),\sum_{j=0}^{N}\bar{v}_j^* h_j(t_k)\right)\right]\right) \\
&= \lim_{N\longrightarrow\infty}\sum_{k=0}^{N}\omega_k P\left(t_k,\sum_{j=0}^{N}\bar{y}_j^* h_j(t_k),\sum_{j=0}^{N}\bar{v}_j^* h_j(t_k)\right) = \lim_{N\longrightarrow\infty} L_N\left(\bar{y}^*,\bar{v}^*\right).
\end{aligned} \tag{38}
$$

Hence,

$$
L(\bar{Y}(.),\bar{V}(.)) = \lim_{N\longrightarrow\infty} L_N\left(\bar{y}^*,\bar{v}^*\right). \tag{39}
$$

On the other hand, by Assumption I, for any optimal solution $(Y^*(.),V^*(.))$ of the problem (9) and (10), there exists a corresponding sequence $(y_j^*,v_j^*)_{i=0}^{\infty}$ such that

$$
\lim_{N\longrightarrow\infty}\left\|Y^*(.)-\sum_{j=0}^{N}y_j^* h_j(.)\right\|_\infty = \lim_{N\longrightarrow\infty}\left\|V^*(.)-\sum_{j=0}^{N}v_j^* h_j(.)\right\|_\infty = 0. \tag{40}
$$

Since $(Y^*(.),V^*(.))$ satisfies constraint (10) sequence $(y_j^*,v_j^*)_{j=0}^{N}$ with $N\longrightarrow\infty$ satisfies constraint (32) (or equivalently (30)). Similar to the relation (39) and the process of achieving it, we have

$$
L\left(Y^*(.),V^*(.)\right) = \lim_{N\longrightarrow\infty} L_N\left(y^*,v^*\right), \tag{41}
$$

where $y^* = (y_0^*,y_1^*,\ldots,y_N^*)$ and $v^* = (v_0^*,v_1^*,\ldots,v_N^*)$. By relations (39) and (41), and optimality of pairs $(\bar{y},\bar{v})$ and $(Y^*(.),V^*(.))$, we achieve

$$
L\left(Y^*(.),V^*(.)\right) \le L(\bar{Y}(.),\bar{V}(t)) = \lim_{N\longrightarrow\infty} L_N(\bar{y},\bar{v}) \le \lim_{N\longrightarrow\infty} L_N\left(y^*,v^*\right) = L\left(Y^*(.),V^*(.)\right), \tag{42}
$$

which tends to $L(Y^*(.),V^*(.)) = L(\bar{Y}(.),\bar{V}(.))$. Thus, $(\bar{Y}(.),\bar{V}(.))$ is an optimal solution for the VOFOC problem (9) and (10). □

5. Numerical Examples

To show the feasibility and validity of the presented scheme, we implement the numerical method mentioned above in three examples and utilize the FMINCON command in MATLAB software and SQP algorithm to solve the corresponding NLP problem (29) and (30). Also, we define the absolute error based on the following relations:

$$
E_Y^N(t) = \left|Y(t)-Y_N(t)\right|, E_V^N(t) = \left|V(t)-V_N(t)\right|, \quad 0\le t\le T, \tag{43}
$$

where (Y,V) and (Y_N,V_N) are the exact and approximate solutions, respectively.

Example 1. Consider the following VOFFOC problem

$$
\begin{aligned}
&\text{Minimize } L = \int_0^1\left(\left(Y(t)-t^2\right)^2+\left(V(t)+t^4\right)^2\right)dt, \\
&\text{subject to } \begin{cases} {}_0^C\mathbb{D}_t^{\varepsilon(t)}Y(t) = Y\left(t^2\right)+V(t)+\dfrac{\Gamma(3)}{\Gamma(3-\varepsilon(t))}t^{2-\varepsilon(t)}, \\ \\ Y(0) = 0, \end{cases}
\end{aligned} \tag{44}
$$

where ϵ: $[0, 1] \longrightarrow [0, 1]$ is an arbitrary continuous function. The functions t^2 and $-t^4$ are the optimal state and optimal control, respectively. The corresponding problem (29) and (30) can be given as

$$\begin{aligned}
&\text{Minimize} \quad L_N(\bar{y}, \bar{v}) = \sum_{k=0}^{N}\left(\left(\bar{y}_k - t_k^2\right)^2 + \left(\bar{v}_k + t_k^4\right)^2\right)\omega_k,\\
&\text{subject to}\begin{cases} \sum_{j=0}^{N} \bar{y}_k F_{k,j+1}^{\varepsilon} - \sum_{j=0}^{N} \bar{y}_j h_j\left({t_k}^2\right) - \bar{v}_k - \dfrac{\Gamma(3)}{\Gamma(3-\varepsilon(t_k))} t_k^{2-\varepsilon(t_k)} = 0, \quad k = 1, 2, \ldots, N,\\[2ex] \bar{y}_0 - Y_0 = 0. \end{cases}
\end{aligned} \tag{45}$$

Figure 1 displays the exact solution and the approximate solutions of our proposed method for various values of N. According to this figure, we can see that the numerical solutions for different values of N correspond to the exact solution. Also, the logarithms of absolute errors for state and control variables for different values of $\epsilon(.)$ at $N = 8$ are shown in Figure 2. Results represent that our method has a relatively good performance.

Example 2. As a second example, we consider the following VOFFOC problem

$$\begin{aligned}
&\text{Minimize}\, L = \int_0^1 \left(\left(Y(t) - t^{1.5}\right)^2 + (V(t) - \sin(\pi\epsilon(t)))^2\right) dt,\\
&\text{subject to} \begin{cases} {}_0^C D_t^{\epsilon(t)} Y(t) = Y\left(\dfrac{t}{2}\right) + V(t) - \dfrac{1}{2\sqrt{2}} t^{1.5} - \sin(\pi\epsilon(t)) + \dfrac{\Gamma(2.5)}{\Gamma(2.5-\epsilon(t))} t^{1.5-\epsilon(t)},\\[2ex] Y(0) = 0, \end{cases}
\end{aligned} \tag{46}$$

where ϵ: $[0, 1] \longrightarrow [0, 1]$ is an arbitrary continuous function. The exact optimal solution of this problem is $(Y^*, V^*) = (t^{1.5}, \sin(\pi\epsilon(t)))$. We employ our approach to approximate the solutions. The obtained results are shown in Figure 3 for $\epsilon(t) = 1 - 0.5(t - t^2)$ and $N = 4, 6, 8$. In Figure 4, the gained approximate control is given for different fractional orders $\epsilon(.)$ and $N = 8$. Moreover, the logarithm of absolute errors is provided in Figures 5 and 6. The results confirm the accuracy and efficiency of the presented approach.

Example 3. Consider the following VOFFOC problem

$$\begin{aligned}
&\text{Minimize}\, L = \int_0^1 \left(Y(t) - t^2 - 1\right)^2 + (V(t) - t)^2,\\
&\text{subject to}\begin{cases} {}_0^C\mathbb{D}_t^{\epsilon(t)} Y(t) = \dfrac{\Gamma(3)}{\Gamma(3-\varepsilon(t))} t^{2-\varepsilon(t)} + Y\left(e^{t-1}\right) + V(t) - e^{2t-2} - t - 1, \quad 0 < t \leq 1,\\[2ex] Y(0) = 1. \end{cases}
\end{aligned} \tag{47}$$

The exact solution for this system is $(Y^*, V^*) = (t^2 + 1, t)$. To solve this problem, we use the proposed method with $N = 5, 7, 9$ for $\epsilon(t) = 1 - 0.8\tanh(t)$ in Figure 7. In Figure 8, the absolute error functions of the state (control) variable with $N = 8$ and different functions of $\epsilon(t)$ are plotted. Based on these figures, we can see that our numerical solutions are excellently in agreement with the exact solution.

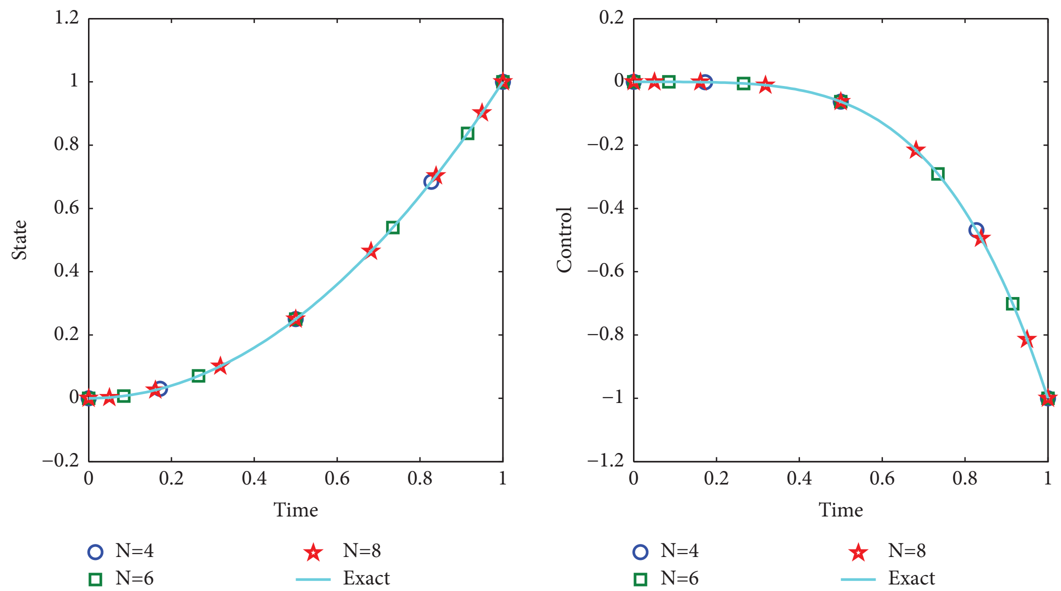

Figure 1: The approximate solutions for Example 1 with $\epsilon(t) = 1 - 0.5\tanh(t)$.

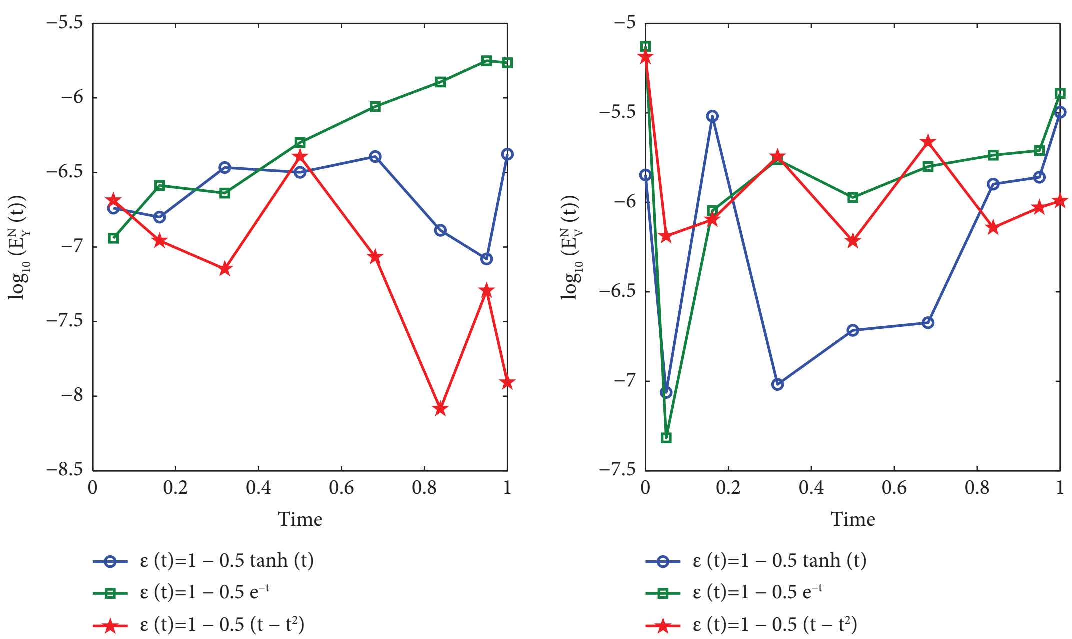

Figure 2: The logarithm of absolute errors for Example 1 with $N = 8$.

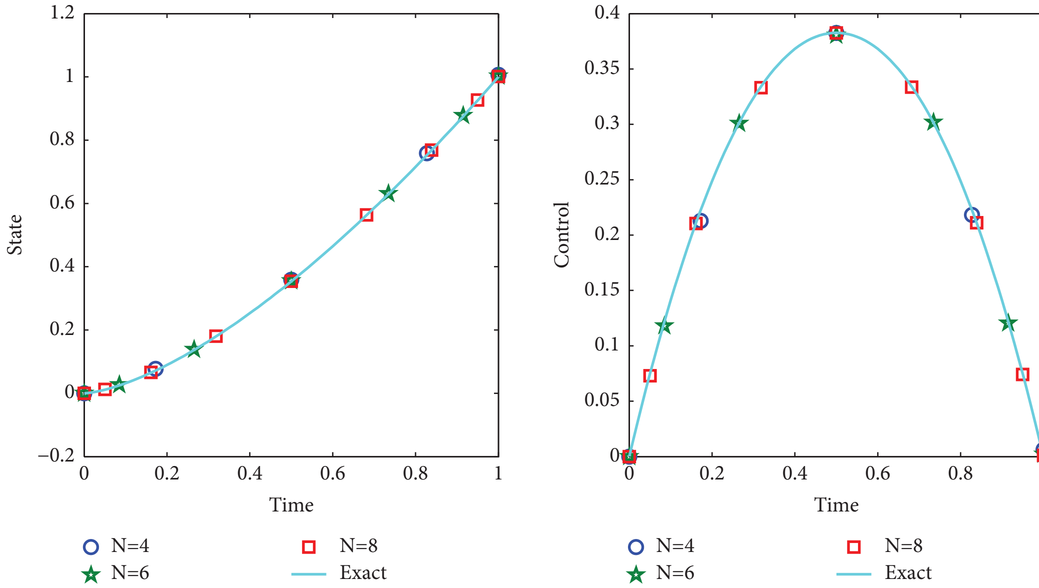

Figure 3: The approximate solutions for Example 2 with $\epsilon(t) = 1 - 0.5(t - t^2)$.

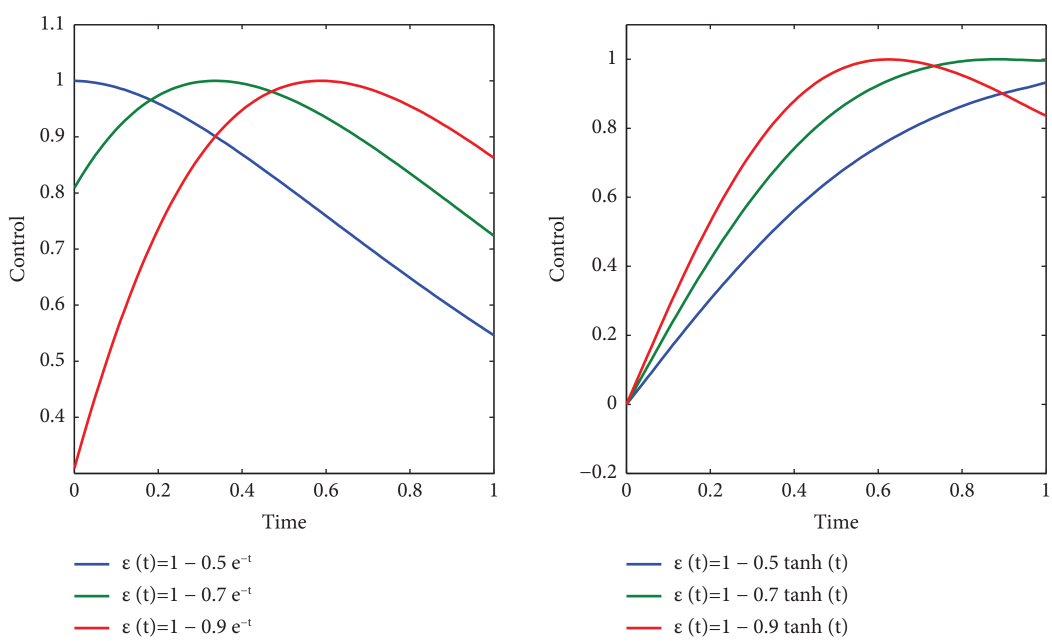

Figure 4: The approximate control $V_N(.)$ for Example 2 with $N = 8$.

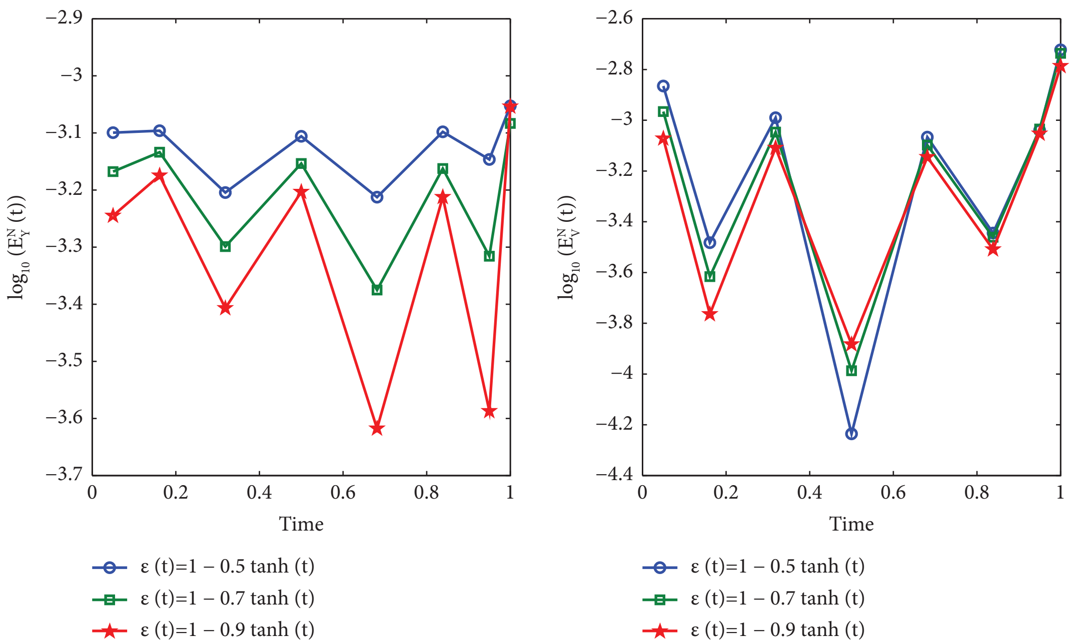

FIGURE 5: The logarithm of absolute errors for Example 2 with $N = 8$.

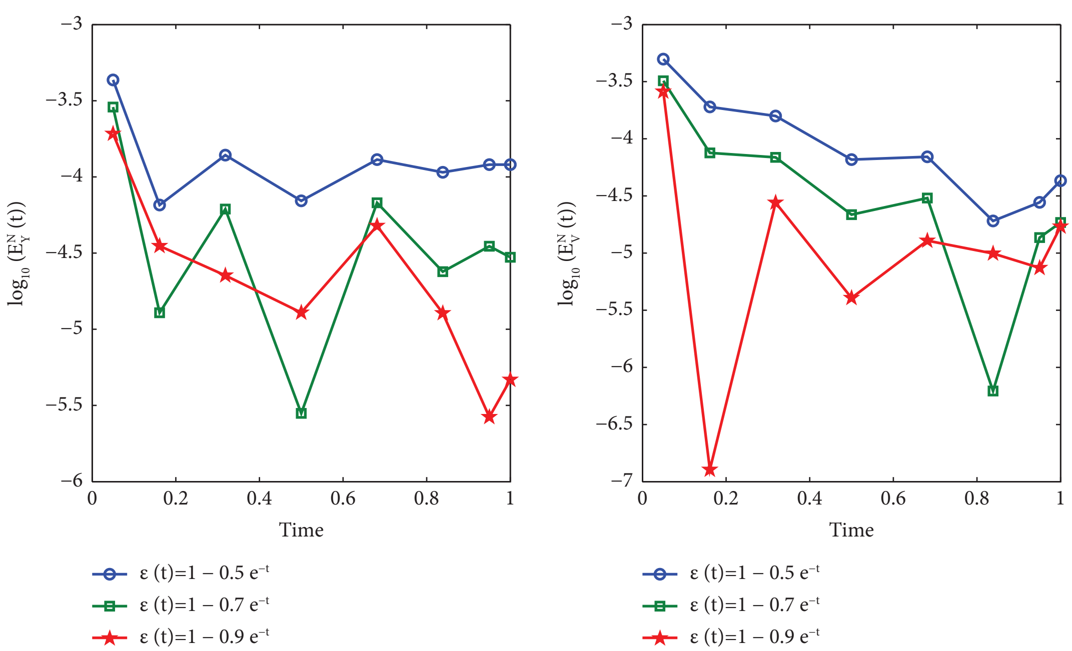

FIGURE 6: The logarithm of absolute errors for Example 2 with $N = 8$.

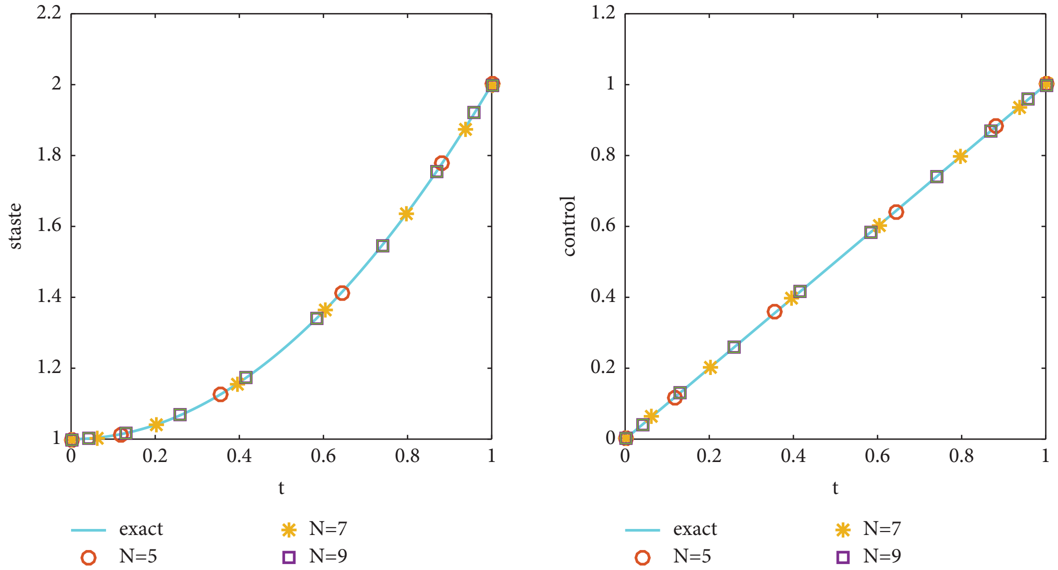

Figure 7: The approximate solutions for Example 3 with $\epsilon(t) = 1 - 0.8\tanh(t)$.

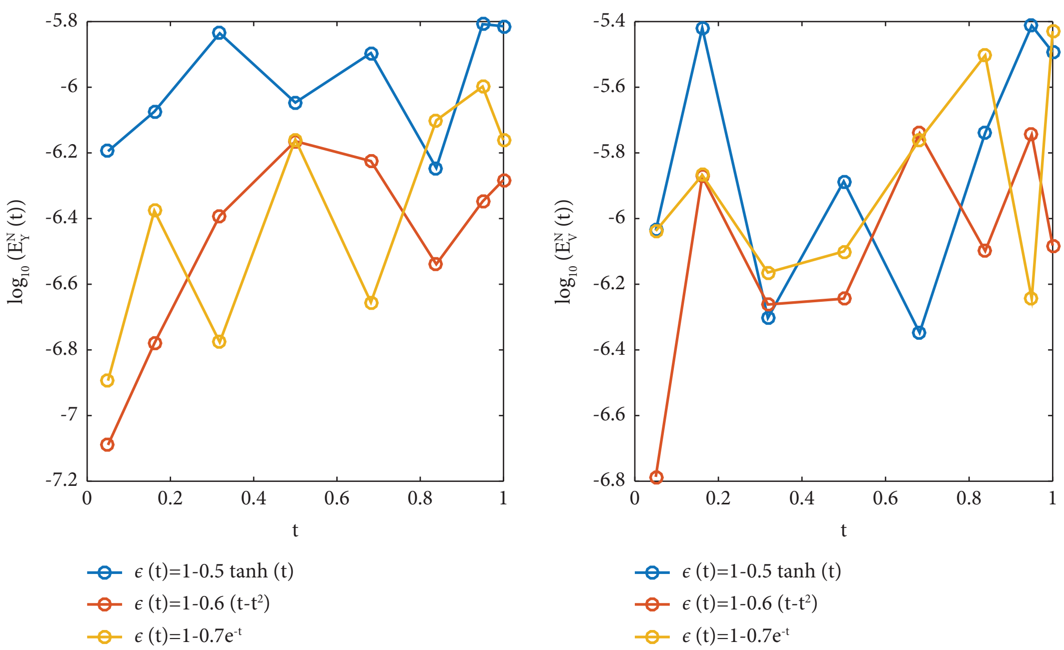

Figure 8: The logarithm of absolute errors for Example 3 with $N = 8$.

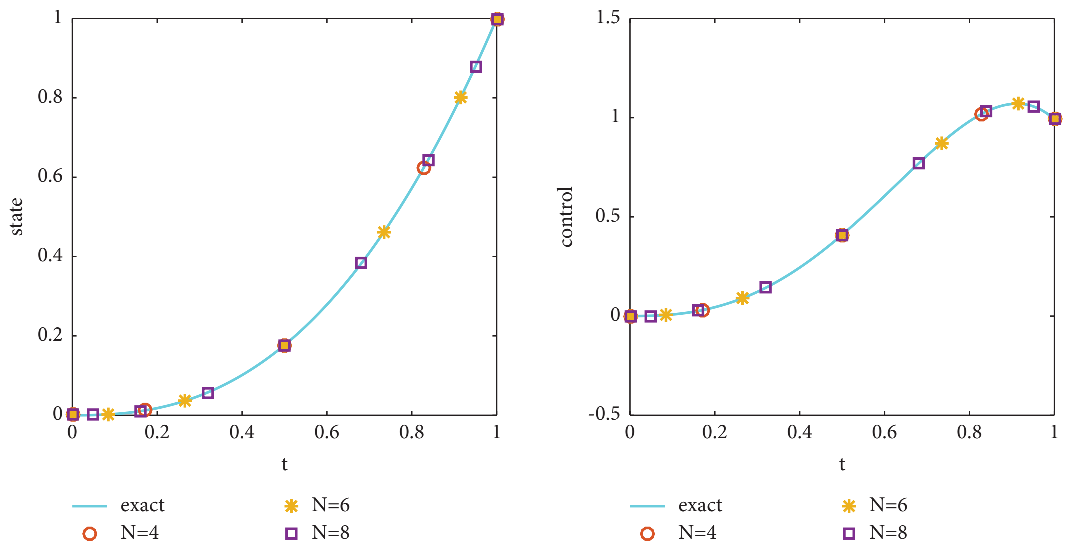

Figure 9: The approximate solutions for Example 4 with $\epsilon(t) = 1 - 0.8\tanh(t)$.

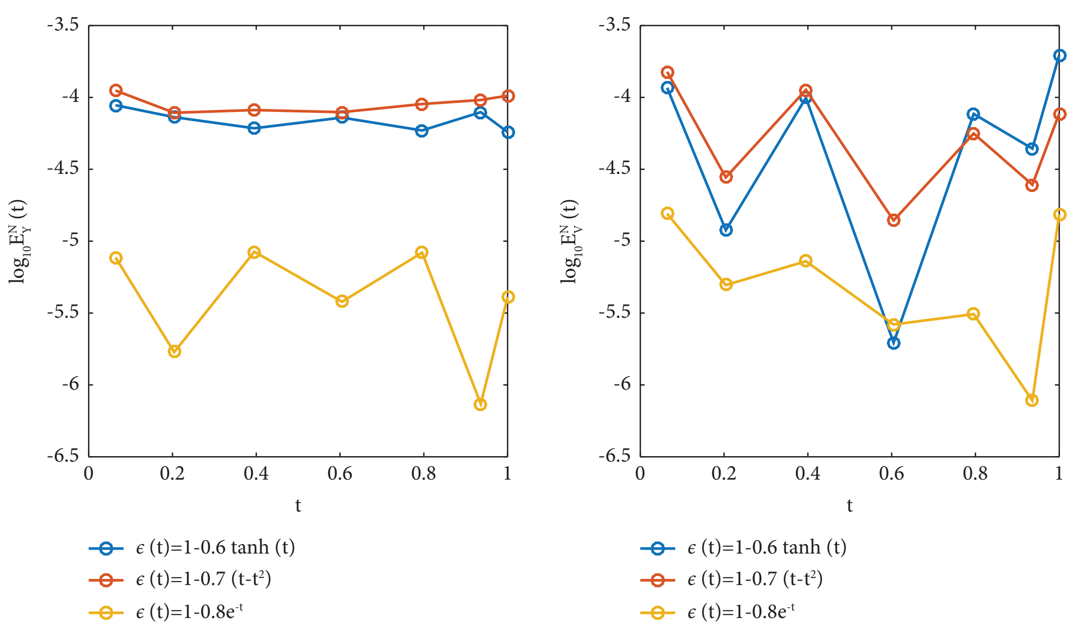

Figure 10: The logarithm of absolute errors for Example 4 with $N = 8$.

Example 4. Consider the following VOFFOC problem

$$\begin{aligned}
&\text{Minimize} \quad L = \int_0^1 \left(Y(t) - t^{2.5}\right)^2 + \left(V(t) + t^6 - \frac{15\sqrt{\pi}\, t^{2.5-\epsilon(t)}}{8\Gamma(3.5 - \epsilon(t))}\right)^2 dt,\\
&\text{subject to} \quad \begin{cases} {}_0^C D_t^{\epsilon(t)} Y(t) = Y\left(t^{(12/5)}\right) + u(t), \\ Y(0) = 0. \end{cases}
\end{aligned} \tag{48}$$

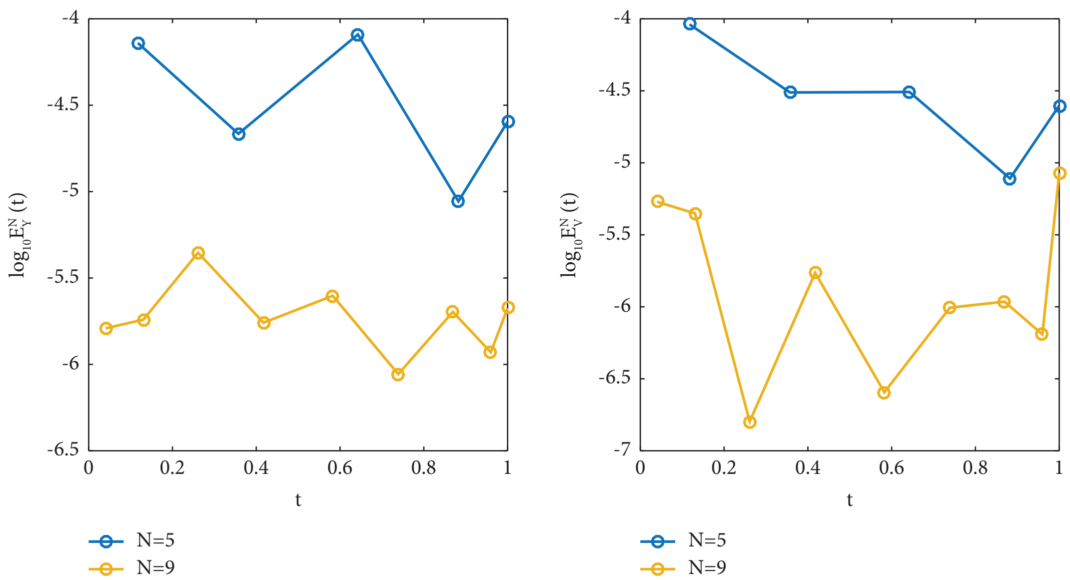

FIGURE 11: The logarithm of absolute errors for Example 4 with different N and $\epsilon(t) = 1 - 0.8e^{-t}$.

The exact solution for this system is $(Y^*, V^*) = (t^{2.5}, -t^6 + 15\sqrt{\pi}\, t^{2.5-\epsilon(t)}/8\Gamma(3.5-\epsilon(t)))$ and $L^* = 0$. Figure 9 illustrates the approximate solutions of our proposed method for $\epsilon(t) = 1 - 0.8\tanh(t)$ with $N = 4, 6, 8$. In Figure 10, the absolute error functions of the state (control) variable with $N = 8$ and different functions of $\epsilon(t)$ are plotted. We can understand from Figure 11 that by increasing the number of collocation points, the error of approximate variables decreases, which indicates the efficiency of the presented method. Based on these figures, we observed an excellent agreement between our numerical solutions and the exact solution.

6. Conclusions and Suggestions

In this paper, we considered a class of optimal control problems under variable-order fractional functional differential equations. We obtained an approximate solution based on the shifted Legendre pseudospectral collection method. This is the first time that the shifted Legendre pseudospectral collection method has been applied to variable-order fractional-functional problems. The proposed method accurately and efficiently calculates matrix differentiation, avoiding singular integral approximation and overcoming challenges of variable-order functional fractional derivatives. By implementing this method, the original optimal control problem was transformed into an optimization problem which is easier to solve. Several numerical examples have been examined. We obtained a high level of agreement between the numerical and exact solutions, indicating that our method has good performance. In future studies, the applicability of the mentioned method on other OC problems, such as OC problems under variable-order fractional-functional differential equations with delay and variable-order fractional integro-differential equations, will be investigated.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. Diethelm, D. Baleanu, and E. Scalas, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012.

[2] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, Amsterdam, Netherlands, 1998.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.

[4] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[5] K. Abuasbeh, A. U. K. Niazi, H. M. Arshad, M. Awadalla, and S. Trabelsi, "Approximate controllability of fractional stochastic evolution inclusions with non-local conditions," *Fractal and Fractional*, vol. 7, no. 6, p. 462, 2023.

[6] A. Khan, A. U. K. Niazi, W. Abbasi, A. Jamil, and J. A. Malik, "Control design for fractional order leader and follower systems with mixed time delays: a resilience-based approach," *Fractal and Fractional*, vol. 7, no. 5, p. 409, 2023.

[7] A. U. K. Niazi, J. Wei, M. Ur Rehman, and D. Jun, "Existence results for hybrid fractional neutral differential equations," *Advances in Difference Equations*, vol. 2017, p. 353, 2017.

[8] H. Dehestani and Y. Ordokhani, "A numerical study on fractional optimal control problems described by Caputo-Fabrizio fractional integro-differential equation," *Optimal Control Applications and Methods*, vol. 44, no. 4, pp. 1873–1892, 2023.

[9] M. H. Heydari and M. Razzaghi, "Extended Chebyshev cardinal wavelets for nonlinear fractional delay optimal control problems," *International Journal of Systems Science*, vol. 53, no. 5, pp. 1048–1067, 2022.
[10] G. Ghanbari and M. Razzaghi, "Numerical solutions for fractional optimal control problems by using generalised fractional-order Chebyshev wavelets," *International Journal of Systems Science*, vol. 53, no. 4, pp. 778–792, 2022.
[11] S. Sabermahani and Y. Ordokhani, "Fibonacci wavelets and Galerkin method to investigate fractional optimal control problems with bibliometric analysis," *Journal of Vibration and Control*, vol. 27, no. 15-16, pp. 1778–1792, 2020.
[12] S. Nemati, P. M. Lima, and D. F. Torres, "A numerical approach for solving fractional optimal control problems using modified hat functions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 78, Article ID 104849, 2019.
[13] A. Lotfi, M. Dehghan, and S. A. Yousefi, "A numerical technique for solving fractional optimal control problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1055–1067, 2011.
[14] S. Momani, O. Abu Arqub, and B. Maayah, "Piecewise optimal fractional reproducing kernel solution and convergence analysis for the Atangana-Baleanu-Caputo model of the Lienard's equation," *Fractals*, vol. 28, no. 08, Article ID 2040007, 2020.
[15] C. Phang, N. F. Ismail, A. Isah, and J. R. Loh, "A new efficient numerical scheme for solving fractional optimal control problems via a Genocchi operational matrix of integration," *Journal of Vibration and Control*, vol. 24, no. 14, pp. 3036–3048, 2018.
[16] M. H. Heydari, M. R. Mahmoudi, Z. Avazzadeh, and D. Baleanu, "Chebyshev cardinal functions for a new class of nonlinear optimal control problems with dynamical systems of weakly singular variable-order fractional integral equations," *Journal of Vibration and Control*, vol. 26, no. 9-10, pp. 713–723, 2020.
[17] M. H. Heydari and Z. Avazzadeh, "A new wavelet method for variable-order fractional optimal control problems," *Asian Journal of Control*, vol. 20, no. 5, pp. 1804–1817, 2018.
[18] H. Hassani, Z. Avazzadeh, and J. A. T. Machado, "Solving two-dimensional variable-order fractional optimal control problems with transcendental Bernstein series," *Journal of Computational and Nonlinear Dynamics*, vol. 14, no. 6, 2019.
[19] A. H. Bhrawy and M. Zaky, "Numerical algorithm for the variable-order Caputo fractional functional differential equation," *Nonlinear Dynamics*, vol. 85, no. 3, pp. 1815–1823, 2016.
[20] R. Almeida, D. Tavares, and D. F. Torres, *The Variable-Order Fractional Calculus of Variations*, Springer International Publishing, Berlin, Germany, 2019.
[21] J. Shen, T. Tang, and L. L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Science and Business Media, Berlin, Germany, 2011.