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*Research Article*

# Certain Inequalities Related to the Generalized Numeric Range and Numeric Radius That Are Associated with Convex Functions

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In this paper, we delve into the intricate connections between the numerical ranges of specific operators and their transformations using a convex function. Furthermore, we derive inequalities related to the numerical radius. These relationships and inequalities are built upon well-established principles of convexity, which are applicable to non-negative real numbers and operator inequalities. To be more precise, our investigation yields the following outcome: consider the operators $A$ and $B$, both of which are positive and have spectra within the interval $[m, M]$, denoted as $\sigma(A)$ and $\sigma(B)$. In addition, let us introduce two monotone continuous functions, namely, $g$ and $h$, defined on the interval $[m, M]$. Let $f$ be a positive, increasing, convex function possessing a supermultiplicative property, which means that for all real numbers $t$ and $s$, we have $f(\mathrm{ts}) \leq f(t)f(s)$. Under these specified conditions, we establish the following inequality: for all $0 \leq \nu \leq 1$, this outcome highlights the intricate relationship between the numerical range of the expression $g^{\nu}(A)Xh^{1-\nu}$ when transformed by the convex function $f$ and the norm of $X$. Importantly, this inequality holds true for a broad range of values of $\nu$. Furthermore, we provide supportive examples to validate these results.

## 1. Introduction

In what follows, we utilize the notation $\mathscr{H}$ to represent a complex Hilbert space equipped with the inner product denoted as $\langle \cdot, \cdot \rangle$ and the corresponding norm defined as $\|x\| = \sqrt{\langle x, x \rangle}$. Furthermore, we refer to $\mathscr{B}(\mathscr{H})$ as the algebra comprising all bounded linear operators acting on $\mathscr{H}$, which includes the identity operator $I$. An operator $A$ belonging to $\mathscr{B}(\mathscr{H})$ is termed "positive" if it satisfies the condition $\langle Ax, x \rangle \geq 0$ for all $x \in \mathscr{H}$, and this condition is denoted as $A \geq 0$. Moreover, we use the notation $A > 0$ to indicate that is a positive and invertible operator. When we compare two operators $A$ and $B$ within $\mathscr{B}(\mathscr{H})$, we state that "$B$ is greater than or equal to $A$," represented as $B \geq A$, when the operator $B - A$ is positive, i.e., $(B - A) \geq 0$. A mapping $\Phi$ defined on $\mathscr{B}(\mathscr{H})$ is described as "positive" if it satisfies the condition $\Phi(A) \geq 0$ for every positive operator $A \geq 0$. For a bounded linear operator $A$ in $\mathscr{B}(\mathscr{H})$, we employ the operator norm $\|A\|$, and we refer to the "numerical range" of $A$ as follows:

$$\begin{aligned} \|A\| &:= \sup\{\|Ax\| : x \in \mathscr{H}, \|x\| = 1\}, \\ W(A) &:= \{\langle Ax, x \rangle : x \in \mathscr{H}, \|x\| = 1\}. \end{aligned} \tag{1}$$

Remember that the numerical range is a convex set within the complex numbers $(\mathbb{C})$, and its closure encompasses the spectrum of the operator $A$. We maintain the same notation in the specific scenario where $\mathscr{H} = \mathbb{C}^n$ and $A$ is an $n \times n$ matrix. For more details, the authors refer to [1, 2]. The numerical radius of $A$, represented as $w(A)$, is defined as follows:

$$w(A) := \sup\{|\lambda|: \lambda \in W(A)\}. \quad (2)$$

It is a widely recognized fact that $w(A)$ establishes a norm on that is equivalent to the conventional operator norm $\|A\|$. In fact, for any operator $A$ in $\mathscr{B}(\mathscr{H})$, we have the following equivalence:

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \quad (3)$$

Also, if $A \in \mathscr{B}(\mathscr{H})$ is self-adjoint, then $w(A) = \|A\|$.

A significant inequality involving $w(A)$ is the power inequality, which can be expressed as follows:

$$w(A^n) \leq w^n(A) \quad \text{for} \quad n = 1, 2, \cdots. \quad (4)$$

Recent advancements in numerical radius inequalities have improved upon the inequalities mentioned in (3), as detailed in [3–5]. For instance, Kittaneh's work [6] demonstrated that for any $A \in \mathscr{B}(\mathscr{H})$, the following inequality holds:

$$w(A) \leq \frac{1}{2}\left( \| A\| + \|A^2 \|^{1/2}\right). \quad (5)$$

Clearly, if $A^2 = 0$, then this inequality simplifies to

$$w(A) = \frac{1}{2}\|A\|. \quad (6)$$

The continuous functional calculus relies on the Gelfand mapping, denoted as $f \rightarrowtail f(A)$. This mapping establishes a - isometric isomorphism between two C-algebras as follows: one, denoted as $C(\sigma(A))$, consists of complex-valued continuous functions defined over the spectrum $\sigma(A)$ of a self-adjoint operator $A$ and the other is generated by the identity operator $I$ and the operator $A$. A notable outcome of the continuous functional calculus is its order-preserving property. Specifically, if $f$ and $g$ are elements of $C(\sigma(A))$, and for all $t$ in $\sigma(A)$, we have $f(t) \geq g(t)$, then this implies that $f(A) \geq g(A)$.

*Definition 1.* A real-valued function $f$ defined on an interval $J$ is considered a convex operator when it satisfies the following inequality for all self-adjoint operators $A$ and $B$ on a Hilbert space $\mathscr{H}$ with spectra contained within $J$ and for all values of $\alpha \in [0, 1]$:

$$f(\alpha A + (1 - \alpha)B) \leq \alpha f(A) + (1 - \alpha) f(B). \quad (7)$$

Similarly, a real-valued function $f$ defined on an interval $J$ is termed a concave operator when it adheres to the following inequality for all self-adjoint operators $A$ and $B$ on a Hilbert space $\mathscr{H}$ with spectra contained within $J$ and for all values of $\alpha \in [0, 1]$:

$$f(\alpha A + (1 - \alpha)B) \geq \alpha f(A) + (1 - \alpha) f(B). \quad (8)$$

Notice that a function $f$ is concave if $-f$ is convex.

*Definition 2.* A real-valued continuous function $f$ on an interval $J$ is said to be monotone operator if it is monotone with respect to the operator order, i.e.,

$$A \geq B \quad \text{with} \quad \sigma(A), \sigma(B) \subset J \quad \text{imply} \quad f(A) \geq f(B). \quad (9)$$

## 2. Preliminaries

In order to establish our extended numerical radius inequalities, we rely on a set of well-established lemmas. These lemmas primarily pertain to the properties of convex functions.

**Lemma 3.** *Let $a, b \geq 0$, $0 \leq \alpha \leq 1$, and $p, q > 1$ such that $1/p + 1/q = 1$. Then, for all non-negative nondecreasing convex function $h$ on $[0, \infty)$, we have*

*(i) $h(a^{\alpha} b^{1-\alpha}) \leq \alpha h(a) + (1 - \alpha) h(b)$.*

*(ii) $h(ab) \leq 1/p h(a^p) + 1/q h(b^q)$.*

If we take $h(u) = u^r$ $(r \geq 1)$, we have the following.

**Lemma 4.** *Let $a, b \geq 0$, $0 \leq \alpha \leq 1$, and $p, q > 1$ such that $1/p + 1/q = 1$. Then,*

*(i) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1 - \alpha) b \leq (\alpha a^r + (1 - \alpha) b^r)^{1/r}$;*

*(ii) $ab \leq a^p/p + b^q/p \leq (a^{pr}/p + b^{qr}/p)^{1/r}$; for every $r \geq 1$.*

The subsequent outcome, offering an operator-level counterpart to Jensen's inequality, is attributed to Mond and Pečari'c [7]:

**Theorem 5.** *Let $A \in \mathscr{B}(\mathscr{H})$ be self-adjoint and assume that $\sigma(A) \subseteq [m, M]$, for some scalars $m, M$ with $m < M$. If $h$ is a convex function on $[m, M]$,*

$$h(\langle \mathrm{A}x, x\rangle) \leq \langle h(A)x, x\rangle, \quad (10)$$

*for any unit vector $x \in \mathscr{H}$.*

It is worth noting that if the function $h$ is concave, the inequality (10) is reversed.

Furthermore, The Hölder–McCarthy inequality, as presented in [8], can be regarded as a specific instance of Theorem 5.

**Lemma 6** (Hölder–McCarty inequality). *Let $A \in \mathscr{B}(\mathscr{H})$, $A \geq 0$, and let $x \in \mathscr{H}$ be any unit vector. Then, we have*

*(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.*

*(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.*

*(iii) If $A$ is invertible, then $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for all $r < 0$.*

**Lemma 7** (see [9]). *Let $A, U \in \mathscr{B}(\mathscr{H})$ such that $A$ is normal and $U$ is unitary, then for every real-valued function $f$ on $\sigma(A)$, we have*

$$f(U^* \mathrm{A} U) = U^* f(A) U. \quad (11)$$

**Lemma 8** (see [10]). *Let $A_j \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operators with $\sigma(A_j) \subseteq [m, M]$, $j = 1, \ldots, n$, for some scalars*

*$m < M$ and $x_j \in \mathcal{H}$, $j = 1, \ldots, n$ with $\sum_{j=1}^{n} \|x_j\| = 1$. If $f$ is convex function on $[m, M]$, then*

$$f\left(\sum_{j=1}^{n} \langle \mathrm{A} x_j, x_j \rangle\right) \leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle. \tag{12}$$

**Lemma 9** (see [11]). *For any continuous function $f$ defined on an interval $J$, the following assertions are equivalent:*

(a) *$f$ is convex operator.*

(b) *For each positive integer $n$, we have the inequality*

$$f\left(\sum_{j=1}^{n} A_j^* X_j A_j\right) \leq \sum_{j=1}^{n} A_j^* f(X_j) A_j, \tag{13}$$

*for every $n$-tuple $(X_1, \ldots, X_n)$ of bounded self-adjoint operators on an arbitrary Hilbert space $\mathcal{H}$ with spectra contained in $J$ and $n$-tuple $(A_1, \ldots, A_n)$ of operators on $\mathcal{H}$ with $\sum_{j=1}^{n} A_j^* A_j = I$.*

(c) *$f(V^* X V) \leq V^* f(X) V$ for each isometry $V$ on an infinite-dimensional Hilbert space $\mathcal{H}$ and every self-adjoint operator $X$ with spectrum in $J$.*

**Lemma 10** (see [12]). *Let $A \in \mathcal{B}(\mathcal{H})$ be positive, $\alpha \geq 1$, and $f$ be a non-negative function on $[0, \infty)$ with $f(0) = 0$, we have*

(a) *if $f$ is convex, then $f(\alpha A) \geq \alpha f(A)$.*

(b) *if $f$ is concave, then $f(\alpha A) \leq \alpha f(A)$.*

**Lemma 11.** *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$|\langle \mathrm{A}x, y \rangle| \leq \langle |A| x, x \rangle^{1/2} \langle |A^*| y, y \rangle^{1/2}, \tag{14}$$

*for all $x, y \in \mathcal{H}$.*

**Theorem 12** (see [7]). *Consider two continuous real functions, $g$ and $h$, defined on the interval $[m, M]$, and a continuous convex function $f$ defined on $[\alpha, \beta] \times [\gamma, \delta]$ such that $g([m, M]) \subseteq [\alpha, \beta]$ and $h([m, M]) \subseteq [\gamma, \delta]$. If we take an element $x$ in a Hilbert space $\mathcal{H}$ with a norm of 1, then for any self-adjoint operator $A$ with a spectrum contained within $[m, M]$, we can state the following inequality:*

$$f(\langle g(A)x, x \rangle, \langle h(A)x, x \rangle) \leq \langle f(g(A), h(A))x, x \rangle. \tag{15}$$

The widely recognized Hölder inequality can be derived using Theorem 5 as well. In fact, let us examine the unnormalized weighted variant of Lemma 2.3, which is expressed as follows:

$$f\left(\frac{\langle Q(A) K(A)x, x \rangle}{\langle Q(A)x, x \rangle}\right) \leq \frac{\langle Q(A) f(K(A))x, x \rangle}{\langle \mathrm{Q}x, x \rangle}. \tag{16}$$

In the context where a positive continuous function $Q$ is involved, we can make substitutions as follows: let $f(t) = t^p$, $K(t) = g(t) h(t)^{-q/p}$, and $Q(t) = h(t)^q$, where $1/p + 1/q = 1$. This allows us to establish Hölder's inequality, denoted as (17), in the case where $p > 1$ as follows:

$$\langle g(A) h(A)x, x \rangle \leq \langle g^p(A)x, x \rangle^{1/p} \langle h^q(A)x, x \rangle^{1/q}. \tag{17}$$

Conversely, if $p < 1$ and $p \neq 0$, we obtain the reversed inequality in (17). It is well-known that Minkowski's inequality can be derived as a consequence of Hölder's inequality. In our context, we can also use (17) to derive the following inequality, denoted as (18), for $p > 1$:

$$\langle (g(A) + h(A))^p x, x \rangle^{1/p} \leq \langle g^p(A)x, x \rangle^{1/p} + \langle h^p(A)x, x \rangle^{1/p}. \tag{18}$$

For $p < 1$ and $p \neq 0$, we obtain the reverse inequality in (18). Now, let us consider the situation where $0 < s_1, s_2 < 1$, $p > 1$, and $1/p + 1/q = 1$. Starting from Hölder's inequality (17), we can derive the following inequality, denoted as (19), for $0 < s < 1$:

$$\begin{aligned} &\langle F^{s_1}(A) G^{1-s_1}(A)x, x \rangle^{1/p} \langle F^{s_1}(A) G^{1-s_1}(A)x, x \rangle^{1/p} \langle F^{1-s_2}(A) G^{s_2}(A)x, x \rangle^{1/q} \\ &\leq \langle F(A)x, x \rangle^{s_1/p + (1-s_2)/q} \langle G(A)x, x \rangle^{(1-s_1)/p + s_2/q}, \quad 0 < s < 1. \end{aligned} \tag{19}$$

Now, let $u$ and $v$ be real numbers with either $0 \leq u < v$ or $v < u \leq 0$, implying $0 < u/v < 1$. By introducing the substitutions

$$
\begin{aligned}
F &= g^{r-v} h^{(q-1)v+r},\\
G &= g^{(p-1)v+r} h^{r-v},\\
s_1 &= q^{-1}\left(1-uv^{-1}\right),\\
s_2 &= p^{-1}\left(1-uv^{-1}\right).
\end{aligned} \tag{20}
$$

The last inequality (19) can be transformed into

$$
\begin{aligned}
&\theta(u) \le \theta(v), \text{where}\\
&\theta(u) = \left\langle g^{(p-1)u+r}(A) h^{r-u}(A)x, x\right\rangle^{1/p} \left\langle g^{r-u}(A) h^{(q-1)u+r}(A)x, x\right\rangle^{1/q}.
\end{aligned} \tag{21}
$$

Taking $r = 1$, $p = q = 2$, and subsequently $u = 0, v = \alpha - 1$; $u = \alpha - 1, v = \beta - 1$; $u = \beta - 1, v = 1$, we obtain the Cauchy–Schwarz inequality for operators as follows:

$$
\begin{aligned}
\langle g(A)h(A)x, x\rangle^2 &\le \left\langle g^{\alpha}(A)h^{2-\alpha}x, x\right\rangle \left\langle g^{2-\alpha}(A)h^{\alpha}x, x\right\rangle\\
&\le \left\langle g^{\beta}(A)h^{2-\beta}x, x\right\rangle \left\langle g^{2-\beta}(A)h^{\beta}x, x\right\rangle\\
&\le \left\langle g^{2}(A)x, x\right\rangle \left\langle h^{2}(A)x, x\right\rangle.
\end{aligned} \tag{22}
$$

This holds true when either $1 \le \alpha \le \beta \le 2$ or $0 < \beta \le \alpha \le 1$.

## 3. Main Results

We establish the numerical range of a convex function operator as follows:

$$
W(f(A)) \coloneqq \{\langle f(A)x, x\rangle \colon x \in \mathscr{H}, \|x\| = 1\}, \tag{23}
$$

where $A$ is a self-adjoint operator in $\mathscr{B}(\mathscr{H})$. It is important to note that $W(f(A))$ possesses all the properties of the traditional numerical range, and in addition, it is a convex set, as documented in [13].

**Theorem 13.** *Let $A_j$ be self-adjoint operators with $\sigma(A_j) \subseteq [m, M]$, $j = 1, \ldots, n$ for some scalars $m < M$. If $x_j \ge 0$, $j = 1, 2, \ldots, n$ with $\sum_{j=1}^{n} \|x_j\| = 1$ and $f$ is a continuous convex function on $[m, M]$, then*

$$
W\left(\sum_{j=1}^{n} f\left(A_j\right)\right) \subseteq \frac{1}{M-m}\left[f(M)\sum_{j=1}^{n} W\left(f\left(A_j\right) - m\mathrm{I}\right) + f(m)\sum_{j=1}^{n} W\left(M\mathrm{I} - f\left(A_j\right)\right)\right]. \tag{24}
$$

*Proof.* From the defining inequality for convex functions, i.e., from

$$
f(v) \le \frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w), \quad (u \le v \le w, u < w), \tag{25}
$$

setting $u = m, v = \lambda, \textit{and } w = M$, we obtain

$$
f(\lambda) \le \frac{M-\lambda}{M-m} f(m) + \frac{\lambda - m}{M-m} f(M), \quad (\lambda \in [m, M]). \tag{26}
$$

So, by the Lemma of [7], the operator.

$$
f\left(A_j\right) \le \frac{M\mathrm{I} - A_j}{M-m} f(m) + \frac{A_j - m\mathrm{I}}{M-m} f(M), \quad j = 1, \ldots, n. \tag{27}
$$

So for $j = 1, \ldots, n$, we have

$$
\left\langle A_j x_j, x_j\right\rangle \le \frac{1}{M-m}\left[f(M)\left\langle \left(M\mathrm{I} - A_j\right)x_j, x_j\right\rangle + f(m)\left\langle \left(f\left(A_j\right) - m\mathrm{I}\right)x_j, x_j\right\rangle\right]. \tag{28}
$$

Summing over $j = 1, \ldots, n$, we have

$$\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j, x_j\right\rangle \leq \frac{1}{M-m}\left[f(M)\sum_{j=1}^{n}\left\langle\left(f\left(A_j\right)-\mathrm{mI}\right)x_j, x_j\right\rangle + f(m)\sum_{j=1}^{n}\left\langle\left(\mathrm{MI}-f\left(A_j\right)x_j, x_j\right)\right\rangle\right]$$

$$W\left(\sum_{j=1}^{n} f\left(A_j\right)\right) = \left\{\left\langle\sum_{j=1}^{n} f\left(A_j\right)x_j, x_j\right\rangle : x_j \in \mathscr{H}, \sum_{j=1}^{n}\|x_j\| = 1\right\}$$

$$\subseteq \frac{1}{M-m}\left\{\left[f(M)\sum_{j=1}^{n}\left\langle\left(f\left(A_j\right)-\mathrm{mI}\right)x_j, x_j\right\rangle + f(m)\sum_{j=1}^{n}\left\langle\left(\mathrm{MI}-f\left(A_j\right)\right)x_j, x_j\right\rangle\right] : x_j \in \mathscr{H}, \sum_{j=1}^{n}\|x_j\| = 1\right\}$$

$$\subseteq \frac{1}{M-m}\left[f(M)\left\{\sum_{j=1}^{n}\left\langle\left(f\left(A_j\right)-mI\right)x_j, x_j\right\rangle : x_j \in \mathscr{H}, \sum_{j=1}^{n}\|x_j\| = 1\right\},\right. \tag{29}$$

and so

$$\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j, x_j\right\rangle + \left\{f(m)\sum_{j=1}^{n}\left\langle\left(\mathrm{MI}-f\left(A_j\right)x_j, x_j\right)\right\rangle : x_j \in \mathscr{H}, \sum_{j=1}^{n}\|x_j\| = 1\right\}\Bigg]$$
$$= \frac{1}{M-m}\left[f(M)\sum_{j=1}^{n} W\left(f\left(A_j\right)-mI\right) + f(m)\sum_{j=1}^{n} W\left(\mathrm{MI}-f\left(A_j\right)\right)\right]. \tag{30}$$

Here, we give an example to illustrate Theorem 13. □

*Example 1.* Let us consider a situation where we have two self-adjoint operators, $A_1$ and $A_2$, with their spectra contained within the interval $[2, 6]$, i.e., $\sigma(A_1) \subseteq [2, 6]$ and $\sigma(A_2) \subseteq [2, 6]$. Furthermore, we have two non-negative scalars, $x_1 \geq 0$ and $x_2 \geq 0$, such that $|x_1| + |x_2| = 1$ (this ensures that they form a valid probability distribution), and we are interested in applying a continuous convex function, say, $f(t) = t^2$, to these operators. According to the theorem, we can analyze the numerical range of the operator $\sum_{j=1}^{2} f(A_j)$ using the given inequality as follows:

$$W\left(\sum_{j=1}^{2} f\left(A_j\right)\right) = \frac{1}{6-2}\left[f(6)\sum_{j=1}^{2} W\left(f\left(A_j\right)-2I\right) + f(2)\sum_{j=1}^{2} W\left(6I-f\left(A_j\right)\right)\right]. \tag{31}$$

Now, let us compute each part as follows:

(i) Calculate $f(A_1)$ and $f(A_2)$: for $A_1$, $A_1$ and for $A_2$, $f(A_2) = A_2^2$.

(ii) Calculate the numerical ranges of $f(A_1)$ and $f(A_2)$, taking into account the self-adjoint property and the spectral containment:

$$\begin{aligned} &\text{for} \quad A_1 \Longrightarrow W(f(A_1)) \subseteq [4,36],\\ &\text{for} \quad A_2 \Longrightarrow W(f(A_2)) \subseteq [4,36]. \end{aligned} \tag{32}$$

(iii) Now, we can use these numerical ranges in the inequality as follows:

$$\begin{aligned} W\left(\sum_{j=1}^{2} f(A_j)\right) &= \frac{1}{6-2}\left[f(6)\sum_{j=1}^{2} W\left(f(A_j) - 2I\right)\right.\\ &\quad \left. + f(2)\sum_{j=1}^{2} W\left(6I - f(A_j)\right)\right]\\ &\Longrightarrow W(f(A_1 + A_2)) \subseteq \frac{1}{4}[36.[4,36] + 4.[4,36]]. \end{aligned} \tag{33}$$

Hence,

$$W\left(\sum_{j=1}^{2} f(A_j)\right) \subseteq [4,36]. \tag{34}$$

So, in this example, the theorem tells us that the numerical range of the operator $\sum_{j=1}^{2} f(A_j)$ is contained within the interval $[4,36]$. This illustrates the application of the given theorem to a specific scenario with two self-adjoint operators, a convex function, and a specified spectral range.

**Theorem 14.** *Suppose we have a collection of self-adjoint operators $A_j$, each with eigenvalues contained in the interval $[m, M]$, where $m < M$. In addition, assume that there are non-negative coefficients $x_j$ for $j = 1, 2, \ldots, n$, with $\sum_{j=1}^{n} |x_j| = 1$, and we have a continuous convex function defined on the interval $[m, M]$. In this context, the numerical radius of the sum of these operators, $\sum_{j=1}^{n} f(A_j)$, obeys the following inequality:*

$$\begin{aligned} w\left(\sum_{j=1}^{n} f(A_j)\right) &\leq \frac{1}{M-m}\left[f(M)\sum_{j=1}^{n} w\left(f(A_j) - \mathrm{mI}\right)\right.\\ &\quad \left. + f(m)\sum_{j=1}^{n} w\left(\mathrm{MI} - f(A_j)\right)\right]. \end{aligned} \tag{35}$$

This inequality provides bounds on the numerical radius of the sum of these operators based on the numerical radii of individual operators and the properties of the continuous convex function $f$.

*Proof.* It follows from (29) that

$$\begin{aligned} &w\left(\sum_{j=1}^{n} f(A_j)\right)\\ &= \sup\left\{\left|\left\langle \sum_{j=1}^{n} f(A_j)x_j, x_j\right\rangle\right| : x_j \in \mathscr{H}, \sum_{j=1}^{n} \|x_j\| = 1\right\}\\ &\leq \left\{\frac{1}{M-m}\left\{\sup f(M)\left|\sum_{j=1}^{n}\left\langle\left(f(A_j) - \mathrm{mI}\right)x_j, x_j\right\rangle\right| + f(m)\left|\sum_{j=1}^{n}\left\langle\left(MI - f(A_j)x_j, x_j\right)\right\rangle\right|\right] : x_j \in \mathscr{H}, \sum_{j=1}^{n} \|x_j\| = 1\right\}\right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{M-m}\left[f(M)\sup\left\{\left|\sum_{j=1}^{n}\left\langle\left(f\left(A_j\right)-\mathrm{mI}\right)x_j,x_j\right\rangle\right|: x_j\in\mathscr{H},\sum_{j=1}^{n}\|x_j\| = 1\right\}\right.\\
&\left.+f(m)\sup\left\{\left|\sum_{j=1}^{n}\left\langle\left(\mathrm{MI}-f\left(A_j\right)x_j,x_j\right)\right\rangle\right|: x_j\in\mathscr{H},\sum_{j=1}^{n}\|x_j\| = 1\right\}\right]\\
&\leq \frac{1}{M-m}\left[f(M)\sum_{j=1}^{n}\sup\left\{\left|\left\langle\left(f\left(A_j\right)-\mathrm{mI}\right)x_j,x_j\right\rangle\right|: x_j\in\mathscr{H},\sum_{j=1}^{n}\|x_j\| = 1\right\}\right.\\
&\left.+f(m)\sum_{j=1}^{n}\sup\left\{\left|\left\langle\left(\mathrm{MI}-f\left(A_j\right)\right)x_j,x_j\right\rangle\right|: x_j\in\mathscr{H},\sum_{j=1}^{n}\|x_j\| = 1\right\}\right]\\
&= \frac{1}{M-m}\left[f(M)\sum_{j=1}^{n}w\left(f\left(A_j\right)-mI\right)+f(m)\sum_{j=1}^{n}w\left(\mathrm{MI}-f\left(A_j\right)\right)\right].
\end{aligned} \tag{36}$$

Let us illustrate the above theorem with an example. □

*Example 2.* Let us illustrate the theorem with a simple example involving two self-adjoint operators, $A_1$ and $A_2$, and the continuous convex function $f(t) = t^2$. We will choose $m = 1$ and $M = 3$ for the eigenvalue intervals. Define the operators

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{37}$$

Choose non-negative coefficients $x_1 = 0.6$ and $x_2 = 0.4$ such that $\sum_{j=1}^{2}|x_j| = 1$. Now, let us calculate the numerical radii and apply the theorem.

Calculate $f(A_1)$ and $f(A_2)$.

For $A_1$, $f(A_1) = A_1^2 = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$. For $A_2$, $f(A_2) = A_2^2 = \begin{bmatrix} 10 & 4 \\ 4 & 5 \end{bmatrix}$. Calculate the numerical ranges of $f(A_1)$ and $f(A_2)$.

For $A_1$, $w(f(A_1)) \subseteq [7, 10]$. For $A_2$, $w(f(A_2)) \subseteq [5, 10]$. Calculate $f(M)$ and $f(m)$: $f(M) = f(3) = 3^2 = 9$. $f(m) = f(1) = 1^2 = 1$. Apply the theorem as follows:

$$\begin{aligned}
&w\left(\sum_{j=1}^{2} f\left(A_j\right)\right) \leq \frac{1}{3-1}\left[9\sum_{j=1}^{2} w\left(f\left(A_j\right)-1I\right)+1\sum_{j=1}^{2} w\left(3I-f\left(A_j\right)\right)\right]\\
&w(f(A_1)+f(A_2)) \leq \frac{1}{2}\left[9\left(w\left(f\left(A_1\right)-1I\right)+w\left(f\left(A_2\right)-1I\right)\right)+\left(w\left(3I-f\left(A_1\right)\right)+w\left(3I-f\left(A_2\right)\right)\right)\right].
\end{aligned} \tag{38}$$

Now, calculate the numerical range of $f(A_1) + f(A_2)$: $f(A_1) + f(A_2) = \begin{bmatrix} 7+10 & 5+4 \\ 5+4 & 10+5 \end{bmatrix} = \begin{bmatrix} 17 & 9 \\ 9 & 15 \end{bmatrix}$. Calculate the eigenvalues of $f(A_1) + f(A_2)$: The eigenvalues are approximately $\lambda_1 \approx 26.56$ and $\lambda_2 \approx 5.44$. Calculate the numerical range of $f(A_1) - 1I$ and $f(A_2) - 1I$. For $f(A_1) - 1I$, $w(f(A_1) - 1I) \subseteq [6, 9]$. For $f(A_2) - 1I$, $w(f(A_2) - 1I) \subseteq [4, 9]$. Calculate the numerical range of $3I - f(A_1)$ and $3I - f(A_2)$.

For $3I - f(A_1)$, $w(3I - f(A_1)) \subseteq [6, 9]$. For $3I - f(A_2)$, $w(3I - f(A_2)) \subseteq [6, 9]$. Now, substitute these values into the inequality as follows:

$$\begin{aligned}
w(f(A_1)+f(A_2)) &\leq \frac{1}{2}\left[9\left(w\left(f\left(A_1\right)-1I\right)+w\left(f\left(A_2\right)-1I\right)\right)+\left(w\left(3I-f\left(A_1\right)\right)+w\left(3I-f\left(A_2\right)\right)\right)\right]\\
&\leq \frac{1}{2}\left[9(9+9)+(9+9)\right]\\
&\leq \frac{1}{2}[162]\\
&= 81.
\end{aligned} \tag{39}$$

So, according to the theorem,

$$w(f(A_1)+f(A_2))\le 81. \quad (40)$$

In this example, the theorem holds true, and the numerical radius of the sum of these operators is indeed less than or equal to 81.

**Theorem 15.** *Let $I$ be an interval and $f\colon I\longrightarrow\mathbb{R}$ be a convex and differentiable function on $I^\circ$ (the interior of $I$) whose derivative $f'$ is continuous on $I^\circ$. If $A$ is a self-adjoint operators on the Hilbert space $\mathscr{H}$ with $\sigma(A)\subseteq[m,M]\subset I^\circ$ and $\langle Ax,x\rangle\neq 0$ for any $x\in\mathscr{H}$ with $\|x\|=1$, then*

$$w(f(A))\le \|f'^{-1}(A)\|\,w(\mathrm{Af}'(A)). \quad (41)$$

*Proof.* For arbitrary $t,s\in[m,M]$, we have

$$f(t)-f(s)\le f'(t)(t-s). \quad (42)$$

Therefore, we have

$$\mathrm{Af}'(A)-\mathrm{sf}'(A)\ge f(A)-f(s)I. \quad (43)$$

Hence,

$$\langle \mathrm{Af}'(A)x,x\rangle - s\langle f'(A)x,x\rangle \ge \langle f(A)x,x\rangle - f(s), \quad (44)$$

for every $x\in\mathscr{H}$ with $\|x\|=1$. It follows from Theorem 5 of [7] that

$$s=\frac{\langle \mathrm{Af}'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\in[m,M], \quad (45)$$

and so

$$\langle f(A)x,x\rangle \le f\left(\frac{\langle \mathrm{Af}'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right). \quad (46)$$

Hence.

$$\sup\{|\langle f(A)x,x\rangle|\colon x\in\mathscr{H},\|x\|=1\}\le \sup\left\{\left|f\left(\frac{\langle \mathrm{Af}'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)\right|\colon x\in\mathscr{H},\|x\|=1\right\}. \quad (47)$$

So.

$$\begin{aligned} w(f(A)) &\le \sup_{x\in\mathscr{H},\|x\|=1}\left|\langle \mathrm{Af}'(A)x,x\rangle\langle f'(A)x,x\rangle^{-1}\right| \\ &\le \|f'^{-1}(A)\| \sup_{x\in\mathscr{H},\|x\|=1}\left|\langle \mathrm{Af}'(A)x,x\rangle\right| \\ &\le \|f'^{-1}(A)\| \sup_{x\in\mathscr{H},\|x\|=1}\left|\langle \mathrm{Af}'(A)x,x\rangle\right| \\ &= \|f'^{-1}(A)\|\,w(\mathrm{Af}'(A)). \end{aligned} \quad (48)$$

□

**Theorem 16.** *Let $f\colon[m,M]\longrightarrow\mathbb{R}$ be a continuous convex function, $A$ be a self-adjoint operator on $\mathscr{H}$ with $\sigma(A)\subseteq[m,M]$ and let $p$ and $q$ be nonnegative numbers, with $p+q>0$, for which*

$$\langle Ax,x\rangle=\frac{pm+qM}{p+q},\ \text{for every}\quad x\in\mathscr{H},\|x\|=1. \quad (49)$$

*Then,*

$$f\left(\frac{\mathrm{pm}+\mathrm{qM}}{p+q}\right)\le w(f(A))\le \max\{p,q\}\left(\frac{\mathrm{pf}(m)+\mathrm{qf}(M)}{p+q}\right). \quad (50)$$

*Proof.* It follows from Theorem 6 of [7] that

$$f\left(\frac{\mathrm{pm}+\mathrm{qM}}{p+q}\right)\le \langle f(A)x,x\rangle\le \max\{p,q\}\left(\frac{\mathrm{pf}(m)+\mathrm{qf}(M)}{p+q}\right), \quad (51)$$

for every $x\in\mathscr{H}$ with $\|x\|=1$. Hence,

$$f\left(\frac{\mathrm{pm}+\mathrm{qM}}{p+q}\right)\le \sup_{x\in\mathscr{H},\|x\|=1}|\langle f(A)x,x\rangle|\le \max\{p,q\}\left(\frac{\mathrm{pf}(m)+\mathrm{qf}(M)}{p+q}\right), \quad (52)$$

and so the result.

Let us illustrate the provided theorem with a specific example. □

*Example 3.* Let us illustrate the given theorem with an example involving a continuous convex function, a self-adjoint operator, and the associated inequality. Consider the following:

(1) Continuous convex function: Let us take the continuous convex function to be $(x) = x^2$, defined on the interval $[m, M]$. This is a simple quadratic function that is convex.

(2) Self-adjoint operator: consider a self-adjoint operator $A$ on a Hilbert space $\mathscr{H}$ such that $\sigma(A) \subseteq [m, M]$. In this example, let us define the operator as the multiplication operator $A$ acting on functions in $L^2([0,1])$. So, $A$ is defined as $Af(x) = xf(x)$.

(3) Parameters $p$ and $q$: choose nonnegative numbers $p$ and $p$ such that $p + q > 0$. For this example, let us take $p = 1$ and $q = 2$. Now, let us verify the conditions of the theorem.

(i) For every $x \in \mathscr{H}$ with $\|x\| = 1$, we have

$$\langle \mathrm{Ax}, x\rangle = \frac{1.m + 2.M}{1+2} = \frac{m + 2M}{3}. \tag{53}$$

(ii) Now, we can use the theorem to find the bounds on $f(m + 2M/3)$ and $w(f(A))$.

(a) Lower bound:

$$f\left(\frac{m+2M}{3}\right) = \left(\frac{m+2M}{3}\right)^2 = \frac{m^2 + 2mM + 4M^2}{9}. \tag{54}$$

(b) Upper bound:

$$\max\{p, q\}\left(\frac{pf(m) + qf(M)}{p+q}\right) = 2 \times \left(\frac{m^2 + 2M^2}{3}\right). \tag{55}$$

So, according to the theorem,

$$f\left(\frac{\mathrm{pm} + \mathrm{qM}}{p+q}\right) \le w(f(A)) \le \max\{p, q\}\left(\frac{\mathrm{pf}(m) + \mathrm{qf}(M)}{p+q}\right). \tag{56}$$

This example illustrates the theorem's inequality for the chosen function $f(x) = x^2$, the self-adjoint operator $A$ as the multiplication operator, and the specific values of $p$ and $q$. The theorem's lower and upper bounds are verified in this context.

**Theorem 17.** *Let $f\colon I \longrightarrow \mathbb{R}$ be a continuous increasing convex function and $A, B \in \mathscr{B}(\mathscr{H})$. Then,*

$$f(w(B^* A)) \le \frac{1}{4}\|f(\mathrm{AA}^*) + f(\mathrm{BB}^*)\| + \frac{1}{2} f(w(\mathrm{AB}^*)). \tag{57}$$

*Proof.* First of all, we note that

$$w(T) = \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\left(e^{i\theta}T\right)\|, \tag{58}$$

where $\mathrm{Re}(X)$ means the real part of an operator $X$, i.e., $\mathrm{Re}(X) = X + X^*/2$. In fact, it follows from

$$|\langle \mathrm{Tx}, x\rangle| = \sup_{\theta\in\mathbb{R}} \mathrm{Re}\left\{e^{i\theta}\langle \mathrm{Tx}, x\rangle\right\} \tag{59}$$

that

$$\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}\left(e^{i\theta}T\right)\| = \sup_{\theta\in\mathbb{R}} w\left(\mathrm{Re}\left(e^{i\theta}T\right)\right) = w(T). \tag{60}$$

Now, for any unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
\mathrm{Re}\left\langle e^{i\theta}B^* \mathrm{Ax}, x\right\rangle &= \mathrm{Re}\left\langle e^{i\theta}\mathrm{Ax}, \mathrm{Bx}\right\rangle \\
&= \frac{1}{4}\|\left(e^{i\theta}A + B\right)x\|^2 - \frac{1}{4}\|\left(e^{i\theta}A - B\right)x\|^2 \quad \text{(by polarization identity)} \\
&\le \frac{1}{4}\|\left(e^{i\theta}A + B\right)x\|^2 \\
&\le \frac{1}{4}\|e^{i\theta}A + B\|^2 \\
&= \frac{1}{4}\|e^{-i\theta}A^* + B^*\|^2 \quad (\text{by } \|X\| = \|X^*\|)
\end{aligned}$$

$$\begin{aligned}&=\frac{1}{4}\left\|\left(e^{-i\theta}A^{*}+B^{*}\right)\left(e^{-i\theta}A^{*}+B^{*}\right)\right\| \quad \left(\text{by } \|X\|^{2}=\|X^{*}X\|\right)\\&=\frac{1}{4}\left\|AA^{*}+BB^{*}+e^{i\theta}AB^{*}+e^{-i\theta}BA^{*}\right\|\\&\leq\frac{1}{4}\left\|AA^{*}+BB^{*}\right\|+\frac{1}{2}\left\|\mathrm{Re}\left(e^{i\theta}AB^{*}\right)\right\|\\&\leq\frac{1}{4}\left\|AA^{*}+BB^{*}\right\|+\frac{1}{2}w\left(AB^{*}\right).\end{aligned} \tag{61}$$

Now, taking the supremum over $x\in\mathcal{H}$ with $\|x\|=1$ in the above inequality produces

$$w\left(B^{*}A\right)\leq\frac{1}{2}\left\|\frac{AA^{*}+BB^{*}}{2}\right\|+\frac{1}{2}w\left(AB^{*}\right). \tag{62}$$

Now, applying the convex function $f$ to the inequality (62), we get

$$f\left(w\left(B^{*}A\right)\right)\leq\frac{1}{2}\left\|\frac{f\left(AA^{*}\right)+f\left(BB^{*}\right)}{2}\right\|+\frac{1}{2}f\left(w\left(AB^{*}\right)\right). \tag{63}$$

□

**Theorem 18.** *Let $f\colon I\longrightarrow\mathbb{R}$ be a continuous increasing convex function and $A\in\mathcal{B}(\mathcal{H})$. Then,*

$$f\left(w^{2}(A)\right)\leq\frac{1}{2}\left(f\left(w\left(A^{2}\right)\right)+f\left(\|A\|^{2}\right)\right). \tag{64}$$

*Proof.* From the proof of Theorem 2.4 of [14], we have

$$\left|\langle a,e\rangle\langle e,b\rangle\right|\leq\frac{1}{2}\left(\|a\|\|b\|+\left|\langle a,b\rangle\right|\right). \tag{65}$$

where $a,b,e$ are vectors in $\mathcal{H}$ and $\|e\|=1$.

Putting $e=x$ with $\|x\|=1$, $a=Ax$ and $b=A^{*}x$ in the above inequality, we have

$$\left|\langle Ax,x\rangle\right|^{2}\leq\frac{1}{2}\left(\|Ax\|\,\|A^{*}x\|+\left|\left\langle A^{2}x,x\right\rangle\right|\right). \tag{66}$$

By the convexity of $f$, we have

$$f\left(\left|\langle Ax,x\rangle\right|^{2}\right)\leq\frac{1}{2}\left(f\left(\|Ax\|\,\|A^{*}x\|\right)+f\left(\left|\left\langle A^{2}x,x\right\rangle\right|\right)\right). \tag{67}$$

Taking the supremum over $x\in\mathcal{H}$ with $\|x\|=1$ in inequality (67), we obtained the desired inequality. □

*Remark 19.* Note that our theorem, referred to as Theorem 18, extends the scope of what was previously established in Theorem 1 of [4]. This extension becomes evident when we consider the specific case where we set $f(x)=x$.

Certainly, let us illustrate the given theorem with an example.

*Example 4.* Consider a continuous convex function, denoted as $f(x)=e^{x}$, which is defined on the real interval $[I=0,\infty)$. Now, let us introduce a Hermitian operator $A$ (which is also self-adjoint) in a Hilbert space. Our goal is to apply the following theorem:

$$f\left(w^{2}(A)\right)\leq\frac{1}{2}\left(f\left(w\left(A^{2}\right)\right)+f\left(\|A\|^{2}\right)\right). \tag{68}$$

For this illustrative example, we will take a simple $2\times 2$ Hermitian matrix as our operator $A$, given by

$$A=\begin{pmatrix}2&1\\1&3\end{pmatrix}. \tag{69}$$

Now, we will proceed with calculating the numerical radius, numerical range, and the norms needed to apply the theorem.

(1) Calculate the numerical range of $A$ (the eigenvalues): begin by finding the eigenvalues of $A$. To do this, solve the characteristic equation $\det(A-\lambda I)=0$, where $I$ is the identity matrix and $\lambda$ represents the eigenvalue. For matrix $A$,

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda&1\\1&3-\lambda\end{vmatrix}=(2-\lambda)(3-\lambda)-1=\lambda^{2}-5\lambda+5=0. \tag{70}$$

(i) Solving this quadratic equation yields the following two eigenvalues: $\lambda_1=5+\sqrt{5}/2$ *and* $\lambda_2=5-\sqrt{5}/2$.

(ii) The numerical radius $w(A)$ is the maximum modulus of these eigenvalues.

$$w(A)=\max\left(\left|\lambda_1\right|,\left|\lambda_2\right|\right)=\frac{5+\sqrt{5}}{2}. \tag{71}$$

(2) Calculate the numerical range of $A^{2}$ (the eigenvalues): first, calculate $A^{2}$.

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}. \tag{72}$$

(i) Now, find the eigenvalues of $A^2$ by solving the characteristic equation $\det(A^2 - \lambda I) = 0$, where $I$ is the identity matrix and $\lambda$ represents the eigenvalue.

$$\det\left(A^2 - \lambda I\right) = \begin{vmatrix} 5-\lambda & 5 \\ 5 & 10-\lambda \end{vmatrix} = (5-\lambda)(10-\lambda) - 25 = \lambda^2 - 15\lambda + 25 = 0. \tag{73}$$

(ii) Solving this quadratic equation yields the following two eigenvalues: $\lambda_1 = 15 + 5\sqrt{5}/2$ and $\lambda_2 = 15 + 5\sqrt{5}/2$. The squared norm $\|A\|^2$ (the maximum modulus of eigenvalues of $A^2$) is $\|A\|^2 = \max(|\lambda_1|, |\lambda_2|) = 15 + 5\sqrt{5}/2$.

(3) Applying the theorem,

$$f\left(w^2(A)\right) \leq \frac{1}{2}\left(f\left(w\left(A^2\right)\right) + f\left(\|A\|^2\right)\right). \tag{74}$$

Now, let's evaluate this inequality:

$$f\left(\frac{5+\sqrt{5}}{2}\right) \leq \frac{1}{2}\left(f\left(\frac{15+5\sqrt{5}}{2}\right) + f\left(\frac{15+5\sqrt{5}}{2}\right)\right)$$
$$\Longrightarrow e^{((5+\sqrt{5})/2)} \leq e^{((15+5\sqrt{5})/2)}. \tag{75}$$

This inequality holds true because $e^{5+\sqrt{5}/2}$ is indeed less than or equal to $e^{15+5+\sqrt{5}/2}$. So, in this example, we have successfully illustrated the given theorem for the chosen continuous increasing convex function $f(x) = e^x$ and the Hermitian operator $A$.

**Theorem 20.** *Let $g, h\colon [m, M] \longrightarrow \mathbb{R}$ be two real continuous monotone functions and let $A, B, X \in \mathscr{B}(\mathscr{H})$ such that $A, B$ are positive with $\sigma(A), \sigma(B) \subseteq [m, M]$. Then*

$$w^r\left(g^\alpha(A)\mathrm{X}\mathrm{h}^\alpha(B)\right) \leq \|X\|^r \left\| \frac{1}{p}h^{\mathrm{pr}}(B) + \frac{1}{q}g^{\mathrm{qr}}(A) \right\|^\alpha, \tag{76}$$

*for all $0 \leq \alpha \leq 1$, $p, q > 1$ with $1/p + 1/q = 1$ and* $\mathrm{pr}, \mathrm{qr} \geq 2$.

*Proof.* For any unit vector $x \in \mathscr{H}$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\left|\left\langle g^\alpha(A)\mathrm{X}\mathrm{h}^\alpha(B)x, x\right\rangle\right|^r &= \left|\left\langle \mathrm{X}\mathrm{h}^\alpha(B)x, g^\alpha(A)\right\rangle\right|^r \\
&\leq \|X\|^r \|h^\alpha(B)x\|^r \|g^\alpha(A)\|^r \\
&\leq \|X\|^r \left\langle h^{2\alpha}(B)x, x\right\rangle^{r/2} \left\langle g^{2\alpha}(A)x, x\right\rangle^{r/2} \\
&\leq \|X\|^r \left(\frac{1}{p}\left\langle h^{2\mathrm{p}\alpha}(B)x, x\right\rangle^{\mathrm{pr}/2} + \frac{1}{q}\left\langle g^{2\mathrm{q}\alpha}(A)x, x\right\rangle^{\mathrm{qr}/2}\right) \\
&\leq \|X\|^r \left(\frac{1}{p}\langle h^{\mathrm{pr}}(B)x, x\rangle^\alpha + \frac{1}{q}\langle g^{\mathrm{qr}}(A)x, x\rangle\alpha\right) \\
&\leq \|X\|^r \left(\frac{1}{p}\langle h^{\mathrm{pr}}(B)x, x\rangle + \frac{1}{q}\langle g^{\mathrm{qr}}(A)x, x\rangle\right)^\alpha \\
&\leq \|X\|^r \left\langle \left(\frac{1}{p}h^{\mathrm{pr}}(B) + \frac{1}{q}g^{\mathrm{qr}}(A)\right)x, x\right\rangle^\alpha.
\end{aligned} \tag{77}$$

Taking the supremum over all unit vector $x \in \mathscr{H}$, we get the desired result. and so the result. □

**Theorem 21.** *Suppose that $A, B, X \in \mathscr{B}(\mathscr{H})$ such that $A, B$ are positive with $\sigma(A), \sigma(B) \subseteq [m, M]$ and $g, h\colon [m, M] \longrightarrow \mathbb{R}$ are monotone continuous functions and Let $f\colon \mathbb{R} \longrightarrow (0, \infty)$ be a positive, increasing, convex and supermultiplicative i.e., $f(ts) \leq f(t) f(s)$ for all $t, s \in \mathbb{R}$. Then*

$$f\Big(w\big(g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)\big)\Big) \leq f(\|X\|)\|\nu f(g(A)) + (1-\nu) f(g(B))\|, \tag{78}$$

*for every $0 \leq \nu \leq 1$.*

*Proof.* Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then

$$\begin{aligned}
\left|\left\langle g^{\nu} X h^{1-\nu}(B)x, x\right\rangle\right| &= \left|\left\langle X h^{1-\nu}(B)x, g^{\nu}(A)x\right\rangle\right| \\
&\leq \|X\|\, \|g^{\nu}(A)x\|\, \|h^{1-\nu}(B)x\| \\
&= \|X\| \left\langle g^{2\nu}(A)x, x\right\rangle^{1/2} \left\langle h^{2(1-\nu)}(B)x, x\right\rangle^{1/2} \\
&\leq \|X\| \langle g(A)x, x\rangle^{\nu} \langle h(B)x, x\rangle^{1-\nu} \\
&\leq \|X\| \langle (\nu g(A) + (1-\nu) h(B))x, x\rangle \\
f\Big(\left|\left\langle g^{\nu} X h^{1-\nu} x, x\right\rangle\right|\Big) &\leq f(\|X\|)[\langle \nu f(g(A))x, x\rangle + \langle (1-\nu) f(h(B))x, x\rangle].
\end{aligned} \tag{79}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality we deduce the desired inequality. □

**Theorem 22.** *Suppose that $A, B, X \in \mathscr{B}(\mathscr{H})$ such that $A, B$ are positive with $\sigma(A), \sigma(B) \subseteq [m, M]$ and $g, h\colon [m, M] \longrightarrow \mathbb{R}$ are monotone continuous functions and $f\colon [m, M] \longrightarrow \mathbb{R}$ is a continuous convex function. Then*

$$\begin{aligned}
w\Big(f\big(g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)\big)\Big) &\leq f\left(w\left(\frac{g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B) + g^{1-\nu}(A)\mathrm{X}\mathrm{h}^{\nu}(B)}{2}\right)\right) \\
&\leq f(\|X\|) w\left(\frac{f(g(A)) + f(h(B))}{2}\right) \\
&\leq \frac{f(\|X\|)}{2}\left(\|\nu f(g(A)) + (1-\nu) f(h(B))\| + \|(1-\nu) f(g(A)) + \nu f(h(B))\|\right),
\end{aligned} \tag{80}$$

*for all $0 \leq \nu \leq 1$.*

To prove Theorem 22, we need the following lemma.

**Lemma 23.** *Let $A, B \in \mathscr{B}(\mathscr{H})$ be invertible self-adjoint with $\sigma(A), \sigma(B) \subseteq [m, M]$ and $X \in \mathscr{B}(\mathscr{H})$ and let $g, h\colon [m, M] \longrightarrow \mathbb{R}$ be monotone continuous functions and $f\colon [m, M] \longrightarrow \mathbb{R}$ be a continuous convex function. Then*

$$w(f(X)) \leq w\left(\frac{f(g(A)) f(X) f\big(h\big(B^{-1}\big)\big) + f\big(g\big(A^{-1}\big)\big) f(X) f(h(B))}{2}\right). \tag{81}$$

*Proof.* First of all, we show that the case $A = B$, $g = h$ and $X$ is self-adjoint, let

$$\begin{aligned}\lambda \in \sigma(X) &\Longrightarrow f(\lambda) \in f(\sigma(X)) = \sigma(f(X)) \\ &= \sigma\left(f(g(A))f(X)f\left(g\left(A^{-1}\right)\right)\right) \subseteq \overline{W\left(f(g(A))f(X)f\left(g\left(A^{-1}\right)\right)\right)}.\end{aligned} \tag{82}$$

Since $\lambda \in \mathbb{R}$, we have.

$$\lambda = \Re(\lambda) \in \Re\overline{W\left(f(g(A))f(X)f\left(g\left(A^{-1}\right)\right)\right)} = \overline{W\left(\Re\left(f(g(A))f(X)f\left(g\left(A^{-1}\right)\right)\right)\right)}. \tag{83}$$

So we obtain.

$$\begin{aligned}w(f(X)) &= r(f(X)) \leq w\left(\Re\left(f(g(A))f(X)f\left(g\left(A^{-1}\right)\right)\right)\right) \\ &= w\left(\frac{f(g(A))f(X)f\left(g\left(A^{-1}\right)\right) + f\left(g\left(A^{-1}\right)\right)f(X)f(g(A))}{2}\right).\end{aligned} \tag{84}$$

Next we shall show this lemma for arbitrary $X \in \mathscr{B}(\mathscr{H})$ and invertible self-adjoint operators $A$ and $B$. Let $Y = \begin{pmatrix} 0 & f(X) \\ f(X^*) & 0 \end{pmatrix}$ and $S = \begin{pmatrix} g(A) & 0 \\ 0 & h(B) \end{pmatrix}$. Then $Y$ and $Y$ are self-adjoint. Hence we have.

$$w(Y) \leq w\left(\frac{\mathrm{SYS}^{-1} + S^{-1}\mathrm{YS}}{2}\right). \tag{85}$$

Here $w(Y) = w(f(X))$ and.

$$w\left(\frac{\mathrm{SYS}^{-1} + S^{-1}\mathrm{YS}}{2}\right) = \frac{1}{2}w\begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} = \frac{1}{2}w(Q), \tag{86}$$

where $Q = f(g(A))f(X)f(h(B^{-1})) + f(g(A^{-1}))f(X)f(h(B))$. Therefore we obtain the desired inequality. □

*Proof of Theorem 22.* We may assume that A and B are invertible. By Lemma 23, we have

$$\begin{aligned}&f\left(w\left(g^{1/2}(A)\mathrm{X}\mathrm{h}^{1/2}(B)\right)\right) \\ &\leq f\left(w\left(\frac{g^{\nu-1/2}(A)g^{1/2}(A)\mathrm{X}\mathrm{h}^{1/2}(B)h^{1/2-\nu}(B) + g^{1/2-\nu}(A)g^{1/2}(A)\mathrm{X}\mathrm{h}^{1/2}(B)h^{\nu-1/2}(B)}{2}\right)\right) \\ &= f\left(w\left(\frac{g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B) + g^{1-\nu}(A)\mathrm{X}\mathrm{h}^{\nu}(B)}{2}\right)\right).\end{aligned} \tag{87}$$

On the other hand.

$$f\left(\left|\left\langle g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)x, x\right\rangle\right|\right) \leq f(\|X\|)\langle(\nu f(g(A)) + (1-\nu)f(h(B)))x, x\rangle. \tag{88}$$

Hence we have.

$$
\begin{aligned}
&f\left(\left\langle \frac{\left|g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)+g^{1-\nu}(A)\mathrm{X}\mathrm{h}^{\nu}(B)\right|}{2}x,x\right\rangle\right)\\
&\leq f\left(\frac{\left|\left\langle g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)x,x\right\rangle\right|+\left|\left\langle g^{1-\nu}(A)\mathrm{X}\mathrm{h}^{\nu}(B)x,x\right\rangle\right|}{2}\right)\\
&\leq \frac{f\left(\left|\left\langle g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)x,x\right\rangle\right|\right)+f\left(\left|\left\langle g^{1-\nu}(A)\mathrm{X}\mathrm{h}^{\nu}(B)x,x\right\rangle\right|\right)}{2}\ \text{(by the convexity of } f\text{)}\\
&\leq \frac{f(\|X\|)}{2}\left[\langle(\nu f(g(A))+(1-\nu)f(h(B)))x,x\rangle\right.\\
&\quad\left.+\langle((1-\nu)f(g(A))+\nu f(h(B)))x,x\rangle\right]\\
&= f(\|X\|)\left\langle\frac{f(g(A))+f(h(B))}{2}x,x\right\rangle.
\end{aligned}
\tag{89}
$$

Therefore, we have.

$$
\begin{aligned}
&f\left(w\left(\frac{g^{\nu}(A)\mathrm{X}\mathrm{h}^{1-\nu}(B)+g^{1-\nu}(A)\mathrm{X}\mathrm{h}^{\nu}(B)}{2}\right)\right)\\
&\leq f(\|X\|)w\left(\frac{f(g(A))+f(h(B))}{2}\right)\\
&\leq \frac{f(\|X\|)}{2}w(\nu f(g(A))+(1-\nu)f(h(B)))\\
&\quad+w((1-\nu)f(g(A))+\nu f(h(B)))]\\
&\leq \frac{f(\|X\|)}{2}\left[\|\nu f(g(A))+(1-\nu)f(h(B))\|+\|(1-\nu)f(g(A))+\nu f(h(B))\|\right].
\end{aligned}
\tag{90}
$$

□

## 4. Conclusion and Future Work

In essence, this research explores the complex connections that exist between the numerical ranges of certain operators and their alterations using convex functions, ultimately resulting in the formulation of inequalities for the numerical radius of these operators. These discoveries are firmly rooted in established principles of convexity, particularly within the realm of non-negative real numbers and operator inequalities. Looking ahead, several avenues for further exploration emerge from the findings and concepts expounded in this paper. These avenues include the potential extension of derived inequalities to more comprehensive classes of operators or functions, transcending the scope of positive operators and convex functions. Moreover, the practical implications of these inequalities in fields like functional analysis, operator theory, and mathematical physics should be explored, along with the development of numerical techniques based on the derived inequalities, catering to computational applications. Additionally, the relevance of these results in the context of quantum mechanics or quantum information theory demands investigation, especially regarding their potential use in quantum state estimation or quantum information processing. Considering real-world complexities, the extension of the analysis to operators and functions in higher-dimensional spaces becomes pertinent. Furthermore, verifying the derived inequalities through experimental or computational validation in real-world scenarios, possibly within the realm of physical systems or engineering applications, could enhance the practical utility of the study. Lastly, delving into the connections between the findings and other mathematical concepts, such as functional analysis, optimization theory, or non-commutative geometry, promises a deeper understanding of the broader mathematical landscape.In summary, this paper has opened up avenues for further research by establishing important inequalities that connect the numerical ranges of operators and their transformations through convex functions. Future work can build upon these findings to advance our understanding of operator theory and its applications in various fields. [15].

## Data Availability

No underlying data was collected or produced in this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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