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Research Article

Flat-Parallel Minkowski Space and β-Change with (α, β)-Metric

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The purpose of this paper is to examine the condition for a Finsler space with a generalized (α, β)-metric to be projectively flat. In addition, we establish that the Finsler space with generalized (α, β)-metric is a flat-parallel Minkowski space and derive the condition under which the β-change for the aforementioned metric is projective. We also explored the projective nature of β-change for various significant Finsler metrics derived from the generalized (α, β)-metric.

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space for the n-dimensional differentiable manifold M^n with a basic function $L(x, y)$. In 1972, Matsumoto [1] introduced the concept of a (α, β)-metric, $L(\alpha, \beta)$. If L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n, then a Finsler metric $L(x, y)$ is called as (α, β)-metric $L(\alpha, \beta)$. The Randers, Kropina, and Matsumoto metrics are the three most interesting instances of (α, β)-metrics.

A Finsler space $F^n = (M^n, L)$ is called projectively flat, or with rectilinear geodesic, if the space is covered by coordinate neighborhoods in which the geodesics can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called rectilinear. The condition for a Finsler space to be projectively flat was studied by Berwald [2] in tensorial form and completed by Matsumoto [3]. Hashiguchi and Ichijyo's paper [4] gives interesting results on the projective flatness of Randers spaces.

We have two important projective invariant tensors: the Weyl tensor W and the Douglas tensor D. A Finsler space with both of these tensors vanishes as a projectively flat space that can be projectively mapped to a locally Minkowski space. Randers spaces, Kropina spaces, and generalized Kropina spaces with $L = (\beta^2/\alpha)$ are examples of Finsler spaces with (α, β)-metric $L(\alpha, \beta)$. Matsumoto [5] demonstrated the criteria for the above spaces to be projectively flat. The concept of projectively flat Finsler space with the (α, β)-metric has been studied by many authors [6–10].

Furthermore, Matsumoto [11, 12] defined a β-change and flat-parallel Minkowski space with (α, β)-metrics in 1988 and 1991, respectively. He dealt flat parallelness of Randers metric, Kropina metric, and their generalized form in his paper [13]. The flat parallelness of the Matsumoto metric was studied by Aikou et al. [14].

In the present paper, we studied the generalized (α, β)-metric [15]. The purpose of this research work is to study the condition for a Finsler space with the generalized (α, β)-metric to be projectively flat using Matsumoto's results. Furthermore, we proved that Finsler space with generalized (α, β)-metric is a flat-parallel Minkowski space and obtained the condition for a projective change $\varphi: \alpha \longrightarrow L = \alpha(1 + (\beta/\alpha))^p$. We also investigated how β-change is projective for various significant Finsler metrics derived from the generalized (α, β)-metric.

We used terminology and notations from Matsumoto's monograph [16] throughout this paper.

2. Preliminaries

Let $F^n = (M^n, L(\alpha,\beta))$ represent an n-dimensional Finsler space and $R^n = (M^n, \alpha)$ represent the corresponding Riemannian space, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$. Let γ^i_{jk} be the Christoffel symbols with respect to α and the covariant differentiation with respect to γ^i_{jk} be indicated by (;). From the differential 1-form $\beta(x,y) = b_i(x)y^i$, we define

$$2r_{ij} = b_{i;j} + b_{j;i}, 2s_{ij} = b_{i;j} - b_{j;i} = \left(\partial_j b_i - \partial_i b_j\right), \\ s^i_j = a^{ir} s_{rj}, b^i = a^{ir} b_r, b^2 = a^{rs} b_r b_s. \tag{1}$$

Then, we consider the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of the Finsler space with the (α,β)-metric $L(\alpha,\beta)$. As is well known, we have

$$G^i_{jk} = \dot{\partial}_k G^i_j, 2G^i = G^i_0 = \left(G^i_r y^r\right), G^i_j = \dot{\partial}_j G^i. \tag{2}$$

Then, the previous paper [12, 17] gives the equation to find the difference $B^i_{jk} = G^i_{jk} - \gamma^i_{jk}$:

$$L_\alpha B^k_{ji} y^j y_k = \alpha L_\beta \left(b_{j:i} - B^k_{ji} b_k\right) y^j. \tag{3}$$

We consider a locally Minkowski space $F^n = (M^n, L)$, that is, M^n admits a covering by coordinate neighborhoods in each of which the fundamental function L is a function of y^i alone. We denote by $\overset{r}{R}_{hijk}$ a Riemannian curvature tensor with respect to γ^i_{jk}.

Definition 1 (see [13]). A locally Minkowski space with (α,β)-metric is called flat parallel, if α is locally flat ($\overset{r}{R}{}^i_{hjk} = 0$) and β is parallel with respect to α ($b_{i;j} = 0$).

Theorem 2 (see [12]). *A $F^n = (M^n, (\alpha,\beta))$ is a locally Minkowski if and only if B^k_{ji} are functions of x alone, and $\overset{r}{R}{}^i_{h\ jk}$ of the Riemannian α is written as*

$$\overset{r}{R}{}^i_{jkm} = \frac{\partial B^i_{jm}}{\partial x^k} - \frac{\partial B^i_{jk}}{\partial x^m} + B^i_{lk} B^l_{jm} - B^i_{lm} B^l_{jk}, \\ \overset{r}{R}{}^i_{hjk} = -\mathcal{U}(jk)\left\{B^i_{hj;k} + B^l_{hj} B^i_{lk}\right\}, \tag{4}$$

where $\mathcal{U}(jk)$ denotes the terms obtained from the preceding terms by interchanging indices j and k and R is Riemannian α.

In this paper, we refer to the contraction of y^i with 0 subscript, and the partial differentiation by α and β of L with α and β subscripts.

According to Theorem 1 of [5], a Finsler space F^n with an (α,β)-metric is projectively flat if and only if the space is covered by coordinate neighborhoods on which $\gamma^i_{jk}(x)$ satisfies

$$\frac{\left(\gamma^i_{00} - \gamma_{000} y^i/\alpha^2\right)}{2} + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 + \left(\frac{L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{C + \alpha r_{00}}{2\beta}\right)\left(\frac{\alpha^2 b^i}{\beta - y^i}\right) = 0, \tag{5}$$

where C is given by

$$C + \left(\frac{\alpha^2 L_\beta}{\beta L_\alpha}\right) s_0 + \left(\frac{\alpha L_{\alpha\alpha}}{\beta^2 L_\alpha}\right)\left(\alpha^2 b^2 - \beta^2\right)\left(\frac{C + \alpha r_{00}}{2\beta}\right) = 0. \tag{6}$$

By the homogeneity of L, we know that $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$; therefore, (6) can be rewritten as

$$\left\{1 + \left(\frac{L_{\beta\beta}}{\alpha L_\alpha}\right)\left(\alpha^2 b^2 - \beta^2\right)\right\}\left(\frac{C + \alpha r_{00}}{2\beta}\right) = \left(\frac{\alpha}{2\beta}\right)\left\{r_{00} - \left(\frac{2\alpha L_\beta}{L_\alpha}\right) s_0\right\}. \tag{7}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (5) and it can be written in the following form:

$$\left\{\frac{1 + L_{\beta\beta}\left(\alpha^2 b^2 - \beta^2\right)}{\alpha L_\alpha}\right\}\left\{\frac{\left(\gamma^i_{00} - \gamma_{000} y^i/\alpha^2\right)}{2} + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0\right\} \\ + \left(\frac{L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\alpha}{2\beta}\right)\left\{r_{00} - \left(\frac{2\alpha L_\beta}{L_\alpha}\right) s_0\right\}\left(\frac{\alpha^2 b^i}{\beta - y^i}\right) = 0. \tag{8}$$

Thus, we have [18].

Theorem 3. *If* $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, *then a Finsler space* F^n *with an* (α, β)*-metric is projectively flat if and only if (8) is satisfied.*

Remark 4. According to [19], if α^2 contains β as a factor, then $b^2 = 0$ and the dimension is equal to two. Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, i.e., $\alpha^2 \not\equiv 0 \pmod{\beta}$.

3. Projectively Flat Finsler Space with Generalized (α, β)-Metric

Let F^n be a Finsler space with the generalized (α, β)-metric given by

$$L = \alpha\left(1 + \frac{\beta}{\alpha}\right)^p, \quad p \neq 0, 1, 2. \tag{9}$$

In this section, we shall find the conditions for F^n with the generalized metric (9) to be projectively flat. The partial derivative with respect to α and β of (9) are given by

$$\begin{aligned} L_\alpha &= \frac{(\alpha+\beta)^{p-1}(\alpha+\beta)(1-p)}{\alpha^P}, L_\beta = p\alpha^{1-p}(\alpha+\beta)^{p-1}, \\ L_{\alpha\alpha} &= \frac{p(1-p)\left(-\beta^2\right)(\alpha+\beta)^{p-2}}{\alpha^{p+1}}, L_{\beta\beta} = p(p-1)\alpha^{1-p}(\alpha+\beta)^{p-2}. \end{aligned} \tag{10}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) = 0$, then we have $\alpha^2(1 - b^2 p + b^2 p^2) + \alpha\beta(2-p) + \beta^2(1-p^2) = 0$ which leads to a contradiction. Thus, we can apply Theorem 3.

Substituting (10) into (8), we have

$$\begin{aligned} &\left\{\alpha^2\left(1 - b^2 p + b^2 p^2\right) + \alpha\beta(2-p) + \beta^2\left(1-p^2\right)\right\}\left\{\left(\alpha^2\gamma^i_{00} - \gamma_{000}y^i\right)(\alpha + \beta(1-p)) \right. \\ &\left. \quad + 2\alpha^4 p s^i_0\right\} - p(1-p)\alpha^2\left\{r_{00}(\alpha + \beta(1-p)) - 2\alpha^2 p s_0\right\}\left(\alpha^2 b^i - y^i\beta\right) = 0. \end{aligned} \tag{11}$$

The above equation is rewritten as a polynomial of sixth degree in α as follows:

$$p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 + \alpha\left(p_5\alpha^4 + p_3\alpha^2 + p_1\right) = 0, \tag{12}$$

where

$$\begin{aligned} p_6 &= 2p\left\{\left(1 - b^2 p + b^2 p^2\right)s^i_0 + p(1-p)b^i s_0\right\}, \\ p_5 &= p(1-p)b^i r_{00} + \left(1 - b^2 p + b^2 p^2\right)\gamma^i_{00} + 2\beta p(2-p)s^i_0, \\ p_4 &= \beta\left(1 - b^2 p + b^2 p^2\right)(1-p)\gamma^i_{00} + \beta(2-p)\gamma^i_{00} + 2\beta^2\left(1-p^2\right)p s^i_0 \\ &\quad - \beta p(1-p)^2 b^i r_{00} + 2\beta p^2(1-p)s_0 y^i, \\ p_3 &= -\left(1 - b^2 p + b^2 p^2\right)\gamma_{000}y^i + \beta^2(2-p)(1-p)\gamma^i_{00} \\ &\quad + \beta^2\left(1-p^2\right)\gamma^i_{00} - \beta p(1-p)y^i r_{00}, \\ p_2 &= -\beta(1-p)\left(1 - b^2 p + b^2 p^2\right)\gamma_{000}y^i - \beta(2-p)\gamma_{000}y^i \\ &\quad + \beta^2(1-p)\gamma^i_{00} - \beta^2\left(1-p^2\right)y^i r_{00}, \\ p_1 &= \beta^2(2-p)(1-p)\gamma_{000}y^i - \beta^2\left(1-p^2\right)\gamma_{000}y^i, \\ p_0 &= \beta^3\left(1-p^2\right)(1-p)\gamma_{000}y^i. \end{aligned} \tag{13}$$

Since $p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0$ and $p_5\alpha^4 + p_3\alpha^2 + p_1$ are rational and α is irrational in y^i, we have

$$p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 = 0, \quad (14)$$

$$p_5\alpha^4 + p_3\alpha^2 + p_1 = 0. \quad (15)$$

The term which does not contain β in (14) is $p_6\alpha^6$. Thus, there exists a homogeneous polynomial W_6 of degree six in y^i such that

$$2p\left\{\left(1 - b^2p + b^2p^2\right)s_0^i + p(1-p)b^i s_0\right\}\alpha^6 = \beta W_6. \quad (16)$$

Since $\alpha^2 \neq 0 (\mathrm{mod} \quad \beta)$, we must have a function $u^i = u^i(x)$ satisfying

$$2p\left\{\left(1 - b^2p + b^2p^2\right)s_0^i + p(1-p)b^i s_0\right\} = u^i\beta. \quad (17)$$

Contracting (17) by b_i, we have

$$2ps_0 = u^i\beta b_i, \quad (18)$$

i.e., $2ps_j = u^i b_i b_j$. Furthermore, contracting this equation by b^j, we have $u^i b_i b^2 = 0$. Substituting this equation into (18), we have $s_0 = 0$. Thus, from (17), we get

$$2p\left(1 - b^2p + b^2p^2\right)s_{ij} = u_i b_{j,} \quad (19)$$

which gives $u_i b_j + u_j b_i = 0$. Contracting this equation by b^j, we have $u^i b^2 = 0$ by virtue of $u_j b^j = 0$. Thus, we get $u_i = 0$. Hence, from (19), we have $s_{ij} = 0$, provided that $2p(1 - b^2p + b^2p^2) \neq 0$.

On the other hand, from (15), we have 1-form $v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \quad (20)$$

Upon replacing $s_0 = 0$, $s_0^i = 0$, and (20) into (11), we have

$$\begin{aligned}&\left\{\alpha^2\left(1 - b^2p + b^2p^2\right) + \alpha\beta(2-p) + \beta^2\left(1-p^2\right)\right\}\left(\gamma_{00}^i - v_0 y^i\right)\\ &\quad - p(1-p)r_{00}\left(\alpha^2 b^i - \beta y^i\right) = 0,\end{aligned} \quad (21)$$

by virtue of $(\alpha + \beta(1-p)) \neq 0$. Then, (21) is written in the form $A\alpha + B = 0$, where

$$\begin{aligned}A &= \beta(2-p)\left(\gamma_{00}^i - v_0 y^i\right),\\ B &= \left\{\alpha^2\left(1 - b^2p + b^2p^2\right) + \beta^2\left(1-p^2\right)\right\}\left(\gamma_{00}^i - v_0 y^i\right)\\ &\quad - p(1-p)r_{00}\left(\alpha^2 b^i - \beta y^i\right).\end{aligned} \quad (22)$$

Since A and B are rational and α is irrational in y^i, we have $A = 0$ and $B = 0$.

First, it follows from $A = 0$ that

$$\gamma_{00}^i - v_0 y^i = 0, \quad 2 - p \neq 0, \quad (23)$$

i.e.,

$$2\gamma_{jk}^i = v_j\delta_k^i + v_k\delta_j^i, \quad (24)$$

which shows that the associated Riemannian space (M^n, α) is projectively flat.

Then, from $B = 0$ and (23), we have

$$-p(1-p)r_{00}\left(\alpha^2 b^i - \beta y^i\right) = 0. \quad (25)$$

Contracting (25) by b_i, we have $-p(1-p)r_{00}(\alpha^2 b^2 - \beta^2) = 0$, from which $r_{00} = 0$, provided $-p(1-p) \neq 0$, i.e., $r_{ij} = 0$. From $s_{ij} = 0$ and $r_{ij} = 0$, we have $b_{i;j} = 0$.

On the other hand, it is easily verified that (11) is a consequence of (23) and $b_{i;j} = 0$. Consequently, we have the following.

Theorem 5. *A Finsler space F^n with the generalized (α, β)-metric (9) is projectively flat if and only if the associated Riemannian space (M^n, α) is projectively flat and $b_{i;j} = 0$.*

4. Flat-Parallel Minkowski Space with Generalized (α, β)-Metric

In [20], Kim and Choi defined a procedure to show that (α, β)-metrics are flat-parallel Minkowski space.

Substituting $B_{ji}^k y^i y_k = P_{i00}$ and $(b_{j:i} - B_{ji}^k b_k)y^i = Q_{i0}$ in (3), we have

$$L_\alpha P_{i00} = \alpha L_\beta Q_{i0}, \quad (26)$$

where the index zero means, as usual, contraction by y^i. It is remarked that for a locally Minkowski space, P_{i00} and Q_{i0} are polynomials in y^i of degree 2 and 1, respectively. If (26) gives $P_{i00} = Q_{i0} = 0$ necessarily, then we have $B_{ji}^k = 0$ and $b_{j:i} = 0$, and (4) shows that $R_{hijk} = 0$. Consequently, the Finsler space with (α, β)-metrics defines a flat-parallel Minkowski space.

In this section, we shall apply the above procedure to the generalized (α, β)-metric (9).

Substituting (10) into (26), we have

$$(\alpha + \beta(1-p))P_{i00} - \alpha^2 pQ_{i0} = 0. \quad (27)$$

Since α is irrational in y^i, (27) leads us to

$$\begin{aligned}P_{i00} &= 0,\\ -\alpha^2 pQ_{i0} + \beta(1-p)P_{i00} &= 0.\end{aligned} \quad (28)$$

Thus, from (28), we have $P_{i00} = Q_{i0} = 0$. Hence, we conclude the following.

Theorem 6. *A Finsler space with the generalized (α, β)-metric (9) is a flat-parallel Minkowski space.*

5. β-Change with Generalized (α, β)-Metric

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler spaces on the same underlying manifold M^n. *If* any geodesic of F^n is a geodesic of $\overline{F}^n$ and vice versa, then F^n is called projective to $\overline{F}^n$ and change σ: $L \longrightarrow \overline{L}$ of the metric is called projective. It is well known that σ is projective if and only if there exists a positively homogeneous function $P(x, y)$ of degree 1 in y^i satisfying $\overline{G} = G^i + Py^i$.

On the other hand, we shall introduce β-change [13] as follows.

Definition 7. Let $L(\alpha, \beta)$ be an (α, β)-metric. The change ϕ: $\alpha \longrightarrow L(\alpha, \beta)$ of the metric is called β-change.

If we denote by R^n the associated Riemannian space with a Finsler space F^n with (α, β)-metric, then the β-change is the change from R^n to F^n. There is a theorem between projective change and β-change as follows:

Theorem 8 (see [13]). *A β-change is projective, if and only if we have*

$$L_\beta \psi_{ij} + L_{\beta\beta} \Omega_{ij} = 0, \tag{29}$$

where $\psi_{ij} = (b_{i;j} - b_{j;i})/2$, $\Omega = (p_i \beta_{;j} - p_j \beta_{;i})/2$ and $\beta_{;j} = b_{i;j} y^j$.

Now, we consider a change φ: $\alpha \longrightarrow L = \alpha(1 + (\beta/\alpha))^p$. Then, from Theorem 8, we can obtain the condition for a change φ to be projective.

Substituting (10) into (29), we have

$$\alpha \psi_{ij} + \beta \psi_{ij} + (p-1)\Omega_{ij} = 0. \tag{30}$$

Since α is an irrational polynomial of y^i, (30) leads us to $\psi_{ij} = 0$. Substituting this into (30), by virtue of $(p-1) \neq 0$, we have $\Omega_{ij} = 0$. Further, contracting this by y^i and using $p_i y^i = 0$ and $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, then it satisfies (30). Thus, we have the following.

Theorem 9. *A change φ: $\alpha \longrightarrow L = \alpha(1 + (\beta/\alpha))^p$ is projective if and only if we have $b_{i;j} = 0$.*

From the above theorem and (30), we have discussed three cases as follows: (i) $p = 1$, (ii) $p = 2$, and (iii) $p = 1/2$.

Case (i): $p = 1$

Let $p = 1$ in (30), then we have $(\alpha + \beta)\psi_{ij} = 0$. Since $(\alpha + \beta) \neq 0$, we have $\psi_{ij} = 0$. From $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies $(\alpha + \beta)\psi_{ij} = 0$. Thus, we have the following.

Corollary 10 (see [13]). *A Randers change $\alpha \longrightarrow \alpha + \beta$ is projective if and only if we have $b_{i;j} = 0$.*

Case (ii): $p = 2$

Let $p = 2$ in (30), then we have

$$(\alpha + \beta)\psi_{ij} + \Omega_{ij} = 0. \tag{31}$$

Since α is irrational in y^i, we have $\psi_{ij} = 0$. Substituting $\psi_{ij} = 0$ in (31), we have $\Omega_{\mathrm{ij}} = 0$. Further, contracting this by y^i and using $p_i y^i = 0$ and $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies (31). Thus, we have the following.

Corollary 11. *A Berwald change $\alpha \longrightarrow (\alpha + \beta)^2/\alpha$ is projective, if and only if we have $b_{i;j} = 0$.*

Case (iii): $p = 1/2$

Let $p = 1/2$ in (30), then we have

$$2(\alpha + \beta)\psi_{ij} - \Omega_{ij} = 0. \tag{32}$$

Since α is irrational in y^i, we have $\psi_{ij} = 0$. Substituting $\psi_{ij} = 0$ in (32), we have $\Omega_{ij} = 0$. Further, contracting this by y^i and using $p_i y^i = 0$ and $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies (32). Thus, we have the following.

Corollary 12. *A square root metric change $\alpha \longrightarrow \sqrt{\alpha(\alpha + \beta)}$ is projective, if and only if we have $b_{i;j} = 0$.*

On the other hand, in Theorem 5, we dealt with the condition that a Finsler space with the generalized (α, β)-metric be projectively flat. By combining Theorems 5 and 9, we can give a more geometrical meaning, similar to Matsumoto's Theorem ([5], Theorem 2). Thus, we have the following.

Corollary 13. *A Finsler space with the generalized (α, β)-metric (9) is projectively flat if and only if a change φ: $\alpha \longrightarrow L = \alpha(1 + (\beta/\alpha))^p$ is projective and the associated Riemannian space with metric α is projectively flat.*

6. Conclusion

The infinitesimal transformations in Finsler geometry, such as conformal, projective, semiprojective, and β-changes, play an important role not only in differential geometry but also in application to other branches of science, especially in the process of geometrization of physical theories. In this paper, we have obtained results concerning the projective flatness and flat parallelness of the generalized (α, β)-metric (9). Further, we have shown that the β-change φ: $\alpha \longrightarrow L = \alpha(1 + (\beta/\alpha))^p$ of the aforementioned metric (9) is projective. Also, we have discussed how β-change is projective for some important Finsler metrics arising from the generalized (α, β)-metric (9).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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