Hindawi
Journal of Mathematics
Volume 2024, Article ID 4461499, 5 pages
https://doi.org/10.1155/2024/4461499

Research Article

New Proof of the Property of Stirling Number Based on Fubini Polynomials

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Received 17 November 2023; Revised 15 March 2024; Accepted 4 May 2024; Published 17 May 2024

Academic Editor: Shaofang Hong

The main purpose of this article is using the elementary methods and the properties of the Fubini polynomials to study the congruence properties of a signless Stirling number of the first kind and solve a conjecture proposed by J. H. Zhao and Z. Y. Chen. Without a doubt, the novel approach employed in this work provides a useful reference for researching the congruence properties of other nonlinear binary recursive sequences.

1. Introduction

For any integer $n \geq 0$, the n-th Fubini number F_n represents the number of ways to partition a set of n elements into weakly ordered subsets (see [1]) or the number of distinct arrangements of sums and integrals in Fubini's theorem (see [2]). The exponential generating function of Fubini numbers is given by (see [3])

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} F_n \cdot \frac{t^n}{n!}. \tag{1}$$

One of the defining recurrence relations for Fubini numbers is given by (see [4])

$$F_n = \sum_{k=0}^{n-1} \binom{n}{k} \cdot F_k, \quad n \geq 1, \tag{2}$$

where $F_0 = 1$, $F_1 = 1$. An appropriate deformation of the above definition yields the formula

$$F_n = -2\sum_{k=0}^{n-1} \binom{n}{k} \cdot (-1)^{n-k} F_k, \quad n \geq 1. \tag{3}$$

In addition, the Fubini polynomials $\{F_n(x)\}$ are determined by the coefficients of the power series expansion of $F(t)$ with respect to t (see [5, 6]). That is,

$$F(t) = \frac{1}{1-x\left(e^t-1\right)} = \sum_{n=0}^{\infty} F_n(x) \cdot \frac{t^n}{n!}. \tag{4}$$

Fubini polynomials and numbers are closely connected with the Stirling numbers. Kim et al. [7] proved the identity

$$F_n(x) = \sum_{k=0}^{n} S_2(n,k) k! x^k, \quad n \geq 0, \tag{5}$$

where $S_2(n,k)$ are the Stirling numbers of the second kind. In this way, we can get easily through $S_2(n,k)$ the initial five terms of $F_n(x)$, respectively, $F_0(x) = 1$, $F_1(x) = x$, $F_2(x) = 2x^2 + x$, $F_3(x) = 6x^3 + 6x^2 + x$, $F_4(x) = 24x^4 + 36x^3 + 14x^2 + x$. It is clear that $F_n = F_n(1)$ for all integers $n \geq 0$.

In recent years, some people had studied the various properties of the Fubini numbers and polynomials in different methods. For instance, Dil et al. [8] proved some properties of geometric and exponential polynomials and numbers by using the Euler–Seidel matrix method. Kim et al. [9, 10] investigated the properties of degenerated Fubini polynomials and higher-order degenerated Fubini polynomials in depth, using generating functions and specific differential operators.

Besides, Diagana et al. [11] proved the following conclusion. Let q and n be positive integers. Then Fubini numbers satisfy the congruence

$$(2^q - 1)F_n \equiv nqF_{n-1} + \sum_{j=1}^{q-1} 2^{q-j-1} j^n \left(\bmod q^2\right). \qquad (6)$$

Furthermore, Zhao et al. [12] introduced a new sequence $a(i, j)$, and they proved the identity

$$\sum_{a_1+a_2+a_3+\cdots+a_i=n} \frac{F_{a_1}(x)}{a_1!} \cdot \frac{F_{a_2}(x)}{a_2!} \cdot \frac{F_{a_3}(x)}{a_3!} \cdots \frac{F_{a_i}(x)}{a_i!} = \frac{1}{(x+1)^{i-1} \cdot (i-1)! \cdot n!} \cdot \sum_{j=1}^{i} a(i, j) \cdot F_{n+j-1}(x), \qquad (7)$$

where the summation is over all i-tuples with nonnegative integer coordinates $(a_1, a_2, a_3, \ldots, a_i)$ such that $a_1 + a_2 + a_3 + \cdots + a_i = n$, and the definition of sequence $\{a(i, j)\}$ is as follows. For any integers $i, j \geq 0$, the sequence $a(i, j)$ is defined as $a(i, j) = 0$ with $j > i$, $a(i, i) = 1$, and $a(i, 1) = (i-1)!$. For $i = 1, 2, \ldots, n$, the sequence $a(i, j)$ satisfies the following recursion:

$$a(i+1, j) = i \cdot a(i, j) + a(i, j-1). \qquad (8)$$

Table 1 provides the values of $\{a(i, j)\}_{j=0}^{i}$ for $0 \leq i \leq 7$.

From these data in the table one can find that $3|a(3, 2)$; $5|a(5, j)$, $j = 2, 3, 4$; $7|a(7, j)$, $j = 2, 3, 4, 5, 6$. So Zhao et al. [12] also proposed the following conjecture. For any odd prime p, one has the congruence

$$a(p, j) \equiv 0 (\bmod \ p), \quad 2 \leq j \leq p - 1. \qquad (9)$$

We were pleasantly surprised to discover that sequence $\{a(i, j)\}$ was exactly signless Stirling number of the first kind (see [13]) when we delved into this conjecture, and the conjecture has been proved using the methods of combinatorial mathematics (see [14]).

This paper as note of [13, 14], where some scholars utilized generating functions and other advanced concepts in their previous proofs, potentially making them challenging for readers to understand. By contrast, we give a very simple and elementary proof for this conjecture; even it does not involve any symbols and concepts of combination. This work gives the congruence of signless Stirling number of the first kind by using the properties of Fubini polynomials and derivatives, which is unprecedented. That is, we have the following result.

Theorem 1. *Let p be an odd prime. Then for any integer j with $2 \leq j \leq p - 1$, we have the congruence*

$$a(p, j) \equiv 0 (\bmod p). \qquad (10)$$

From this theorem and the recursive formula $a(i+1, j) = i \cdot a(i, j) + a(i, j-1)$ we may immediately deduce the following result.

Corollary 2. *Let p be an odd prime. Then for any integer j with $3 \leq j \leq p$, we have*

$$a(p+1, j) \equiv 0 (\bmod p). \qquad (11)$$

2. Two Simple Lemmas

To complete the proof of our main conclusion, we need two elementary lemmas. Of course, the proofs of these lemmas need some knowledge of elementary number theory or combinatorial analysis; all these kinds of simple knowledge can be found in references [15, 16]. Firstly, we have the following lemma.

Lemma 3. *For any positive integer j, we have the identities*

$$F_j^{(j)}(0) = (j!)^2, \qquad F_{j+1}^{(j)}(0) = \frac{j(j+1)}{2} \cdot (j!)^2. \qquad (12)$$

Proof. For the convenience of writing, we define the binary function $F(t, x)$ as

$$F(t, x) = \frac{1}{1 - x\left(e^t - 1\right)} = \sum_{n=0}^{\infty} F_n(x) \cdot \frac{t^n}{n!}. \qquad (13)$$

From the definition of the partial derivatives we have identity

$$\frac{\partial F(t, x)}{\partial x} = \frac{e^t - 1}{\left(1 - x\left(e^t - 1\right)\right)^2} = \sum_{n=1}^{\infty} F_n'(x) \cdot \frac{t^n}{n!}, \qquad \frac{\partial^2 F(t, x)}{\partial x^2} = \frac{2\left(e^t - 1\right)^2}{\left(1 - x\left(e^t - 1\right)\right)^3} = \sum_{n=2}^{\infty} F_n''(x) \cdot \frac{t^n}{n!}. \qquad (14)$$

Generally, for any integer $j \geq 1$, we also have

$$\frac{\partial^j F(t, x)}{\partial x^j} = \frac{j!\left(e^t - 1\right)^j}{\left(1 - x\left(e^t - 1\right)\right)^{j+1}} = \sum_{n=j}^{\infty} F_n^{(j)}(x) \cdot \frac{t^n}{n!}. \qquad (15)$$

Taking $x = 0$ in formula (15) we have

$$j!\left(e^t - 1\right)^j = \sum_{n=j}^{\infty} F_n^{(j)}(0) \cdot \frac{t^n}{n!}. \qquad (16)$$

Using the power series expansion

$$\left(e^t - 1\right)^j = \left(\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!}\right)^j = t^j \cdot \left(\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}\right)^j, \qquad (17)$$

Table 1: Values of $a(i, j)$.

$a(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
$i=0$	1							
$i=1$		1						
$i=2$		1	1					
$i=3$		2	3	1				
$i=4$		6	11	6	1			
$i=5$		24	50	35	10	1		
$i=6$		120	274	225	85	15	1	
$i=7$		720	1764	1624	735	175	21	1

then comparing the coefficients of t^j and t^{j+1} on both sides of formula (16) yields

$$F_j^{(j)}(0) = (j!)^2,$$
$$F_{j+1}^{(j)}(0) = \frac{j(j+1)}{2} \cdot (j!)^2. \tag{18}$$

This proved Lemma 3. □

Lemma 4. *For any positive integer n, we have the identity*

$$F_n(x) = x \cdot \sum_{j=0}^{n-1} \binom{n}{j} \cdot F_j(x). \tag{19}$$

Proof. This can be easily proved by the definition of $F_n(x)$. In fact from the definition of $F_n(x)$ and the power series expansion of the functions, we have

$$\begin{aligned} 1 &= \left(\sum_{n=0}^{\infty} F_n(x) \cdot \frac{t^n}{n!} \right) \cdot \left(1 - x\left(e^t - 1\right)\right) \\ &= \sum_{n=0}^{\infty} F_n(x) \cdot \frac{t^n}{n!} - xt \cdot \left(\sum_{n=0}^{\infty} F_n(x) \cdot \frac{t^n}{n!} \right) \cdot \left(\sum_{m=0}^{\infty} \frac{t^m}{(m+1)!} \right) \\ &= \sum_{n=0}^{\infty} F_n(x) \cdot \frac{t^n}{n!} - xt \cdot \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{F_j(x)}{j!} \cdot \frac{1}{(n-j+1)!} \right) \cdot t^n. \end{aligned} \tag{20}$$

For $n \geq 1$, comparing the coefficients of t^n in formula (20) we have

$$\frac{F_n(x)}{n!} = x \sum_{j=0}^{n-1} \frac{F_j(x)}{j!} \cdot \frac{1}{(n-j)!}, \tag{21}$$

or

$$F_n(x) = x \cdot n! \cdot \sum_{j=0}^{n-1} \frac{F_j(x)}{j!} \cdot \frac{1}{(n-j)!} = x \cdot \sum_{j=0}^{n-1} \binom{n}{j} \cdot F_j(x). \tag{22}$$

This proves Lemma 4.

3. Proof of the Theorem

In this section, we use formula (7) to complete the proof of our theorem. Firstly, taking $n = 1$ and $i = p$ in formula (7) and observing that $F_0(x) = 1$, $F_1(x) = x$, $a(i, 1) = (i-1)!$, and $a(i, i) = 1$, we have the identity

$$\sum_{j=1}^{p} a(p, j) \cdot F_j(x) = x \cdot (x+1)^{p-1} \cdot p!. \tag{23}$$

We know from Lemma 4 that

$$F_p(x) = x \cdot \sum_{j=0}^{p-1} \binom{p}{j} \cdot F_j(x). \tag{24}$$

Now combining formula (23) and (24), we obtain

$$\begin{aligned} &\sum_{j=2}^{p-1} a(p, j) \cdot F_j(x) \\ &\quad = x \cdot (x+1)^{p-1} \cdot p! - x \cdot (p-1)! - x \cdot \sum_{j=0}^{p-1} \binom{p}{j} \cdot F_j(x). \end{aligned} \tag{25}$$

From Lemma 4 we know that $F_n(x)$ is a n-degree polynomial of x; then $F_j^{(k)}(x) = 0$, if $k > j$. Let polynomial $g(x) = x \cdot (x+1)^{p-1}$. Then finding the $(p-1)$-order derivative of x on both sides of formula (25) and taking $x = 0$, from Lemma 3, we have the identity

$$\begin{aligned} a(p, p-1) \cdot ((p-1)!)^2 &= g^{(p-1)}(0) \cdot p! - (p-1) \cdot \sum_{j=0}^{p-1} \binom{p}{j} \cdot F_j^{(p-2)}(0) \\ &\equiv (p-1) \cdot \binom{p}{p-1} \cdot F_{p-1}^{(p-2)}(0) \equiv 0 \,(\bmod p). \end{aligned} \tag{26}$$

Since $((p-1)!, p) = 1$, from formula (26) we have the congruence

$$a(p, p-1) \equiv 0 \,(\bmod p). \tag{27}$$

Now finding the $(p-2)$-order derivative of x on both sides of (25) and then taking $x = 0$, from formula (27) and Lemma 3, we have the congruence

$$\begin{aligned} a(p,p-2)\cdot F_{p-2}^{(p-2)}(0) &\equiv g^{(p-2)}(0)\cdot p! - (p-2)\sum_{j=0}^{p-1}\binom{p}{j}\cdot F_j^{(p-3)}(0) \\ &\equiv -(p-2)\sum_{j=p-3}^{p-1}\binom{p}{j}\cdot F_j^{(p-3)}(0)\,(\bmod p). \end{aligned} \tag{28}$$

Note that $(F_{p-2}^{(p-2)}(0),p) = (((p-2)!)^2,p) = 1$ and $p \mid \binom{p}{j} (j = p-3, p-2, p-1)$. From formula (28), we may immediately deduce the congruence

$$a(p,p-2) \equiv 0\,(\bmod p). \tag{29}$$

If congruence holds for all integers j with $2 \le j \le p-1$, that is,

$$a(p,j) \equiv 0\,(\bmod p), \tag{30}$$

then from formula (25) and the methods of proving formulas (27) and (29), we can easily deduce the congruence

$$a(p,j-1) \equiv 0\,(\bmod p). \tag{31}$$

In fact finding the j-order derivative of x on both sides of formula (25) and then taking $x = 0$, from Lemma 3, we have the congruence

$$\begin{aligned} a(p,j)\cdot(j!)^2 &\equiv g^{(j)}(0)\cdot p! - j \\ &\quad\cdot\sum_{i=0}^{p-1}\binom{p}{i}\cdot F_i^{(j-1)}(0) \equiv 0\,(\bmod p), \end{aligned} \tag{32}$$

and thus

$$a(p,j) \equiv 0\,(\bmod p). \tag{33}$$

This completes the proof of theorem by mathematical induction.

4. Conclusion

This paper's principal contribution is to solve a conjecture proposed by Zhao and Chen in [10] with a completely new method. Specifically, let p be an odd prime. Then we have the congruence

$$a(p,j) \equiv 0\,(\bmod p), \quad \text{for all integers } 2 \le j \le p-1. \tag{34}$$

This work gives the congruence of signless Stirling number of the first kind by using the properties of Fubini polynomials and derivatives, which is unprecedented. The method of theorem proving is innovative and skillful. This method has some reference value for researching combinatorial theory and properties of other binary sequences.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have equally contributed to this work. All authors have read and approved the final manuscript.

Acknowledgments

This work was supported by the N. S. F. (12126357) of P. R. China.

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