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Research Article

Global Well-Posedness and Convergence Results to a 3D Regularized Boussinesq System in Sobolev Spaces

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We consider a regularized periodic three-dimensional Boussinesq system. For a mean free initial temperature, we use the coupling between the velocity and temperature to close the energy estimates independently of time. This allows proving the existence of a global in time unique weak solution. Also, we establish that this solution depends continuously on the initial data. Moreover, we prove that this solution converges to a Leray-Hopf weak solution of the three-dimensional Boussinesq system as the regularizing parameter vanishes.

1. Introduction

Motivated by ([1]) and references therein, we consider the regularization to the periodic three-dimensional Boussinesq system (Bq_α) given by

$$\partial_t v - \nu\Delta v + (u\cdot\nabla)u = -\nabla p + \theta e_3, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{1}$$

$$\partial_t \theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{2}$$

$$v = u - \alpha^2\Delta u, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{3}$$

$$\operatorname{div} u = \operatorname{div} v = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3, \tag{4}$$

$$(u,\theta)|_{t=0} = \left(u^0,\theta^0\right), \quad x\in\mathbb{T}^3, \tag{5}$$

where the unknown velocity, the unknown pressure, and the unknown temperature are, respectively, the three-dimensional vector u, the scalars p, and the scalar θ. The parameters $\nu,\kappa,\alpha>0$ denote, respectively, the viscosity, the thermal conductivity of the fluid, and the regularizing parameter, $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is the three-dimensional torus, u^0 is a given divergence-free initial velocity, and θ^0 is a given mean free initial temperature. The vector $e_3 = (0, 0, 1)^T$.

The periodic three-dimensional Boussinesq system models geophysical fluids such as oceanographic turbulence and atmospheric fronts as well as the Rayleigh–Benard convection [2]. More physical application for the Boussinesq system can be found in [3] and related references. It is known that available mathematical methods do not allow proving the global well-posedness of the three-dimensional fluid equations such as the Boussinesq system, especially in Sobolev spaces which are energy spaces frequently used in real word applications. To make practical advances in this field, researchers took the way of regularisation. In this framework, the idea in [4] was to suggest a particular closure model for the Navier-Stokes equations by approximating the Reynolds stress tensor. This model was simplified in [5] and a mathematical study was performed therein. Existence and uniqueness results in [5] were improved in [6].

The closest reference to our manuscript is [1], where the author proved that a weak solution exists to (Bq_α), $\alpha>0$. This solution depends continuously on initial data and it converges to a weak solution of $(Bq_{\alpha=0})$, as the regularizing parameter $\alpha \longrightarrow 0$. However, it is clear that in Theorem 1 of [1], the right-hand side of the energy estimate depends on time. Thus, the solution belongs to $L^{\infty}_{loc}(\mathbb{R}_+, H^1(\mathbb{T}^3))$ as it will blow up when $T \longrightarrow \infty$. This local aspect is due to the classical arguments based on a brutal application of Cauchy–Schwarz inequality while taking the scalar product of the Buoyancy force θe_3 with the velocity field u. Thus, this was not a global in-time solution but it should be called a large time solution. This insufficiency appeared widely in the literature and is still appearing both in the three-dimensional case and the two-dimensional case which is supposed to be well understood, see among a wide literature [1, 7–12] and references therein. Here, we overcome this insufficiency for the range of mean free initial temperature and we make two improvements that are interesting from an applicable point of view.

The first is to obtain a global in-time weak solution, under minimal regularity requirements. This is the main contribution of this paper. For physicists and engineers, global in-time solutions are closely related to durable in time operating of machines, systems, and networks. Thus, physicists and engineers usually try to start with suitable initial data to avoid blowup in finite time. For mathematician, global in-time solutions open the way to study the long time behavior [13, 14], the existence of the attractors [15], the asymptotic stability [16], and in general, all topics requiring $t \longrightarrow \infty$. In numerical analysis of nonlinear system, although the numerical discretization is generally local in time, the existence of a global in-time solution gives the possibility to extend such numerical discretization, by translation in time. Also, based on [1] and using continuity in time, we deduce that our global solution is continuously dependent on the initial data and in particular, it is unique. First, we recall that the uniqueness of weak solution in energy spaces is still an open problem for three-dimensional fluid equations. In the literature, such uniqueness is the main target behind any regularisation. Second, we note that from an applied mathematical point of view, we seek for a nearby solution to arise from nearby initial data. Otherwise, we will never believe in any computer calculations, for example. For physicists and engineers, it is interesting that when starting with an initial state, the system described by a given partial differential equation should evolve towards an only one future state.

The second is that our solution converges to a global in-time Leray-Hopf type weak solution of the three-dimensional Boussinesq system, as the regularizing parameter $\alpha \longrightarrow 0$. Convergence result is one of the main features of the α-regularisation. First, in practical situations, it allows to consider systems with $\alpha > 0$ as small as required and fully profit from uniqueness and continuous dependence, while keeping nearby a weak solution of the three-dimensional Boussinesq system. Second, from a theoretical point of view, it is indeed a different mathematical method to prove the existence of a weak solution to the three-dimensional Bousssinesq system. This solution is the existing limit. Similar results were proved, as the Rossby number vanishes, in [17, 18] for example.

Let us mention that starting with a mean free initial temperature, such as sinusoidal initial heating sources, is frequent in natural phenomenons and compulsory in many real word applications; see [19] and the multitude references therein in the case of industrial applications or [20] for applications in medicine and health sciences. In [21, 22], authors used the mean free condition to investigate the long time behavior of the solution and to prove an exponential stability result for the periodic 3D Navier-Stokes equations, in critical Sobolev spaces.

Given a Banach space $(X, \|.\|_X)$, the Bochner space $L^p([0,T], X)$ is the space of all functions such that

$$\|u\|_{L^p([0,T],X)} = \left(\int_0^T \|u(t)\|_X^p \mathrm{dt}\right)^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty,$$
$$\|u\|_{L^\infty([0,T],X)} = \operatorname{ess}\sup_{t\in[0,T]} \|u(t)\|_X < \infty. \tag{6}$$

If we denote by s a real number, by $\widehat{u}$ the Fourier transform of u and by $\mathcal{S}'(\mathbb{T}^3)$ the Schwartz space, then the homogeneous Sobolev spaces are given by

$$\dot{H}^s(\mathbb{T}^3) = \left\{\widehat{u} \in \mathcal{S}'(\mathbb{T}^3), \sum_{k\in\mathbb{Z}^3} |k|^{2s} |\widehat{u}(k)|^2 < \infty\right\}, \tag{7}$$

and endowed with the natural norm $\|u\|_{\dot{H}^s(\mathbb{T}^3)} = (\sum_{k\in\mathbb{Z}^3} |k|^{2s} |\widehat{u}(k)|^2)^{1/2}$.

The paper is organized as follows. In the following section, we will prove that a continuous global in time weak solution exists and depends continuously on the initial data and in particular, it is unique. In the last section, we will establish that this solution converges to a global in time Leray-Hopf type solution, as the regularizing parameter $\alpha \longrightarrow 0$.

2. Existence Results

In the following, we give formal estimates for a Galerkin approximating scheme to system (Bq_α). We omit the approximating system and the index of the approximating sequence. Interested readers can see [1] for full details. Taking the inner product in $L^2(\mathbb{T}^3)$ of (1) with u and (2) with θ, we obtain

$$\frac{1}{2}\frac{d}{\mathrm{dt}}\left(\|u\|_{L^2}^2 + \alpha^2 \|\nabla u\|_{L^2}^2\right) + \nu\left(\|\nabla u\|_{L^2}^2 + \alpha^2 \|\Delta u\|_{L^2}^2\right) = \langle \theta e_3, u\rangle_{L^2}, \tag{8}$$

and

$$\frac{1}{2}\frac{d}{\mathrm{dt}}\|\theta\|_{L^2(\mathbb{T}^3)}^2 + \kappa \|\nabla\theta\|_{L^2(\mathbb{T}^3)}^2 = 0. \tag{9}$$

Integrating (2) with respect to x, we infer that the first Fourier coefficient of θ is conserved during time, that is, $C_0(\theta(t)) = C_0(\theta^0)$, $\forall t > 0$. Since the initial temperature θ^0 is mean free, it follows that $C_0(\theta^0) = 0$. So, $C_0(\theta(t)) = 0$, $\forall\, t > 0$. Thus,

$$\langle \theta e_3, u\rangle_{L^2} = \sum_{k\neq(0,0,0)} \widehat{\theta}_n(k)\widehat{u}_n^3(k). \tag{10}$$

Applying the Cauchy–Schwarz inequality and Young inequality to (10), it holds

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2\right) + \nu\left(\|\nabla u\|_{L^2}^2 + \alpha^2\|\Delta u\|_{L^2}^2\right) \leq \frac{1}{2\nu}\|\theta\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u\|_{L^2}^2. \tag{11}$$

Integrating (9) and (11) with respect to time and summing, it follows that

$$\|\theta(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2 + 2\kappa\int_0^t \|\nabla\theta(\tau)\|_{L^2}^2 + \nu\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau + 2\nu\alpha^2\int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq \left\|\theta^0\right\|_{L^2}^2 + \left\|u^0\right\|_{L^2}^2 + \alpha^2\left\|\nabla u^0\right\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|\theta\|_{L^2}^2 d\tau. \tag{12}$$

By Poincaré inequality, one has

$$\int_0^t \|\theta\|_{L^2}^2 d\tau \leq \int_0^t \|\nabla\theta\|_{L^2}^2 d\tau. \tag{13}$$

Above, we have a unitary Poincaré constant. In fact,

$$\|\theta\|_{L^2(\mathbb{T}^3)}^2 = \sum_{k\in\mathbb{Z}^3, k\neq(0,0,0)} |\widehat{\theta}(k)|^2 \leq \sum_{k\in\mathbb{Z}^3, k\neq(0,0,0)} |k|^2|\widehat{\theta}(k)|^2 = \|\nabla\theta\|_{L^2(\mathbb{T}^3)}^2, \tag{14}$$

where we used successively that θ is mean free and that $|k| \geq 1$.

The integral with respect to time of (9) gives

$$\int_0^t \|\theta\|_{L^2}^2 d\tau \leq \left\|\theta^0\right\|_{L^2}^2. \tag{15}$$

Finally, we are able to close the estimates independently on time as follows:

$$\|\theta(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2 + 2\kappa\int_0^t \|\nabla\theta(\tau)\|_{L^2(\mathbb{T}^3)}^2 + \nu\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau + 2\nu\alpha^2\int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C\left(\alpha, \nu, \kappa, u^0, \theta^0\right), \tag{16}$$

where $C(\alpha, \nu, \kappa, u^0, \theta^0) = \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + (1 + (1/2\nu\kappa))\|\theta^0\|_{L^2}^2$. A standard compactness argument finishes the proof of the existence part in Theorem 1. To do so, we take the limit using Aubin compactness lemma [23]. Continuity in time of the existing weak solution to (Bq_α) can be proved in a classical manner as in the case of the weak solutions to three-dimensional Navier-Stokes equations [24]. Also, details were provided in [7] for the case of the strong solution to (Bq_α).

In [1], the author established the continuous dependence of the weak large time solution with respect to the initial data on $[0; T], T > 0$. In particular, he deduced that this large time solution was unique. In our case, as the global solution of (Bq_α) is continuous in time, continuous dependence on initial data and uniqueness follow over $\mathbb{R}_+$. In conclusion, we have the following theorem.

Theorem 1. *Let $\theta^0 \in L^2(\mathbb{T}^3)$ be a mean free scalar function and let $u^0 \in \dot{H}^1(\mathbb{T}^3)$ be a divergence-free vector field. Then, there exists a global in-time weak solution $(u_\alpha, \theta_\alpha)$ of system (Bq_α) such that u_α belongs to $C(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate (16) and depends continuously on the initial data. In particular, it is unique.*

3. Convergence Results

In this section, we will prove the following theorem.

Theorem 2. *Let $\theta_0 \in L^2(\mathbb{T}^3)$ be a mean free scalar function, $u_0 \in \dot{H}^1(\mathbb{T}^3)$ be a divergence-free vector field and $(u_\alpha, \theta_\alpha)$ the solutions of system (Bq_α), and $v_\alpha = u_\alpha - \alpha^2\Delta u_\alpha$. Then, there are subsequences u_{α_k}, v_{α_k}, and θ_{α_k}, a scaler function $\breve{\theta}$, and a divergence-free vector field $\breve{u}$ belonging both of them to $L^\infty(\mathbb{R}_+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$ such that as $\alpha_k \longrightarrow 0^+$, the following holds.*

(1) The sequence u_{α_k} converges to $\breve{u}$ and θ_{α_k} converges to $\breve{\theta}$ weakly in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$ and v_{α_k} converges to $\breve{u}$ weakly in $L^2(\mathbb{R}_+, L^2(\mathbb{T}^3))$.

(2) The sequence u_{α_k} converges to $\breve{u}$ and θ_{α_k} converges to $\breve{\theta}$ strongly in $L^2(\mathbb{R}_+, L^2(\mathbb{T}^3))$ and v_{α_k} converges to $\breve{u}$ strongly in $L^2(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{T}^3))$.

(3) The sequence $(u_{\alpha_k}, \theta_{\alpha_k})$ converges to $(\breve{u}, \breve{\theta})$ weakly in $(L^2(\mathbb{T}^3))^2$ and uniformly over $\mathbb{R}_+$. Furthermore, $(\breve{u}, \breve{\theta})$ is a Leary-Hopf-type weak solution of the Boussinesq system (Bq_0) and satisfies for all $t \in \mathbb{R}_+$ the energy inequality

$$\|\breve{u}(t)\|^2_{L^2(\mathbb{T}^3)} + \|\breve{\theta}(t)\|^2_{L^2(\mathbb{T}^3)} + 2\int_0^t \left(\nu\|\nabla\breve{u}(t)\|^2_{L^2(\mathbb{T}^3)} + \kappa\|\nabla\breve{\theta}(\tau)\|^2_{L^2(\mathbb{T}^3)}\right)d\tau \leq \left\|u^0\right\|^2_{L^2(\mathbb{T}^3)} + \left(1 + \frac{1}{2\nu\kappa}\left\|\theta^0\right\|^2_{L^2(\mathbb{T}^3)}\right). \tag{17}$$

3.1. Proof of Statement (1). Since the regularizing parameter $\alpha \longrightarrow 0^+$, there exists some fixed α_0 such that $0 < \alpha \leq \alpha_0$. Taking $\alpha = \alpha_0$ in the right-hand side of (16), we obtain for all $t \in \mathbb{R}_+$,

$$\begin{aligned}&\left\|\theta_\alpha(t)\right\|^2_{L^2} + \left\|u_\alpha(t)\right\|^2_{L^2} + \alpha^2\left\|\nabla u_\alpha(t)\right\|^2_{L^2} + 2\kappa\int_0^t \left\|\nabla\theta_\alpha(\tau)\right\|^2_{L^2}d\tau \\ &+ \nu\int_0^t \left\|\nabla u_\alpha(\tau)\right\|^2_{L^2}d\tau + 2\nu\alpha^2\int_0^t \left\|\Delta u_\alpha(\tau)\right\|^2_{L^2}d\tau \\ &\leq \left\|u^0\right\|^2_{L^2} + \alpha_0^2\left\|\nabla u^0\right\|^2_{L^2} + \left(1 + \frac{1}{2\nu\kappa}\right)\left\|\theta^0\right\|^2_{L^2}.\end{aligned} \tag{18}$$

Above, we added the index α to make precision that the temperature and the velocity depend implicitly on α. By (18), both of θ_α and u_α are uniformly bounded in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$. Hence, the Banach–Alaoglu theorem [25] applied in the framework of Hilbert spaces allows to extract subsequences $(\theta_{\alpha_k})_k$, $(u_{\alpha_k})_k$, and $(v_{\alpha_k})_k$ such that

$$\begin{aligned}\left(\theta_{\alpha_k}, u_{\alpha_k}, v_{\alpha_k}\right) \rightharpoonup (\theta, u, v) \text{ weakly in } \left(L^2\left(\mathbb{R}_+, \dot{H}^1\right)^2 \right.\\ \times \left(L^2\left(\mathbb{R}_+, L^2\right), \text{as } \alpha_k \longrightarrow 0^+.\end{aligned} \tag{19}$$

3.2. Proof of Statement (2). To deal with the strong convergence, we will apply the Aubin–Lions lemma [23]. This necessitates uniform estimates of the time derivatives of θ_{α_k}, of u_{α_k}, and of v_{α_k} in the appropriate spaces.

In the following, K is a real positive constant that may change from line to line. For all positive time, θ_{α_k} is bounded independently of α, in $L^2([0,T], \dot{H}^1(\mathbb{T}^3))$, $T > 0$. As the Sobolev spaces form a decreasing chain and by definition of the homogeneous Sobolev norm, $\Delta\theta_{\alpha_k}$ is uniformly bounded with respect to α in $L^2([0,T], \dot{H}^{-1}(\mathbb{T}^3))$. Using Sobolev norm properties and Sobolev product laws, it holds that

$$\begin{aligned}\int_0^T \left\|\operatorname{div}\theta_{\alpha_k}u_{\alpha_k}\right\|^2_{\dot{H}^{-3/2}} &\leq \int_0^T \left\|\theta_{\alpha_k}\right\|^2_{L^2}\left\|u_{\alpha_k}\right\|^2_{\dot{H}^1} \\ &\leq \|\theta\|^2_{L^\infty_T(L^2)}\|u\|^2_{L^2_T(\dot{H}^1)} \\ &\leq \|\theta\|^2_{L^\infty_{\mathbb{R}_+}(L^2)}\|u\|^2_{L^2_{\mathbb{R}_+}(\dot{H}^1)}.\end{aligned} \tag{20}$$

The above estimates of the diffusion and the advection terms lead to

$$\left\|\frac{d}{\mathrm{dt}}\theta_{\alpha_k}\right\|_{L^2_T(\dot{H}^{-3/2})} \leq K, \tag{21}$$

where $\dot{H}^{-3/2}$ is the dual space of the homogeneous Sobolev space $\dot{H}^{3/2}(\mathbb{T}^3)$ and $L^2_T(\dot{H}^{-3/2})$ is the Bochner space as defined in the introduction. Applying the operator $(I - \alpha^2\Delta)^{-1}$ to the velocity equation (1), we obtain $\forall (x,t)$ in $\mathbb{R}^+ \times \mathbb{T}^3$,

$$\begin{aligned}\frac{d}{\mathrm{dt}}u_{\alpha_k} = \nu\Delta u_{\alpha_k} - \left(I - \alpha^2\Delta\right)^{-1}\left(u_{\alpha_k}\cdot\nabla\right)u_{\alpha_k} \\ - \left(I - \alpha^2\Delta\right)^{-1}\nabla p_{\alpha_k} + \left(I - \alpha^2\Delta\right)^{-1}\theta_{\alpha_k}e_3.\end{aligned} \tag{22}$$

In the following, we will be conformed to the statement of the Aubin lemma [23] and consider a time $T > 0$. As u_{α_k} is bounded independently of α in $L^2([0,T], \dot{H}^1(\mathbb{T}^3))$, the dissipation Δu_{α_k} will be so in the space $L^2([0,T], \dot{H}^{-1}(\mathbb{T}^3))$. For the other terms, we mention that the operator $(I - \alpha^2\Delta)^{-1}$ is bounded from $H^{-2}(\mathbb{T}^3)$ into $L^2(\mathbb{T}^3)$ and that by frequency calculations, we have $|||(I - \alpha^2\Delta)^{-1}||| \leq 1$, where we denote by $|||.|||$, the norm of the operator. Also, as θ_{α_k} is bounded independently of α in $L^2([0,T], \dot{H}^1(\mathbb{T}^3))$, then $\|(I - \alpha^2\Delta)^{-1}\theta e_3\|_{L^2(\dot{H}^3)} \leq K$. As for the convection term, Sobolev norm properties, Sobolev product laws, and classical computations lead to

$$\begin{aligned}\int_0^T \left\|\left(I - \alpha^2\Delta\right)^{-1}\operatorname{div}\left(u_{\alpha_k}\otimes u_{\alpha_k}\right)\right\|^2_{L^2} &\leq \int_0^T \left\|\operatorname{div}\left(u_{\alpha_k}\otimes u_{\alpha_k}\right)\right\|^2_{H^{-2}} \\ &\leq \|u\|^2_{L^\infty_{\mathbb{R}_+}(L^2)}\|u\|^2_{L^2_{\mathbb{R}_+}(\dot{H}^1)}.\end{aligned} \tag{23}$$

It is standard to rewrite the pressure in terms of the velocity and the temperature. Also, one applies the divergence operator and the Riesz transform to obtain

$$p = -\Delta^{-1}\left(\sum_{i,j=1}^{3} \partial_i\partial_j\left(u^i u^j\right) - \partial_3\theta\right). \tag{24}$$

Using the precedent bounds of the temperature and the velocity, it holds that

$$\left\|\left(I-\alpha^2\Delta\right)^{-1}\nabla p\right\|_{L^2\left(\dot{H}^{-1}\right)} \le K. \tag{25}$$

So, equation (22) implies that

$$\left\|\frac{d}{dt}u_{\alpha_k}\right\|_{L^2_T\left(\dot{H}^{-1}\right)} \le K. \tag{26}$$

Remark 3. It is clear that in (21) and (26) as in [1], the constant $K = K(\alpha_0, \nu, \kappa, u^0, \theta^0)$. Thus, it is uniform with respect to α. However, the most interesting feature in the present paper is the fact that K is independent of the time T. This makes these estimates valid for all time T. Especially, as time goes to infinity. This was not the case of convergence result in [1], where estimates for convergence results blow up, as $t \longrightarrow +\infty$.

By Aubin–Lions lemma, we extract subsequences relabeled u_k and θ_k that converge strongly in $L^2([0,T], L^2(\mathbb{T}^3))$ and in $L^2([0,T], \dot{H}^{-1}(\mathbb{T}^3))$, respectively. Since

$$\begin{aligned}\left\|v_k - u_k\right\|^2_{L^2\left([0,T],\dot{H}^{-1}\right)} &= \alpha^4\int_0^T\left(\sum_{k\in\mathbb{Z}^3}|k|^{-2}\left|\widehat{\Delta u_k}\right|^2\right)\\ &= \alpha^4\left\|u_k\right\|^2_{L^2\left([0,T],\dot{H}^{1}\right)},\end{aligned} \tag{27}$$

we deduce that v_k converges strongly to u in $L^2([0,T], \dot{H}^{-1})$ because u_k belongs to $L^2([0,T], \dot{H}^{1})$.

3.3. Proof of Statement (3). For the first result, since (u_k, θ_k) converges strongly to (u,θ) in $(L^2(\mathbb{R}_+), L^2)^2$, then by the Cauchy–Schwarz inequality, it converges weakly for almost every $t \in \mathbb{R}_+$. In particular, this holds for the supremum. That is, $(u_k(t), \theta_k(t))$ converges to $(u(t), \theta(t))$ weakly in $L^2(\mathbb{T}^3)$ and uniformly over $\mathbb{R}_+$. To prove the second result, we recall that the time derivatives are uniformly bounded with respect to α, as proved above. We apply the Banach–Alaoglu theorem, in Hilbert spaces, to deduce that

$$\begin{aligned}&\left(\frac{d}{dt}\theta_k, \frac{d}{dt}u_k\right) \rightharpoonup \left(\frac{d}{dt}\theta, \frac{d}{dt}u\right)\text{weakly in } L^2\left([0,T], \dot{H}^{-1}\left(\mathbb{T}^3\right)\right), \text{as } k \longrightarrow +\infty\\ &\frac{d}{dt}v_k \rightharpoonup \frac{d}{dt}u \text{ weakly in } L^2\left([0,T], \dot{H}^{-2}\left(\mathbb{T}^3\right)\right), \text{as } k \longrightarrow +\infty.\end{aligned} \tag{28}$$

Let $\Lambda \in \dot{H}^2$ be a divergence-free vector field and $\Xi \in \dot{H}^1$ a scalar mean free test function. We take the inner product and we integrate with respect to time to obtain

$$\begin{aligned}&\langle\theta_k(t), \Xi\rangle_{H^{-1}} - \langle\theta_k(0), \Xi\rangle_{H^{-1}} - \int_0^t (\theta_k, \Delta\Xi)_{L^2}d\tau + \int_0^t (B(u_k, \theta_k), \Xi)_{L^2}d\tau = 0\\ &\langle v_k(t), \Lambda\rangle_{H^{-2}} - \langle v_k(0), \Lambda\rangle_{H^{-2}} - \int_0^t (v_k, \Delta\Lambda)_{L^2}d\tau + \int_0^t \langle\widetilde{B}(u_k, v_k), \Lambda\rangle_{H^{-2}}d\tau - \int_0^t (\theta_k e_3, \Lambda)_{L^2}d\tau = 0.\end{aligned} \tag{29}$$

To deal with the nonlinear terms, we use a standard compactness argument to obtain $\widetilde{B}(u_k, v_k) \longrightarrow B(u,u)$ and $B(u_k, \theta_k) \longrightarrow B(u,\theta)$. Taking the limit, it follows that

$$\begin{aligned}&\langle\theta(t), \Xi\rangle_{H^{-1}} - \langle\theta(0), \Xi\rangle_{H^{-1}} - \int_0^t (\theta, \Delta\Xi)_{L^2}d\tau + \int_0^t (B(u,\theta), \Xi)_{L^2}d\tau = 0,\\ &\langle u(t), \Lambda\rangle_{H^{-2}} - \langle u(0), \Lambda\rangle_{H^{-2}} - \int_0^t (u, \Delta\Lambda)_{L^2}d\tau + \int_0^t \langle B(u,u), \Lambda\rangle_{H^{-2}}d\tau - \int_0^t (\theta e_3, \Lambda)_{L^2}d\tau = 0.\end{aligned} \tag{30}$$

The solution $(u(t), \theta(t))$ satisfies the energy inequality (17), as we can take the lower limit when $\alpha_k \longrightarrow 0$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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