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Research Article

The Weak (Gorenstein) Global Dimension of Coherent Rings with Finite Small Finitistic Projective Dimension

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The small finitistic dimension of a ring is determined as the supremum projective dimensions among modules with finite projective resolutions. This paper seeks to establish that, for a coherent ring R with a finite weak (resp. Gorenstein) global dimension, the small finitistic dimension of R is equal to its weak (resp. Gorenstein) global dimension. Consequently, we conclude some new characterizations for (Gorenstein) von Neumann and semihereditary rings.

1. Introduction

In this paper, we assume all rings are commutative with identity, and all modules are unitary. Let R be a ring, and M an R-module. As usual, we use $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$ to represent the classical projective dimension and flat dimension of M, respectively. The weak dimension of R is defined as $\mathrm{wdim}(R) = \sup\{\mathrm{fd}_R(M) \mid M \text{ is an } R-\text{module}\}$, and $T(R)$ denotes the total quotient ring of R.

The G-dimension was initially introduced, by Auslander and Bridger [1], for commutative Noetherian rings. This concept was subsequently expanded to modules over any ring by Enochs and Jenda [2, 3] through the introduction of Gorenstein projective, injective, and flat modules. The investigation of homological dimensions based on these modules was pursued in [4].

Let us consider a ring R. A module M is termed Gorenstein projective, for short G-projective, if there exists an exact sequence of projective modules

$$\mathscr{Q}: \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \cdots, \tag{1}$$

such that M is isomorphic to the image of the map $Q_0 \longrightarrow Q^0$, and the functor $\mathrm{Hom}_R(-, Q)$ maintains the exactness of $\mathscr{Q}$ whenever Q is a projective module. This sequence $\mathscr{Q}$ is termed a complete projective resolution.

Similarly, a module M is is termed Gorenstein flat, for short G-flat, if there exists an exact sequence of flat modules:

$$\mathscr{F}: \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots, \tag{2}$$

such that M is isomorphic to the image of the map $F_0 \longrightarrow F^0$, and the functor $I\otimes_R -$ preserves the exactness of $\mathscr{F}$ whenever I is an injective module. The sequence $\mathscr{F}$ is called a complete flat resolution.

Gorenstein projective and flat dimensions, denoted by $\mathrm{Gpd}(-)$ and $\mathrm{Gfd}(-)$, respectively, are defined based on resolutions ([4, 5]).

The weak Gorenstein global dimension of a ring R is defined as follows:

$$\mathrm{wGdim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \text{ is an } R-\text{module}\}. \tag{3}$$

It is important to observe that for a given ring R, the weak Gorenstein global dimension $\mathrm{wGdim}(R)$ is bounded above by the weak dimension $\mathrm{wdim}(R)$, and the two coincide if $\mathrm{wdim}(R)$ is finite.

Let R be a ring and M be a module. An exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{4}$$

where P_i is finitely generated projective modules, is called a finite projective resolution ($f\,pr$ for short) of M.

The small finitistic dimension of a ring R, denoted $\mathrm{fPD}(R)$, is defined to be the supremum of projective dimensions of modules with $f\,pr$. In the case of a Noetherian local ring R, Auslander and Buchweitz in [6] showed that $\mathrm{fPD}(R)$ coincides with the depth of R. It is evident that $\mathrm{fPD}(R)=0$ if and only if any module M with an $f\,pr$ is projective. Equivalently, this condition holds if and only if M is projective whenever there exists an exact sequence $0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$, where Q_0 and Q_1 are finitely generated projective modules.

In the context of a coherent ring R, the small finitistic projective dimension $\mathrm{fPD}(R)$ assumes a more tractable form, namely,

$$\mathrm{fPD}(R)=\sup\{\mathrm{pd}_R(M) \mid M \text{ is finitely presented and } \mathrm{pd}_R(M)<\infty\}. \tag{5}$$

Similarly, for the weak global dimension of a coherent ring R, a nice description is given by

$$\mathrm{wdim}(R)=\sup\{\mathrm{pd}_R(M) \mid M \text{ is finitely presented and } \mathrm{pd}_R(M)<\infty\}. \tag{6}$$

Similarly, we define the small finitistic Gorenstein projective dimension of a ring R, as follows:

$$\mathrm{fGPD}(R)=\sup\{\mathrm{Gpd}_R(M) \mid M \text{ is a module with fpr and } \mathrm{Gpd}_R(M)<\infty\}. \tag{7}$$

The close relation between the small finitistic projective dimension, the weak global dimension, and the weak Gorenstein global dimension renders it natural to track the possible values of $\mathrm{wdim}(R)$ and $\mathrm{wGdim}(R)$ for a given value of $\mathrm{fPD}(R)$.

The aim of this paper is to answer the following question:

Question. For a coherent ring with $\mathrm{fPD}(R)=n$, what values can the weak (resp. Gorenstein) global dimension R take?

We begin by establishing the equality of $\mathrm{fPD}(R)$ and $\mathrm{wdim}(R)$ for a coherent ring R (in Theorem 1). This leads to new characterizations of von Neumann regular and semihereditary rings. A particular focus is on rings with zero fPD. It has been demonstrated that when R is Noetherian with zero Krull dimension, $\mathrm{fPD}(R)=0$ (in ([6], Theorem 1.6)). Interestingly, this result extends beyond the Noetherian assumption, as proved in [7], Proposition 3.14. However, Example 2 illustrates that the converse implication is not valid. Theorem 10 establishes the equality of $\mathrm{fPD}(R)$ and $\mathrm{fGPD}(R)$, both bounded by $\mathrm{wGdim}(R)$. In addition, in the case of a coherent ring R, these three dimensions coincide. Consequently, we found new characterizations for rings with small wGdim. Finally, Proposition 17 presents a new characterization of quasi-Frobenius rings through the utilization of Nagata rings.

2. The Weak (Gorenstein) Global Dimension of Coherent Rings with Finite Small Finitistic Projective Dimension

Generally, for a ring R, $\mathrm{fPD}(R)\le \mathrm{wdim}(R)$, with equality when R is local, coherent, and regular, as shown in [8], Lemma 3.1. A ring is said to be regular if every finitely generated ideal of R has finite projective dimension, as defined in [8]. This concept has been extensively explored in the context of coherent rings. Notably, coherent rings having finite weak global dimension are regular. Nevertheless, it is essential to note that there exist coherent rings, including local ones, possessing an infinite weak global dimension while maintaining regularity.

The first main result of this paper drops the "local" condition in Glaz's result [8], Lemma 3.1.

Theorem 1. *Let R be a regular coherent ring. Then, $\mathrm{fPD}(R)=\mathrm{wdim}(R)$.*

Proof. Consider a coherent regular ring R. It is evident that $\mathrm{fPD}(R)\le \mathrm{wdim}(R)$. Now, let's set $\mathrm{fPD}(R)=n$. Consider a finitely generated ideal I of R. Then, since R is regular, $\mathrm{pd}_R(R/I)<\infty$. As R is coherent

$$n = \mathrm{fPD}(R) = \sup\{\mathrm{pd}_R(M) \mid M \text{ is finitely presented and } \mathrm{pd}_R(M) < \infty\}. \tag{8}$$

Consequently, $\mathrm{pd}_R(R/I) \leq n$. Hence, $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ for any module M. By ([9], Theorems 2.6.1 and 2.6.3), $\mathrm{wdim}(R) \leq n$. Therefore, $\mathrm{wdim}(R) \leq \mathrm{fPD}(R)$. Consequently, we conclude that $\mathrm{wdim}(R) = \mathrm{fPD}(R)$, as desired.

Let J be a finitely generated ideal of a ring R. If $\mathrm{Hom}_R(R/J, R) = 0$, then J is called semiregular. When R is the only finitely generated semiregular ideal of R, then R is called a *DQ* ring. It is proven, in ([10], Proposition 2.2), that a ring R is a *DQ* ring if and only if $\mathrm{fPD}(R)$ is zero. Hence, fPD mesures how far a ring to be *DQ*.

In [9], Glaz introduced the concept of P-rings. A ring R is a P-ring (or has the property (P)) if $\mathrm{ann}_R(I) \neq (0)$ for each finitely generated proper ideal I of R. Glaz pioneered the exploration of the homological properties of local P-rings and demonstrated that a local ring R is a P-ring if and only if fPD $(R) = 0$. The aforementioned result has been further generalized in ([11], Theorem 1) to apply to arbitrary rings (not necessarily local).

We conclude the following corollaries. □

Corollary 2. *If R is a ring, then the following are equivalent:*

(1) R is a von Neumann regular ring (i.e., wdim$(R) = 0$).

(2) R is coherent, fPD$(R) = 0$, and wdim$(R) < \infty$.

(3) R is a coherent P-ring andwdim$(R) < \infty$.

(4) R is coherent regular with fPD$(R) = 0$.

(5) R is a coherent regular P-ring.

(6) R is a coherent regular DQ ring.

Corollary 3. *If R is a ring, then the following are equivalent:*

(1) R is a semihereditary ring (i.e., R is coherent and wdim$(R) \leq 1$).

(2) R is coherent, fPD$(R) \leq 1$, and wdim$(R) < \infty$.

(3) R is coherent regular with fPD$(R) \leq 1$.

Remark 4. It is established in ([12], Corollary 3.2) that, for a ring R, if $\mathrm{fPD}(R) = 0$ then finitey generated flat modules are projective. However, this assertion does not hold in general. Take, for instance, a von Neumann regular ring R which is not semisimple. Since R is not Noetherian, R has a nonfinitely generated ideal I. The module R/I is finitely generated flat that is not projective since it is not of finitely presented.

Auslander and Buchsbaum, in ([6], Theorem 1.6), established that for a Noetherian ring R, $\mathrm{fPD}(R)$ is less than or equal to the Krull dimension of R, denoted by $\dim(R)$. Consequently, when $\dim(R) = 0$, it implies that $\mathrm{fPD}(R) = 0$. However, this conclusion holds true even in cases where R is not necessarily Noetherian as shown by Wang, Zhou, and Chen in ([7], Proposition 3.14).

In ([13], Problem 1b), Cahen et al. asked if $\mathrm{fPD}(R)$ is always zero for a total ring of quotients R. Rings with zero Krull dimension constitute a subclass of total rings of quotients where fPD is indeed zero. However, a recent study in [7] provided a negative answer to this question.

Note that a ring R with $\mathrm{fPD}(R) = 0$ does not need to be coherent or have finite $\mathrm{wdim}(R)$.

Example 1

(1) Let R_1 be a nonsemisimple quasi-Frobenius ring. Then, R_1 is Noetherian, $\mathrm{fPD}(R_1) = 0$, and $\mathrm{wdim}(R_1) = \infty$.

(2) Let R_2 be a noncoherent ring with zero Krull dimension (see, for instance, ([14], Example 2.8),). Then, by ([7], Proposition 3.14), $\mathrm{fPD}(R_2) = 0$.

(3) Set $R = R_1 \times R_2$. Let J be a proper finitely generated ideal of R generated by $\{(x_i, y_i)\}_{i=1,\ldots,n}$. Set $J_1 = \sum_{i=1}^n R_1 x_i$ and $J_2 = \sum_{i=1}^n R_2 y_i$. Clearly, $J = J_1 \times J_2$. Since J is proper, $J_1 \neq R_1$ or $J_2 \neq R_2$. Suppose, for example, that $J_1 \neq R_1$. Then, since R_1 is a P-ring, there exists a nonzero element $z \in R_1$ such that $zJ_1 = (0)$. Hence, $(z, 0)J_1 \times J_2 = (0)$. So, R is a P-ring, and then $\mathrm{fPD}(R) = 0$. However, R is not coherent since R_2 is not, and $\mathrm{wdim}(R) = \sup\{\mathrm{wdim}(R_1), \mathrm{wdim}(R_2)\} = \infty$.

Recall that a ring R is called McCoy (or satisfies Property (A)) if $\mathrm{ann}_R(I) \neq (0)$ for each finitely generated ideal I consisting of zero divisors of R. McCoy rings include Noetherian rings, rings with Krull dimension zero, and graded rings (in particular, polynomial rings). Next, we give a new characterization of McCoy rings.

Proposition 5. *A ring R is McCoy if and only if fPD$(T(R)) = 0$. In particular, if* R *is a total ring of quotients then R is McCoy if and only if fPD$(R) = 0$.*

Proof. By ([15], Corollary 2.6), R is McCoy if and only if $T(R)$ is McCoy.

Clearly, if $\mathrm{fPD}(T(R)) = 0$, then $T(R)$ is a P-ring, and, in particular, it is McCoy. Consequently, R is McCoy. Conversely, if R is McCoy, then so is $T(R)$. Since $T(R)$ is a total ring of quotients, every proper ideal of $T(R)$ consists only of zero divisors. Hence, $T(R)$ is a P-ring, as desired.

A ring R with $\mathrm{fPD}(R) = 0$, even Noetherian, does not necessarily have a zero Krull dimension, as shown by the next example. □

Example 2. Let k be a field. Consider the additive group

$$A = \frac{k[[x]](+)k[[x]]}{(x)} := \frac{k[[x]] \oplus k[[x]]}{(x)}, \tag{9}$$

equipped with multiplication defined as

$$\left(P_1,\overline{Q_1}\right)\left(P_2,\overline{Q_2}\right)=\left(P_1P_2,P_1.\overline{Q_2}+P_2.\overline{Q_1}\right)=\left(P_1P_2,\overline{P_1Q_2}+\overline{P_2Q_1}\right). \tag{10}$$

This forms a commutative ring with unity $1_A=(1,\overline{0})$, known as the trivial extension of $k[[x]]$ by the $k[[x]]$-module $k[[x]]/(x)$. According to ([16], Theorems 3.2 and 4.8), $\dim(A)=\dim(k[[x]])=1$, and A is a local Noetherian ring since $k[[x]]$ is local Noetherian, and $k[[x]]/(x)$ is a finitely generated $k[[x]]$-module. Moreover, the maximal ideal of A is $\underline{M}=(x)\oplus k[[x]]/(x)$. For each $(P,\overline{Q})\in M$, we have $(P,\overline{Q})(0,\overline{1})=(0,\overline{0})$. Consequently, every nonunit element of A is a zero divisor, establishing A as a total ring of quotients. According to ([10], Proposition 2.3), A is a DQ ring, and thus, $\mathrm{fPD}(A)=0$.

Let R be a ring, and consider an ideal I of R. In accordance with [17], I is designated as a GV-ideal if it is finitely generated, and the natural homomorphism $R\longrightarrow\mathrm{Hom}_R(I,R)$ is an isomorphism. Let $GV(R)$ denotes the set of GV-ideals of R. Consider a module M and set

$$\mathrm{tor}_{GV}(M)=\{x\in M\mid Ix=0 \text{ for some } I\in GV(R)\}. \tag{11}$$

It is evident that $\mathrm{tor}_{GV}(M)$ forms a submodule of M. A module M is said to be GV-torsion-free (resp. GV-torsion) if $\mathrm{tor}_{GV}(M)=0$ (resp. $\mathrm{tor}_{GV}(M)=M$). A GV-torsion-free module M is said to be a w-module if $\mathrm{Ext}^1_R(R/I,M)=0$ for any $I\in GV(R)$. When every ideal of R is a w-ideal, we say that R is DW.

The notion of DW rings is related to rings with small finitistic projective dimension ≤ 1. Let R be a ring. Wang et al. in ([10], Proposition 2.2 and Theorem 3.2) proved that $\mathrm{fPD}(R)=0$ is equivalent to R being a DW ring and $R=Q_0(R)$ (where $Q_0(R)$ is the ring of finite fractions of R). It is also proved, in ([18], Corollary 3.7), that R is a DW ring if and only if $\mathrm{fPD}(R)\leq 1$. Hence, we can rewrite Corollary 3 as follows:

Proposition 6. *If R is a ring, then the following are equivalent:*

(1) R is a semi-hereditary ring.

(2) R is a coherent DW ring with $\mathrm{wdim}(R)<\infty$.

(3) R is a coherent regular DW ring.

In particular, R is a Prüfer domain (i.e., a semihereditary domain) if and only if R is a coherent regular DW domain.

In the previous result, the particular case is exactly ([19], Proposition 3.1 (2) $\Longleftrightarrow$ (6)). Recall that a ring R is called a Prüfer ring if every finitely generated regular ideal is invertible. Over a domain, the two definitions of Prüfer domains coincide. It is also well known that semihereditary rings are Prüfer rings. Hence, coherent regular DW rings are Prüfer rings. However, as mentioned in [19], Prüfer rings need not be regular. For example, $\mathbb{Z}/4\mathbb{Z}$ is a local Noetherian Prüfer ring with infinite (weak) global dimension, and so it is not a regular ring.

Using Corollary 3 and Proposition 6, we conclude the following corollary:

Corollary 7. *If R is a domain, then the following are equivalent:*

(1) R is a Dedekind domain.

(2) R is Noetherian, $\mathrm{fPD}(R)\leq 1$, and R has a finite global dimension.

(3) R is Noetherian regular DW domain.

Remark 8. Recall the classical definition of regularity for Noetherian rings: A local Noetherian ring R is regular if it has a finite global dimension. A general Noetherian ring R is regular if it is locally regular. It's important to note that for a Noetherian ring, the two definitions of regularity, the one provided in [8] and the classical one, coincide. Therefore, Corollary 7 is partially ([20], Proposition 3.6).

Now, we define a Gorenstein analogue for the $\mathrm{fPD}(--)$.

Definition 9. Let R be a ring and M be a module. The small finitistic Gorenstein projective dimension of a ring R, denoted $\mathrm{fGPD}(R)$, is defined as follows:

$$\mathrm{fGPD}(R)=\sup\left\{\mathrm{Gpd}_R(M)\mid M \text{ is a module with fpr and } \mathrm{Gpd}_R(M)<\infty\right\}. \tag{12}$$

The next result compares fPD(−) with fGPD(−), and wGdim(−).

Theorem 10. *Let R be a ring. Then,*

(1) $\mathrm{fPD}(R)=\mathrm{fGPD}(R)\leq \mathrm{wGdim}(R)$.

(2) If R is coherent with $\mathrm{wGdim}(R)<\infty$, then $\mathrm{fGPD}(R)=\mathrm{fPD}(R)=\mathrm{wGdim}(R)$.

The following lemmas are required.

Lemma 11. *Let X be a finitely generated G-projective module. There is a short exact sequence $0\longrightarrow X\longrightarrow P\longrightarrow X'\longrightarrow 0$, where P is a finitely generated projective module and X' is a finitely generated G-projective module.*

Proof. This is exactly ([21], Lemma 2.9) with the precision that in the proof X' can be taken to be finitely generated. ☐

Lemma 12. *Let R be a ring and M be a module with finite G-projective dimension $n\geq 1$. If M has a $f$pr, then there exists an epimorphism $\epsilon\colon G_0\twoheadrightarrow M$, where G_0 is a G-projective* *module with fpr, and $K = \ker(\epsilon)$ is module with fpr and $pd_R(K) \le n-1$.*

Proof. Since M has a fpr, we can consider an exact sequence

$$0 \longrightarrow N \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{13}$$

where all P_i is finitely generated projective modules and N is a module with fpr. According to ([4], Theorem 2.20), N is G-projective. Using Lemma 11, we obtain an exact sequence:

$$0 \longrightarrow N \longrightarrow Q_0 \longrightarrow \cdots \longrightarrow Q_{n-1} \longrightarrow G \longrightarrow 0, \tag{14}$$

where all Q_i is finitely generated projective and G is a G-projective module with fpr, and such that the functor $\mathrm{Hom}(-, Q)$ maintains the exactness of this sequence when Q is projective.

This enables the construction of homomorphisms $Q_i \longrightarrow P_{n-1-i}$ for $i = 0, \ldots, n-1$ and $G \longrightarrow M$ such that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & Q_0 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Q_{n-1} \arrow[r] \arrow[d, dotted] & G \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & N \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \tag{15}$$

By ([22], Proposition 1.4.14), we get an exact sequence.

$$0 \longrightarrow N \longrightarrow N \oplus Q_0 \longrightarrow P_{n-1} \oplus Q_1 \longrightarrow \ldots \longrightarrow P_0 \oplus G \longrightarrow M \longrightarrow 0, \tag{16}$$

resulting in the exactness of the sequence

$$0 \longrightarrow Q_0 \longrightarrow P_{n-1} \oplus Q_1 \longrightarrow \ldots \longrightarrow P_0 \oplus G \longrightarrow M \longrightarrow 0. \tag{17}$$

It is worth noting that $P_0 \oplus G$ has a fpr. Consequently, the kernel K of ϵ: $P_0 \oplus G \longrightarrow M$ satisfies $\mathrm{pd}_R(K) \le n-1$ and has a fpr (by ([8], Theorem 2.1.2)). □

Proof of Theorem 13

(1) We claim the inequality $\mathrm{fPD}(R) \le \mathrm{fGPD}(R)$. Let us assume $\mathrm{fGPD}(R) = n < \infty$. Since every R-module has a fpr is finitely presented with finite projective dimension, by ([4], Proposition 2.27), we have $\mathrm{fPD}(R) \le \mathrm{fGPD}(R)$.

Now, we aim to establish $\mathrm{fGPD}(R) \le \mathrm{wGdim}(R)$. Let us assume $\mathrm{wGdim}(R) = n < \infty$. Consider a module M with a fpr and finite G-projective dimension. Since every R-module has a fpr is infinitely presented, by ([23], Theorem 3.3), we have $\mathrm{Gpd}_R(M) = \mathrm{Gfd}_R(M) \le n$. This implies $\mathrm{fGPD}(R) \le \mathrm{wGdim}(R)$, as desired.

We claim $\mathrm{fGPD}(R) \le \mathrm{fPD}(R)$. We may assume $\mathrm{fPD}(R) = m < \infty$. Consider a module M with fpr s and finite G-projective dimension. According to Lemma 12, there exists an exact sequence $0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$, where G is G-projective and K is a module with fpr and finite projective dimension. Hence, $\mathrm{pd}_R(K) \le m$. By Lemma 11, there exists a short exact sequence $0 \longrightarrow G_0 \longrightarrow Q \longrightarrow G_1 \longrightarrow 0$, where Q is finitely generated projective and G_1 is finitely generated G-projective. The pushout diagram with exact rows and columns is given by

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & G_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, dotted] & G_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, dotted] \arrow[d] & N \arrow[r] \arrow[d] & G_1 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{18}$$

Clearly, N has a fpr and a finite projective dimension. Thus, $\mathrm{pd}_R(N) \le m$. Using the short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow G_1 \longrightarrow 0$ and ([4], Theorem 2.22), for each integer $i > m$, we obtain

$$0 = \mathrm{Ext}^i_R(N, T) \longrightarrow \mathrm{Ext}^i_R(M, T) \longrightarrow \mathrm{Ext}^{i+1}_R(G_1, T) = 0, \tag{19}$$

for all projective modules T. Once again, by ([4], Theorem 2.22), $\mathrm{Gpd}_R(M) \leq m$. Consequently, $\mathrm{fGPD}(R) \leq \mathrm{fPD}(R)$.

(2) Let n be a positive integer. Recall that a ring R is said to be n-FC if R is coherent and $\mathrm{FP0id}_R(R) \leq n$ (the FP-injective dimension of R). Suppose that a ring R is coherent with $\mathrm{wGdim}(R) = n < \infty$. Using ([24], Theorem 3.8) and ([20], Theorem 10), we get that

$$n = \mathrm{wGdim}(R) = \sup\{\mathrm{fd}_R(I) \mid I \text{ is an injective } R - \text{module}\} = \mathrm{FP0id}_R(R). \quad (20)$$

Thus, R is a n-FC ring. In accordance with ([25], Theorem 7), we conclude that $\mathrm{fGPD}(R) = \mathrm{wGdim}(R)$. □

Corollary 14. *If R is a ring. Then, $fPD(R) = 0$ if and only if every finitely generated projective submodule of a G-projective module is a direct summand.*

Proof. This follows from ([21], Lemma 2.8) and the fact that $\mathrm{fPD}(R) = 0$ if and only if R is a P-ring.

Recall that a ring R is called G-von Neumann regular (resp. G-semi-hereditary) if $\mathrm{wGdim}(R) = 0$ (resp. R is coherent and $\mathrm{wGdim}(R) \leq 1$). □

Corollary 15. *If R is a ring, then the following are equivalent:*

(1) R is a G-semi-hereditary ring.

(2) R is coherent, $fPD(R) \leq 1$, and $wGdim(R) < \infty$.

Let Q denote the set of finitely generated semiregular ideals of a ring R. The concept of the ring of finite fractions for a ring R, denoted as $Q_0(R)$, was explored, by Lucas in [26]:

$$Q_0(R) = \{P \in T(R[X]) \mid IP \subseteq R \text{ for some } I \in Q\}. \quad (21)$$

The inclusions $R \subseteq T(R) \subseteq Q_0(R)$ were established, and in the case where R is an integral domain, it was shown that $Q_0(R)$ serves as the quotient field of R. Moreover, in [27], it was demonstrated that $Q_0(R) = R$ if and only if every finitely generated semiregular ideal of R is a GV-ideal. For further details, please refer to [10, 26, 27].

Theorem 16. *If R is a ring, then the following are equivalent:*

(1) R is Gorenstein von Neuman regular.

(2) R is coherent, $fPD(R) = 0$, and $wGdim(R) < \infty$.

(3) R is a coherent P-ring and$wGdim(R) < \infty$.

(4) R is a coherent DQ ring with $wGdim(R) < \infty$.

(5) R is G-semi-hereditary and $fPD(R) = 0$.

(6) R is a G-semi-hereditary P-ring.

(7) R is a DQ G-semi-hereditary ring.

(8) R is a G-semi-hereditary ring and $Q_0(R) = R$.

Proof. The equivalence between (1), (2), (3), (4), (5), (6), and (7) follows immediately from Theorem 10 and the fact that $\mathrm{fPD}(R) = 0$ if and only if R is DQ if and only if R is a P-ring.

(7) $\Longrightarrow$ (8) This follows from ([10], Proposition 2.2).

(8) $\Longrightarrow$ (7) Suppose R is a G-semi-hereditary ring such that $Q_0(R) = R$. Then, by Theorem 10, $\mathrm{fPD}(R) \leq 1$. Using ([10], Corollary 3.3), R is a DQ ring.

Consider a commutative ring R and a polynomial $f \in R[x]$. The content of f, denoted by $c(f)$, refers to the ideal of R generated by the coefficients of f.

Set $S = \{P \in R[X] \mid c(P) = R\}$, the set of polynomials with unit content. The Nagata ring $R(X)$ is obtained by localizing the polynomial ring $R[X]$ with respect to S; that is $R(X) = S^{-1}(R[X])$. □

Proposition 17. *If R is a ring, then the following are equivalent:*

(1) R is quasi-Frobenius.

(2) R is a Noetherian G-semi-hereditary ring and $R(X) = T(R[X])$.

Proof. Note that R is quasi-Frobenius if and only if R is a Noetherian G-von Neumann regular (by ([28], Theorem 2.2) and ([21], Corollary 2.11)). Moreover, if R is Noetherian (and thus a McCoy ring), then, by ([26], Theorem 3.2), $R = Q_0(R)$ is equivalent to $R(X) = T(R[X])$. Hence, the desired conclusion follows from Theorem 16. □

Data Availability

The study of this article does not require any data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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