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Research Article

On Some New Sequence Spaces and Their Duals

Damla Barlak

Department of Statistics, Dicle University, Diyarbakır, Türkiye

Correspondence should be addressed to Damla Barlak; dyagdiran@hotmail.com

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In this study, we defined some new sequence spaces using regular Tribonacci matrix. We examined some properties of these spaces such as completeness, Schauder basis. We have identified $\alpha-$, $\beta-$, and $\gamma-$duals of the newly created spaces.

1. Introduction and Preliminaries

Let us we denote the space of all real or complex sequence by w. We write the sequence spaces of all convergent, null, bounded, and absolutely $p-$ summable sequences by c, c_0, l_∞, and l_p, respectively. Also we will denote the space of all bounded, convergent, and absolutely convergent series with bs, cs, and l_1, respectively. The space $l_p\,(1 \leq p < \infty)$ is Banach space with $x_p = (\sum_{k=0}^{\infty}|x_k|^p)^{1/p}$ and c, c_0, and l_∞ are Banach spaces with $x_\infty = \sup_k |x_k|$.

Let X be a linear metric space. A function q: $X \longrightarrow \mathbb{R}$ is called a paranorm, if

$(P1)$ $q(x) \geq 0$ for $\in X$

$(P2)$ $q(-x) = q(x)$ for all $x \in X$

$(P3)$ $q(x+y) \leq q(x) + q(y)$ for all $x, y \in X$

$(P4)$ If (λ_n) is a sequence of scalars with $\lambda_n \longrightarrow \lambda$ as $n \longrightarrow \infty$ and (x_n) is a sequence of vectors with $q(x_n - x) \longrightarrow 0$ as $n \longrightarrow \infty$, then $q(\lambda_n x_n - \lambda x) \longrightarrow 0$ as $n \longrightarrow \infty$

A paranorm q, where $q(x) = 0$ implies $x = \theta$, is termed as a total paranorm, and the combination (X, q) is referred to as a total paranormed space. It is widely recognized that the metric of any linear metric space is represented by some total paranorm (see ([1], Theorem 10.4.2, page 183)). To gain a better understanding of the theory of paranormed spaces, you can refer to these valuable articles (see Barlak [2], Zengin Alp [3], İlkhan et al. [4], and many others).

Let $p = (p_k)$ be a bounded sequence of real numbers such that $p_k > 0$, $\sup_{k \in \mathbb{N}} p_k = P$, and $S = \max\{1, P\}$. For any $\zeta \in \mathbb{R}$ and $k \in \mathbb{N}$, it has been established in [2] that

$$|\zeta|^{p_k} \leq \max\left\{1, |\zeta|^S\right\}. \quad (1)$$

Throughout this study we will assume that $p_k^{-1} + (p_k')^{-1} = 1$ provided that $1 < \inf p_k < P < \infty$. Maddox [5, 6] introduced the linear spaces $c(p), c_0(p)$, $l_\infty(p)$, and $l(p)$ by

$$\begin{aligned}
c(p) &= \left\{z = (z_k) \in w\colon \lim_{k \longrightarrow \infty} \left|z_k - l\right|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\right\},\\
c_0(p) &= \left\{z = (z_k) \in w\colon \lim_{k \longrightarrow \infty} \left|z_k\right|^{p_k} = 0\right\},\\
l_\infty(p) &= \left\{z = (z_k) \in w\colon \sup_{k \longrightarrow \infty} \left|z_k\right|^{p_k} < \infty\right\},\\
l(p) &= \left\{z = (z_k) \in w\colon \sum_k \left|z_k\right|^{p_k} < \infty\right\}.
\end{aligned} \quad (2)$$

The linear spaces $c(p), c_0(p)$, $l_\infty(p)$, and $l(p)$ are complete spaces paranormed by

$$q_\infty(z) = \sup_{k \in \mathbb{N}} \left|z_k\right|^{p_k/S}, \text{iff } \inf_{k \in \mathbb{N}} p_k > 0, \quad (3)$$

and

$$q(z) = \left(\sum_k |z_k|^{p_k} \right)^{1/S}, \tag{4}$$

respectively.

Let $A = (a_{rk})$ be an infinite matrix of real or complex numbers and X, Y be subsets of w. We write $A_r(x) = \sum_k a_{rk} x_k$ and $Ax = A_r(x)$ for $r, k \in \mathbb{N}$. For a sequence space X, the matrix domain of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w: \ Ax \in X\}, \tag{5}$$

which is also a sequence space. We denote with (X,Y) the class of all matrices A such that $A: X \longrightarrow Y$.

Recently, the literature focused on the creation of new sequence spaces through the matrix domain and the investigation of their algebraic and topological properties, and the study of matrix transformations has expanded. To enhance comprehension of the theory concerning sequence spaces, you can refer to these valuable articles (see Altay et al. [7], Gürdal [8], Şahiner and Gürdal [9], Gürdal and Şahiner [10], Et and Esi [11], Aiyub et al. [12], and many others).

The investigations into Tribonacci numbers were initially undertaken by a 14-year-old student Mark Feinberg [13] in 1963. Let $(t_k)_{k\in\mathbb{N}}$ be the sequence of Tribonacci numbers defined by the third-order recurrence relation $t_k = t_{k-1} + t_{k-2} + t_{k-3}$ for $k \geq 3$, with initial values $t_0 = t_1 = 1$ and $t_2 = 2$.

Hence, the initial elements of the Tribonacci sequence are 1,1,2,4,7,13,24, ... Some fundamental characteristics of the Tribonacci sequence are as follows:

$$\begin{aligned} &\lim_{k\to\infty} \frac{t_k}{t_{k+1}} = 0,54368901\ldots \\ &\lim_{k\to\infty} \frac{t_{k+1}}{t_k} = 1,83929\ldots \\ &\sum_{n=0}^{k} t_n = \frac{t_{k+2} + t_k - 1}{2}, \quad \text{for } k \geq 0. \end{aligned} \tag{6}$$

Afterwards, there has appeared much research with some arguments related of Tribonacci sequence (see Bruce [14], Choi [15], Kılıç [16], Pethe [17], Scott [18], and many others).

Yaying and Hazarika [19] defined the regular matrix $T = (t_{rk})$ involving Tribonacci numbers as follows:

$$t_{rk} = \begin{cases} \dfrac{2t_k}{t_{r+2} + t_r - 1}, & \text{if } 0 \leq k \leq r, \\ 0, & \text{if } k > r. \end{cases} \tag{7}$$

Equivalently,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} & \frac{4}{15} & \frac{7}{15} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{8}$$

The authors have defined the Tribonacci sequence spaces $X(T)$ as the set of all sequences z for which their transformations under T, denoted as Tz, belong to the spaces l_p and l_∞.

$$X(T) = \{z = (z_k) \in w: \ (Tz) \in X\}, \tag{9}$$

where $X = l_p$, $1 \leq p < \infty$, or $X = l_\infty$.

We would like to mention that the sequences $z = (z_k)$ and $y = (y_k)$ are related by

$$y_r = (Tz)_r = \sum_{k=0}^{r} \frac{2t_k}{t_{r+2} + t_r - 1} z_k, \tag{10}$$

for each $r \in \mathbb{N}$.

In later times, Yaying and Kara [20] introduced the Tribonacci sequence spaces $X(T)$ with the following definitions:

$$X(T) = \{z = (z_k) \in w: \ (Tz) \in X\}, \tag{11}$$

where $X = c$ or c_0.

In a more recent study, Dağli and Yaying [21] have defined some new paranormed sequence spaces using regular Tribonacci matrix.

Now, we give definition of new sequence spaces.

Let $u = (u_r)$ be any fixed sequence of nonzero complex numbers and $p = (p_r)$ be the bounded sequence real numbers. We have defined the following sequence spaces:

$$\begin{aligned} c(T,p,u) &= \left\{ z=(z_r)\in w\colon \lim_{r\to\infty}\left|u_r\sum_{n=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}z_k\right|^{p_r} \text{exists}\right\},\\ c_0(T,p,u) &= \left\{ z=(z_r)\in w\colon \lim_{r\to\infty}\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}z_k\right|^{p_r} =0\right\},\\ l_\infty(T,p,u) &= \left\{ z=(z_r)\in w\colon \sup_{r\in\mathbb{N}}\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}z_k\right|^{p_r} <\infty\right\},\\ l(T,p,u) &= \left\{ z=(z_r)\in w\colon \sum_{r}\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}z_k\right|^{p_r} <\infty\right\}. \end{aligned} \tag{12}$$

Using (5), we may redefine these sequence spaces by $c(T,p,u) = (c(p,u))_T$, $c_0(T,p,u) = (c_0(p,u))_T$, $l_\infty(T,p,u) = (l_\infty(p,u))_T$, and $l(T,p,u) = (l(p,u))_T$.

Remark 1. If we take $u = (1,1,1,\ldots)$ and $p = (1,1,1,\ldots)$, we obtain that the sequence spaces $c(T,p,u)$, $c_0(T,p,u)$, and $l_\infty(T,p,u)$ reduce to the sequence spaces $c(T)$, $c_0(T)$, and $l_\infty(T)$, respectively. Also if $u = (1,1,1,\ldots)$ and $p_r = p$ for all $r\in\mathbb{N}$, we obtain that the sequence space $l(T,p,u)$ reduces to $l_p(T)$.

In this paper, we examined some properties of these spaces such as completeness, Schauder basis. We establish that the novel sequence spaces $c(T,p,u), c_0(T,p,u)$, $l_\infty(T,p,u)$, and $l(T,p,u)$ are linearly isomorphic to the spaces $c(p)$, $c_0(p)$, $l_\infty(p)$, and $l(p)$, correspondingly.

2. Main Results

Now, let us give the completeness of the sequence spaces $c_0(T,p,u)$ and $l(T,p,u)$.

Theorem 2. *The sequence spaces $c_0(T,p,u)$ and $l(T,p,u)$ are complete linear metric spaces paranormed as follows:*

$$q_\infty(z) = \sup_{r\in\mathbb{N}}\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}z_k\right|^{p_r/S}, \tag{13}$$

and

$$q_p(z) = \left(\sum_r\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_m-1}z_k\right|^{p_m}\right)^{1/S}, \tag{14}$$

respectively, where $0\le p_r\le P<\infty$. It is obvious that the spaces $c(T,p,u)$ and $l_\infty(T,p,u)$ are paranormed spaces with q_∞ when $\inf p_r>0$, $c(T,p,u)=c(T)$ and $l_\infty(T,p,u) = l_\infty(T)$.

Proof. We will demonstrate the claim solely for $l(T,p,u)$ with the remaining cases following similar proofs. Let $z = (z_k)$, $y = (y_k)\in l(T,p,u)$, and it follows from Maddox [27, page 30] that

$$\begin{aligned} &\left(\sum_r\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}(z_k+y_k)\right|^{p_r}\right)^{1/S}\\ &\le\left(\sum_r\left|u_r\sum_{k=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}z_k\right|^{p_r}\right)^{1/S}\\ &+\left(\sum_r\left|u_r\sum_{n=0}^{r}\frac{2t_k}{t_{r+2}+t_r-1}(y_k)\right|^{p_r}\right)^{1/S}. \end{aligned} \tag{15}$$

Derived from (1) and (15), we ascertain the linearity of $l(T,p,u)$ concerning scalar multiplication and coordinate-wise addition. Additionally, it is evident that $q_p(\theta)=0$ and $q_p(-z)=q_p(z)$ for all z in $l(T,p,u)$. Based on (1) and (15), we establish the subadditivity of q_p as well as $q_p(\zeta z)\le\max\{1,|\zeta|\}q_p(z)$ for any $\zeta\in\mathbb{R}$.

Let $\{z^m\}$ be any sequence in $l(T,p,u)$ such that $q_p(z^r-z)\longrightarrow 0$ and (ζ^r) be any sequence in $\mathbb{R}$ such that $(\zeta^r)\longrightarrow\zeta$. With the help of the subadditivity of q_p, we can write

$$q_p(z^r)\le q_p(z)+q_p(z^r-z), \tag{16}$$

from which one can attain the boundedness of $q_p(z^r)$ and the fact that

$$q_p(\zeta_k z^r - \zeta z) = \left(\sum_r \left|u_r \sum_{k=0}^{r} \frac{2t_k}{t_{k+2} + t_k - 1}(\zeta_k z^r - \zeta z_k)\right|^{p_r}\right)^{1/S}$$
$$\leq |\zeta_k z^r - \zeta| q_p(z^r) + |\zeta| q_p(z^r - z) \longrightarrow 0, \quad (\text{as } r \longrightarrow \infty). \tag{17}$$

This provides the continuity of scalar multiplication. Consequently, q_p is a paranorm on $l(T,p,u)$. To demonstrate the completeness of $l(T,p,u)$, let $\{v^i\}$ be any Cauchy sequence in $l(T,p,u)$ such that $v^i = (v_0^i, v_1^i, v_2^i, \ldots)$ for every $i \in \mathbb{N}$. For a given $\varepsilon > 0$, there exists an integer $r_0(\varepsilon) \in \mathbb{N}$ such that

$$q_p(v^i - v^j) < \varepsilon, \tag{18}$$

for all $i, j \geq r_0(\varepsilon)$. By utilizing the definition q_p, we have

$$T_r(v^i - v^j) \leq \left(\sum_r \left|T_r(v^i) - T_r(v^j)\right|^{p_r}\right)^{1/S} < \varepsilon, \tag{19}$$

for every $i, j \geq r_0(\varepsilon)$, and this gives that $\{T_r(v^0), T_r(v^1), T_r(v^2), \ldots\}$ is a Cauchy sequence of real numbers for every fixed $r \in \mathbb{N}$. In view of the fact that $\mathbb{R}$ is complete, we get $T_r(v^i) \longrightarrow T_r(v)$, as $i \longrightarrow \infty$ for each fixed $r \in \mathbb{N}$. Considering these infinitely numerous limits $T_0(v), T_1(v), T_2(v), \ldots,$ let us establish the sequence $\{T_0(v), T_1(v), T_2(v), \ldots\}$. It arises from (18) that

$$\sum_{r=0}^{k} \left|T_r(v^i) - T_r(v^j)\right|^{p_r} \leq q_p(v^i - v^j)^S < \varepsilon^S, \tag{20}$$

for all fixed $k \in \mathbb{N}$ and $i, j \geq r_0(\varepsilon)$. If the limit is taken for $k \longrightarrow \infty$ and $j \longrightarrow \infty$ in (20), $q_p(v^i - v^j) < \varepsilon$ is obtained. We consider $\varepsilon = 1$ in (20) so that $i \geq r_0(1)$. Afterwards, we apply Minkowski's inequality, and we get that

$$\left(\sum_r |T_r(v)|^{p_r}\right)^{1/S} \leq q_p(v^i - v) + q_p(v^i) \leq 1 + q_p(v^i), \tag{21}$$

for every fixed $i \in \mathbb{N}$. Therefore, we have $v \in l(T,p,u)$. In view of the fact that $q_p(v^i - v) < \varepsilon$ for all $i \geq r_0(\varepsilon)$, we have $v^i \longrightarrow v$, as $i \longrightarrow \infty$. As a result, $l(T,p,u)$ is complete. □

Theorem 3. *The sequence spaces $c(T,p,u), c_0(T,p,u)$, $l_\infty(T,p,u)$, and $l(T,p,u)$ are linearly isomorphic to the spaces $c(p)$, $c_0(p)$, $l_\infty(p)$, and $l(p)$ correspondingly where $0 < p_r \leq S < \infty$.*

Proof. We will establish the claim exclusively for $l(T,p,u)$ while the others can be similarly demonstrated. To achieve this, we need to establish the existence of a linear transformation between $l(T,p,u)$ and $l(p)$ that satisfies the properties of being injective, surjective, and preserving paranorm. Let $H: l(T,p,u) \longrightarrow l(p)$ be a transformation such that $Hz = ((Tz)_r)$ for $z \in l(T,p,u)$.

The linearity of H is evident due to the inherent linearity found in all matrix transformations. Furthermore, the injectiveness of the transformation H is established by the fact that if $Hz = \theta$, then it follows that $z = \theta$. If we denote the sequence $z = (z_r)$ for $r \in \mathbb{N}$ as

$$z_r = \sum_{j=r-1}^{r} (-1)^{r-j} \frac{t_{r+2} + t_r - 1}{2t_j} y_j, \tag{22}$$

for any sequence $y = (y_r) \in l(p)$, then we have

$$q_p(z) = \left(\sum_r \left|u_r \sum_{k=0}^{r} \frac{2t_k}{t_{r+2} + t_r - 1} z_k\right|^{p_r}\right)^{1/S}$$
$$= \left(\sum_r \left|u_r \sum_{k=0}^{r} \frac{2t_k}{t_{r+2} + t_r - 1} (-1)^{k-j} \frac{t_{k+2} + t_k - 1}{2t_j} y_j\right|^{p_r}\right)^{1/S} \tag{23}$$
$$= \left(\sum_r |y_r|^{p_r}\right)^{1/S} = q(z) < \infty,$$

from which we get $z \in l(T, p, u)$. Therefore, since H is surjective and preserves the paranorm, this concludes the proof.

Let us construct Schauder bases for the sequence spaces $c(T, p, u)$, $c_0(T, p, u)$, and $l(T, p, u)$.

A sequence $a = (a_n)$ in X is recognized a Schauder basis for X if and only if there is a unique sequence of scalars (α_n) such that $g(x - \sum_{n=0}^{m} \alpha_n \delta_n) \longrightarrow 0$ as $m \longrightarrow \infty$. Then we write

$$x = \sum_n \alpha_n a_n. \tag{24}$$

We are ready to provide a Schauder basis for the recently defined paranormed sequence spaces. □

Theorem 4. *Let us define the sequence* $b^{(k)} = (b_r^{(k)})$ *in* $l(T, p, u)$ *as follows:*

$$b_r^{(k)} = \begin{cases} (-1)^{r-k} \dfrac{t_{k+2} + t_k - 1}{2t_r}, & \text{if } r-1 \le k \le r, \\ \\ 0, & \text{if } 0 \le k < r-1 \text{ or } k > r, \end{cases} \tag{25}$$

where $r \in \mathbb{N}$ *is fixed. Then*

(i) The set $\{e, b^{(k)}\}$ *is a Schauder basis for the space* $c(T, p, u)$ *and any* z *in* $c(T, p, u)$ *is solely determined by*

$$z = \zeta e + \sum_k (y_k - \zeta) b^{(k)}, \tag{26}$$

where $\zeta = \lim_{r \longrightarrow \infty} y_r = \lim_{r \longrightarrow \infty} (Tz)_r$.

(ii) The sequence $b^{(k)}$ *is a Schauder basis for the spaces* $l(T, p, u)$ *and* $c_0(T, p, u)$ *and any* z *in* $l(T, p, u)$ *is uniquely determined by*

$$z = \sum_k y_k b^{(k)}, \tag{27}$$

where $y_k = (Tz)_k$ *for each* $k \in \mathbb{N}$.

Proof. We will establish the claim solely for $l(T, p, u)$ with the other cases following analogous proofs.

It is obvious that

$$T b^{(k)} = e^{(k)} \in l(p), \tag{28}$$

for $0 < p_k \le P < \infty$. Let $z \in l(T, p, u)$ and denote

$$z^{[v]} = \sum_{k=0}^{r} (Tz)_k b^{(k)}, \tag{29}$$

for each nonnegative integer v. By employing (28) and (29), we derive

$$\begin{aligned} T z^{[v]} &= \sum_{k=0}^{r} (Tz)_k T b^{(k)} \\ &= (Tz)_k e^{(k)}, \end{aligned} \tag{30}$$

and

$$\left(T\left(z - z^{[v]}\right)\right)_r = \begin{cases} 0, & \text{if } 0 \le r \le v, \\ (Tz)_r, & \text{if } r > v. \end{cases} \tag{31}$$

Now, for a given $\varepsilon > 0$ there exists an integer v_0 such that

$$\left(\sum_{r \ge v} \left| (Tz)_r \right|^{p_r} \right)^{1/S} < \frac{\varepsilon}{2}, \tag{32}$$

for all $v \ge v_0$. This provides us with the information that

$$\begin{aligned} q_p\left(z - z^{[v]}\right) &= \left(\sum_{r \ge v} \left| (Tz)_r \right|^{p_r} \right)^{1/S} \\ &\le \left(\sum_{r \ge v_0} \left| (Tz)_r \right|^{p_r} \right)^{1/S} < \frac{\varepsilon}{2} < \varepsilon, \end{aligned} \tag{33}$$

for all $v \ge v_0$. This results in a representation like (27). To show the uniqueness of (27), another representation of (27) is $z = \sum_k y' b^{(k)}$. Then we write

$$(Tz)_r = \sum_k y_k' \left(T b^{(k)}\right)_r = \sum_k y_k' e_r^{(k)} = y_r', \ (r \in \mathbb{N}). \tag{34}$$

Therefore, representation of (27) is unique. □

3. The $\alpha-$, $\beta-$, and $\gamma-$Duals

In this section, we identified $\alpha-, \beta-$, and $\gamma-$duals of the sequence spaces $c(T, p, u), c_0(T, p, u)$, $l_\infty(T, p, u)$, and $l(T, p, u)$.

Now, we will provide some lemmas for our investigations. Let $A = (a_{rk})$ represent an infinite matrix of real or complex numbers and $\mathcal{N}$ denote the family of all finite subsets of $\mathbb{N}$.

Lemma 5 (see [22]). *The subsequent statements are valid:*

(i) Suppose that $1 < p_k \le P < \infty$ *for every* $k \in \mathbb{N}$. *Then* $A = (a_{rk}) \in (l(p), l_1)$ *iff there is an integer* $R > 1$ *such that*

$$\sup_{M \in \mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{r \in M} a_{rk} R^{-1} \right|^{p_k'} < \infty. \tag{35}$$

(ii) Suppose that $0 < p_k \le 1$ *for every* $k \in \mathbb{N}$. *Then* $A = (a_{rk}) \in (l(p), l_1)$ *iff*

$$\sup_{M\in\mathcal{N}}\sup_{k\in\mathbb{N}}\left|\sum_{r\in M}a_{rk}\right|^{p_k}<\infty. \tag{36}$$

Lemma 6 (see [23]). *The subsequent statements are valid:*

(i) *Suppose that $1<p_k\leq P<\infty$ for every $k\in\mathbb{N}$. Then $A=(a_{rk})\in(l(p),l_\infty)$ iff there is an integer $R>1$ such that*

$$\sup_{r\in\mathbb{N}}\sum_{k=0}^{\infty}\left|a_{rk}R^{-1}\right|^{p_k'}<\infty. \tag{37}$$

(ii) *Suppose that $0<p_k\leq 1$ for each $k\in\mathbb{N}$. Then $A=(a_{rk})\in(l(p),l_\infty)$ iff*

$$\sup_{r,k\in\mathbb{N}}\left|a_{rk}\right|^{p_k}<\infty. \tag{38}$$

(iii) *Suppose that $1<p_k\leq P<\infty$ for each $k\in\mathbb{N}$. Then $A=(a_{rk})\in(l(p),c)$ iff (37) and (38) hold and*

$$\lim_{r\longrightarrow\infty}a_{rk}=\beta_r, \tag{39}$$

for all $k\in\mathbb{N}$, also holds.

Theorem 7. *Let $w_k=1/|u_k|$, and consider the sets H_i, $1\leq i\leq 5$, defined by*

$$\begin{aligned}
H_1&=\bigcup_{R>1}\left\{h=(h_k)\in w:\ \sup_{M\in\mathbb{N}}\sum_{k=0}^{\infty}\left|\sum_{r\in M}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_rR^{-1/p_k}\right|w_k<\infty\right\},\\
H_2&=\left\{h=(h_k)\in w:\ \sum_{r=0}^{\infty}\left|\sum_{k=0}^{\infty}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_r\right|w_k<\infty\right\},\\
H_3&=\bigcap_{R>1}\left\{h=(h_k)\in w:\ \sup_{N\in\mathcal{N}}\sum_{r=0}^{\infty}\left|\sum_{k\in N}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_rR^{1/p_k}\right|w_k<\infty\right\},\\
H_4&=\bigcup_{R>1}\left\{h=(h_k)\in w:\ \sup_{M\in\mathbb{N}}\sum_{k=0}^{\infty}\left|\sum_{r\in M}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_rR^{-1}\right|^{p_k'}w_k<\infty\right\},\\
H_5&=\left\{h=(h_k)\in w:\ \sup_{M\in\mathcal{N}}\sup_{k\in\mathbb{N}}\left|\sum_{r\in M}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_rR^{-1}\right|^{p_k}w_k<\infty\right\}.
\end{aligned} \tag{40}$$

Then,

(i) *$[c(T,p,u)]^\alpha=H_1\cap H_2$ and $[c_0(T,p,u)]^\alpha=H_1$,*

(ii) *$[l_\infty(T,p,u)]^\alpha=H_3$ and $[l(T,p,u)]^\alpha=\begin{cases}H_4, & 1<p_k\leq P<\infty,\\ H_5, & 0<p_k\leq 1.\end{cases}$*

Proof. We will establish the claim exclusively for $l(T,p,u)$ while the others can be similarly demonstrated. In view of (22), we see the equality

$$h_rx_r=\sum_{k=r-1}^{r}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_ry_k=(A(t)y)_r \tag{41}$$

holds for $h=(h_k)\in w$, where $A(t)=(a^t_{rk})$ is triangle defined as

$$a^t_{rk}=\begin{cases}\displaystyle\sum_{k=r-1}^{r}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_r, & r-1\leq k\leq r,\\ 0, & \text{otherwise.}\end{cases} \tag{42}$$

Therefore, $hx=(h_kx_k)\in l_1$ whenever $x\in l(T,p,u)$ iff $A(t)y\in l_1$ whenever $y\in l(p)$. This indicates that $h=(h_k)\in[l(T,p,u)]^\alpha$ iff $A(t)\in(l(p),l_1)$. Hence, by employing Lemma 5, we observe that

$$\exists R>1 \ni \sup_{M\in\mathbb{N}}\sum_{k=0}^{\infty}\left|\sum_{r\in M}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_rR^{-1}\right|^{p_k'}w_k<\infty, 1<p_k\le P<\infty,$$
$$\sup_{M\in\mathcal{N}}\sup_{k\in\mathbb{N}}\left|\sum_{r\in M}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}h_rR^{-1}\right|^{p_k}w_k<\infty, 0<p_k\le 1. \quad (43)$$

This indicates that

$$[l(T,p,u)]^{\alpha}=\begin{cases}H_4, & 1<p_k\le P<\infty,\\ H_5, & 0<p_k\le 1.\end{cases} \quad (44)$$

□

Theorem 8. *Let* $w_k=1/|u_k|$, *and consider the sets* H_i, $6\le i\le 10$, *defined by*

$$H_6=\bigcup_{R>1}\left\{h=(h_k)\in w: \sum_{k=0}^{\infty}\left|\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}\right|R^{-1/p_k}w_k<\infty\right\},$$
$$H_7=\bigcap_{R>1}\left\{h=(h_k)\in w: \sum_{k=0}^{\infty}\left|\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}R\right|^{1/p_k}w_k<\infty\right\}\text{and}\left\{\left(\frac{t_{k+2}+t_k-1}{2t_k}h_k\right)R^{1/p_k}w_k\right\}\in c_0,$$
$$H_8=\bigcap_{R>1}\left\{h=(h_k)\in w: \sum_{k=0}^{\infty}\left|\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}\right|R^{1/p_k}w_k<\infty\right\}\text{and}\left\{\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}R^{1/p_k}w_k\right\}\in l_\infty, \quad (45)$$
$$H_9=\bigcup_{R>1}\left\{h=(h_k)\in w: \sum_{k=0}^{\infty}\left|\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}R^{-1}\right|^{p_k'}w_k<\infty\right\}\text{and}\left\{\left(\frac{t_{k+2}+t_k-1}{2t_k}h_k\right)^{p_k'}w_k\right\}\in l_\infty,$$
$$H_{10}=\left\{h=(h_k)\in w: \left\{\left(\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}\right)^{p_k}w_k\right\}\in l_\infty\right\}\text{and}\left\{\left(\frac{t_{k+2}+t_k-1}{2t_k}h_k\right)^{p_k}w_k\right\}\in l_\infty.$$

Then,

(i) $[c(T,p,u)]^{\beta}=H_6\cap c_s$ *and* $[c(T,p,u)]^{\gamma}=H_6\cap b_s$,

(ii) $[c_0(T,p,u)]^{\beta}=[c_0(T,p,u)]^{\gamma}=H_6$,

(iii) $[l_\infty(T,p,u)]^{\beta}=H_7$ *and* $[l_\infty(T,p,u)]^{\gamma}=H_8$,

(iv) $[l(T,p,u)]^{\beta}=[l(T,p,u)]^{\gamma}=\begin{cases}H_9, & 1<p_k\le P<\infty,\\ H_{10}, & 0<p_k\le 1.\end{cases}$

Proof. We will establish the claim exclusively for $l(T,p,u)$ while the others can be similarly demonstrated. We will only demonstrate the assertion for $l(T,p,u)$ with the remaining cases being proven in a similar manner.

For $h=(h_k)\in w$, we can write the following equation:

$$\sum_{k=0}^{r}h_rx_r=\sum_{k=0}^{r}h_r\left[\sum_{k=r-1}^{n}(-1)^{r-k}\frac{t_{k+2}+t_k-1}{2t_r}y_k\right]$$
$$=\sum_{k=0}^{r-1}\left(\frac{h_k}{t_k}-\frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2}+t_k-1}{2}y_k+\frac{t_{r+2}+t_r-1}{2t_r}y_rh_r=(D(t)y)_r, r\in\mathbb{N}, \quad (46)$$

where $D(t) = (d^t_{rk})$ is a triangle defined as

$$d^t_{rk} = \begin{cases} \left(\frac{h_k}{t_k} - \frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2} + t_k - 1}{2}, & 0 \le k < r, \\ \frac{t_{r+2} + t_r - 1}{2t_r} h_r, & k = r, \\ 0, & \text{otherwise}. \end{cases} \quad (47)$$

In the light of (46), we see that $hx = (h_k x_k) \in cs$ whenever $x \in l(T, p, u)$ iff $D(t) y \in c$ whenever $y \in l(p)$. This indicates that $h = (h_k) \in [l(T, p, u)]^\beta$ iff $D(t) \in (l(p), c)$. Hence, by employing Lemma 6, we observe that

$$\begin{aligned} &\sum_{k=0}^{\infty} \left| \left(\frac{h_k}{t_k} - \frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2} + t_k - 1}{2} R^{-1} \right|^{p'_k} w_k < \infty, \quad 1 < p_k \le P < \infty, \\ &\left\{ \left(\frac{t_{k+2} + t_k - 1}{2t_k} h_k\right)^{p'_k} w_k \right\} \in l_\infty, \quad 1 < p_k \le P < \infty, \end{aligned} \quad (48)$$

and

$$\begin{aligned} &\left\{ \left(\left(\frac{h_k}{t_k} - \frac{h_{k+1}}{t_{k+1}}\right)\frac{t_{k+2} + t_k - 1}{2} \right)^{p_k} w_k \right\} \in l_\infty, \quad 0 < p_k \le 1, \\ &\left\{ \left(\frac{t_{k+2} + t_k - 1}{2t_k} h_k\right)^{p_k} w_k \right\} \in l_\infty, \quad 0 < p_k \le 1. \end{aligned} \quad (49)$$

This indicates that

$$[l(T, p, u)]^\beta = \begin{cases} H_9, & 1 < p_k \le P < \infty, \\ H_{10}, & 0 < p_k \le 1. \end{cases} \quad (50)$$

One can derive the γ-dual of the space $l(T, p, u)$ using a comparable method. In order to prevent redundant repetition, we will forgo presenting the proof. □

4. Conclusion

Maddox [5, 6] introduced the linear spaces $c(p), c_0(p)$, $l_\infty(p)$, and $l(p)$. Recently, the literature focused on the creation of new sequence spaces through the matrix domain and the investigation of their algebraic and topological properties, and the study of matrix transformations has expanded. Yaying and Kara [20] introduced the Tribonacci sequence spaces. In this study, we defined some new sequence spaces using regular Tribonacci matrix. We examined some properties of these spaces such as completeness, Schauder basis. We have identified $\alpha-, \beta-$, and $\gamma-$duals of the newly created spaces. In the future, new sequence spaces can be defined by taking this study into consideration.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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