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Research Article

Fractional Mixed Weighted Convolution and Its Application in Convolution Integral Equations

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The convolution integral equations are very important in optics and signal processing domain. In this paper, fractional mixed-weighted convolution is defined based on the fractional cosine transform; the corresponding convolution theorem is achieved. The properties of fractional mixed-weighted convolution and Young's type theorem are also explored. Based on the fractional mixed-weighted convolution and fractional cosine transform, two kinds of convolution integral equations are considered, the explicit solutions of fractional convolution integral equations are obtained, and the computational complexity of solutions are also analyzed.

1. Introduction

The convolution is a powerful mathematical tool that plays a crucial role in various fields such as applied mathematics, harmonic analysis, integral equation solving, signal processing, image processing, and neural networks [1–6]. It enables signal filtering, feature extraction, and system response analysis functions, making it highly significant for advanced signal processing and pattern recognition realization.

The convolution integral equations arise in many branches of natural science and have important applications in various fields such as engineering mechanics, dynamic theory, mathematical theory of the spatial-temporal spread of pandemic, especially when solving the problems of optical systems and the digital signal processing domain [7–13]. In recent years, the convolution integral equations have been studied extensively by many researchers [14–20]. Tuan [14] studied solvability in close form and estimated the boundedness solutions of some classes for integral differential equations of the Barbashin type and the Frcdhohn type integral equation. Askhabov [15] studied various classes of nonlinear convolution-type integral equations appearing in the theory of feedback systems. Sun et al. [16] studied the existence and noethericity of solutions for two classes of singular convolution integral equations with Cauchy kernels in the nonnormal type case. And the solutions for some singular convolution integral equations are discussed in [17–20].

However, all the studies mentioned above are based on Fourier analysis theory, which seriously limits its application scope because of its nonlocality. These limitations force people to find some improvement methods. In recent years, many scholars devote themselves to extending Fourier analysis to fractional domain and study fractional convolution integral equations. K. Razminia and A. Razminia [21] studied fractional diffusion equation (FDE) using the convolution integral. Li et al. [22] analyzed the solvability of the convolution equations by convolution operator for a two-dimensional fractional Fourier transform in polar coordinates. Feng and Wang [23] discussed explicit solutions of the convolution-type integral equations using the generalized fractional convolution. In [24, 25], the author studied the convolution-type integral equations based on the fractional Laplace convolutions. As far as we are concerned, on the one hand, compared with the wide applications of convolution equation in the Fourier domain, the fractional convolution equation is less studied. On the other hand, the mixed-weighted convolution equations in fractional domain have not been studied yet. Hence, it is therefore interesting and worthwhile to investigate fractional convolution integral equations in depth, and how to obtain solutions to these convolution equations is one of the meaningful issues of equation theory.

In this paper, we investigate two types of fractional mixed-weighted convolution integral equations. The contribution of this study is threefold: (1) we propose two kinds of fractional mixed-weighted convolution that enable the processing and analysis of input data by selecting appropriate weighting coefficients, thereby achieving the desired processing effect. (2) The proposed fractional mixed-weighted convolution for the fractional cosine transform can be expressed by classical convolution. (3) We apply these newly developed convolution structures to discuss solutions for the convolution integral equations, which can be efficiently computed using FFT and which exhibit lower computational complexity compared to methods employed in the FRFT domain.

The rest of this paper is organized as follows: Section 2 presents preliminaries. In Section 3, the fractional mixed-weighted convolution for the fractional cosine transform is proposed, and the corresponding convolution theorem is derived. The important relation between the mixed-weighted fractional convolution and the classical convolution is established, and properties and Young's type theorem are further investigated. In Section 4, two kinds of fractional mixed-weighted convolution equations are discussed, explicit solutions for these convolution equations are given, and the computation complexity of solutions are analyzed. Conclusions are summarized in Section 5.

2. Preliminaries

In this section, we introduce some definitions and important properties of the fractional Fourier transform, fractional cosine transform, and corresponding convolution operation.

The fractional Fourier transform (FRFT) [26] is defined as follows:

$$\left(F^{\alpha} f\right)(u) = \int_{\mathbb{R}} f(t) K^{\varphi}(t,u) e^{-\mathrm{j}tu\csc\varphi} \mathrm{d}t, \tag{1}$$

here

$$K^{\varphi}(t,u) = \begin{cases} A_{\varphi} e^{j\left(t^2+u^2/2\right)\cot\varphi}, & \varphi \neq k\pi, \\ \delta(t-u), & \varphi = 2k\pi, \\ \delta(t+u), & \varphi = (2k-1)\pi, \end{cases} \tag{2}$$

where $A_{\varphi} = \sqrt{1 - j\cot\varphi/2\pi}, \varphi = \pi/2\alpha$. Whenever $\varphi = (2k-1)\pi/2, k \in Z$, (1) reduces to the classical Fourier transform (FT) as follows:

$$(Ff)(u) = \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} f(t) e^{-\mathrm{j}tu} \mathrm{d}t. \tag{3}$$

Based on the FRFT and the FT, fractional cosine transform (FRCT) [27, 28] are defined as follows:

$$\left(F_c^{\alpha} f\right)(u) = 2\int_{\mathbb{R}_+} f(t) K^{\varphi}(t,u) \cos(\csc\varphi \cdot \mathrm{tu}) \mathrm{d}t, \quad u > 0, \tag{4}$$

where $\varphi = \pi\alpha/2$. The inverse transform of FRCT is given by

$$f(t) = 2\int_{\mathbb{R}_+} F_c^{\alpha}(u) K^{-\varphi}(t,u) \cos(\csc\varphi \cdot tu) \mathrm{d}u, \quad t > 0, \tag{5}$$

when $\varphi = (2k-1)\pi/2, k \in Z$, the fractional cosine transform (FRCT) is reduced to Fourier cosine transform (FCT) [29].

$$\left(F_c f\right)(u) = \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}_+} f(t) \cos(\mathrm{tu}) \mathrm{d}t, \quad u > 0, \tag{6}$$

where $\left(F_c f\right)(u)$ denotes the FCT. From (5) and (6), the FRCT can be expressed by FCT as follows:

$$\left(F_c^{\alpha} f\right)(u) = 2\sqrt{2\pi} A_{\varphi} \widetilde{F}_c(\widetilde{f}(t))(\csc\varphi \cdot u), \tag{7}$$

and $(\widetilde{F}_c f)(u) = e^{ju^2/2\cot\varphi} \left(F_c f\right)(u)$, $\widetilde{f}(t) = e^{jt^2/2\cot\varphi} f(t)$. From (7), we can realize the calculation of FRCT (see Figure 1). For N point of samples, FRCT has the same computational complexity as FCT, that is, $O(1/2N\log_2^N)$, which is very important in practical applications.

The classical convolution operation [30] is given by

$$h(t) * f(t) = \int_{\mathbb{R}} h(\tau) f(t-\tau) \mathrm{d}\tau, \tag{8}$$

which satisfies the following convolution theorem:

$$F[(h * f)(t)](u) = (Fh)(u)(Ff)(u), \tag{9}$$

where $*$ denotes the classical convolution operation.

The fractional cosine convolution, denoted by $(h \underset{F_c^{\alpha}}{*} f)(t)$ was recently defined in [28].

$$\begin{aligned}\left(h \underset{F_c^{\alpha}}{*} f\right)(t) = A_{\varphi} e^{-\mathrm{j}t^2/2\cot\varphi} \int_{\mathbb{R}_+} \widetilde{h}(\tau)[\widetilde{f}(|t-\tau|) \\ + \widetilde{f}(t+\tau)] \mathrm{d}\tau,\end{aligned} \tag{10}$$

and the corresponding convolution theorem for FRCT is satisfied

$$F_c^{\alpha}\left[\left(h \underset{F_c^{\alpha}}{*} f\right)(t)\right](u) = e^{-\mathrm{j}u^2/2\cot\varphi} \left(F_c^{\alpha} h\right)(u) \left(F_c^{\alpha} f\right)(u), \quad u > 0. \tag{11}$$

3. Fractional Mixed-Weighted Convolution and Convolution Theorem for FRCT

Convolution is an integral transform, which is very important in optical systems and signal processing, especially in solving convolution integral equations. This section primarily provides the definition of fractional mixed-weighted convolution for the fractional cosine transform and derives the corresponding convolution theorem. Additionally, it explores the relationship between the proposed convolution and existing convolutions, as well as investigates the properties and Young's type theorem of fractional mixed-weighted convolution.

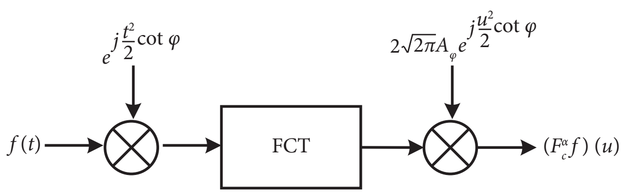

Figure 1: The calculation process of FRCT.

3.1. Fractional Mixed-Weighted Convolution for FRCT. In this subsection, we give fractional mixed-weighted convolution for the fractional cosine transform, the relationship between proposed convolution and classical convolution is given.

Definition 1. For any two functions $h(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $f(t) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, fractional mixed-weighted convolution operation of $h(t)$ and $f(t)$ for fractional cosine transform is defined as follows:

$$\left(h \overset{\gamma}{*}_{\alpha} f\right)(t) = D_{\varphi} e^{-\mathrm{j}t^2/2\cot\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} I(s, v, t) h(s) \tilde{f}(v) \mathrm{d}s\, \mathrm{d}v, \quad (12)$$

where $\gamma = e^{-u}\cos u, u > 0, D_{\varphi} = \sqrt{1/2\pi}|\csc\varphi|/2\pi, \tilde{f}(t) = e^{\mathrm{j}t^2/2\cot\varphi} f(t)$, and

$$\begin{aligned} I(s, v, t) = {} & \frac{1 + js}{(1 + js)^2 + [1 + (v + t)\csc\varphi]^2} + \frac{1 + js}{(1 + js)^2 + [1 + (v - t)\csc\varphi]^2} \\ & + \frac{1 + js}{(1 + js)^2 + [1 - (v - t)\csc\varphi]^2} + \frac{1 + js}{(1 + js)^2 + [1 - (v + t)\csc\varphi]^2}. \end{aligned} \quad (13)$$

Based on the Definition 1, the fractional mixed-weighted convolution operation for the FRCT can be implemented in Figure 2.

Remark 2. According to Definition 1, when $\varphi = (2k - 1)\pi/2$, $k \in Z$, the fractional mixed-weighted convolution operation $(h \overset{\gamma}{*}_{\alpha} f)(t)$ reduces to mixed-weighted convolution operation $(h *^{\gamma} f)(t)$ in the Fourier domain.

Remark 3. Let $h(t) \in L_1(\mathbb{R}, \sqrt{1 + t^2})$ and $f(t) \in L_1(\mathbb{R}_+)$, then the fractional mixed-weighted convolution operations $(h \overset{\gamma}{*}_{\alpha} f)(t)$ can be expressed by classical convolution $*$ as follows:

$$\begin{aligned} \left(h \overset{\gamma}{*}_{\alpha} f\right)(t) = {} & D_{\varphi} e^{-\mathrm{j}t^2/2\cot\varphi} \cdot \left\{ \int_{\mathbb{R}} h(s) \left[\tilde{f}(v) * \frac{(1 + js)\mathrm{sign}v}{(1 + js)^2 + v^2\csc^2\varphi} \right] \left(t + (\sin\varphi)^2\right) \mathrm{d}s \right. \\ & \left. + \int_{\mathbb{R}} h(s) \left[\tilde{f}(v) * \frac{(1 + js)\mathrm{sign}v}{(1 + js)^2 + v^2\csc^2\varphi} \right] \left(t - \sin^2\varphi\right) \mathrm{d}s \right\}. \end{aligned} \quad (14)$$

Based on Definition 1, we will give the fractional mixed-weighted convolution theorem associated with the fractional cosine transform.

3.2. Fractional Mixed-Weighted Convolution Theorem for FRCT. In this section, the fractional mixed-weighted convolution theorem for the fractional cosine transform is derived.

Theorem 4. *Let $h(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $f(t) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, the fractional mixed-weighted convolution $(h \overset{\gamma}{*}_{\alpha} f)(t) \in L_1(\mathbb{R}_+)$ satisfies the following factorization property*

$$F_c^{\alpha}\left[\left(h \overset{\gamma}{*}_{\alpha} f\right)(t)\right](u) = e^{-u}\cos u\, (\mathrm{Fh})(u) \left(F_c^{\alpha} f\right)(u), \quad u > 0. \quad (15)$$

Proof. We first prove the existence of the convolution operation $(h \overset{\gamma}{*}_{\alpha} f)(t)$. From Definition 1, we have

$$\left\| \left(h \overset{\gamma}{*}_{\alpha} f\right)(t) \right\|_{L_1(\mathbb{R}_+)} \leq D_{\varphi} \int \mathbb{R}_+ \int \mathbb{R} \int \mathbb{R}_+ |I(s, v, x) h(s) \tilde{f}(v) \mathrm{d}s\, \mathrm{d}v| \mathrm{d}t, \quad (16)$$

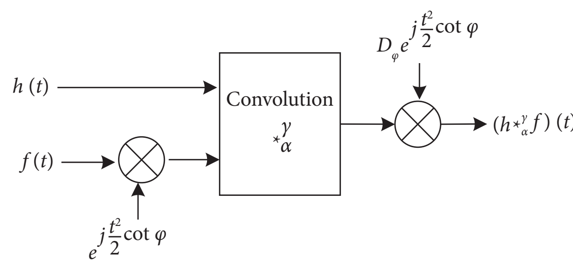

Figure 2: Implementation of the fractional mixed-weighted convolution in time domain.

Since

$$\begin{aligned}\int_{\mathbb{R}_+}\left|\frac{1+js}{(1+js)^2+[1+(v+t)\csc\varphi]^2}\right|dt &= \int_{\mathbb{R}_+}\frac{\sqrt{1+s^2}}{\sqrt{\left([1+(v+t)\csc\varphi]^2-s^2+1\right)^2+4s^2}}dt\\ &\le \int_{\mathbb{R}_+}\frac{\sqrt{1+s^2}}{\sqrt{\left([1+(v+t)\csc\varphi]^2-s^2\right)^2+s^2+1}}dt\\ &\le \int_{1+v\csc\varphi}^{\infty}\frac{\sqrt{1+s^2}}{\sqrt{(t-s)^2+1+s^2}}dt\le\pi.\end{aligned}\tag{17}$$

The same estimation is obtained for the other three integrals in a similar manner

$$\begin{aligned}&\int_{\mathbb{R}_+}\left|\frac{1+js}{(1+js)^2+[1+(v-t)\csc\varphi]^2}\right|dt\le\pi,\\ &\int_{\mathbb{R}_+}\left|\frac{1+js}{(1+js)^2+[1-(v-t)\csc\varphi]^2}\right|dt\le\pi,\\ &\int_{\mathbb{R}_+}\left|\frac{1+js}{(1+js)^2+[1-(v+t)\csc\varphi]^2}\right|dt\le\pi.\end{aligned}\tag{18}$$

According to (12), (13), (16), and (17), we have

$$\left\|\left(h*_\alpha^\gamma f\right)(t)\right\|_{L_1(R_+)}\le 4\pi D_\varphi\int_{\mathbb{R}_+}|h(s)|ds\int_{\mathbb{R}_+}|f(v)|dv<\infty,\tag{19}$$

therefore, $(h*_\alpha^\gamma f)(t)\in L_1(\mathbb{R}_+)$. Next, we prove the convolution Theorem 4. We have

$$\begin{aligned}\cos u\cos(\csc\varphi\cdot vu)\cos(\csc\varphi\cdot tu) &= \frac{1}{4}[\cos(u(1+\csc\varphi\cdot(v+t)))\\ &+\cos(u(1+\csc\varphi\cdot(v-t)))\\ &+\cos(u(1-\csc\varphi\cdot(v+t)))\\ &+\cos(u(1-\csc\varphi\cdot(v-t)))],\end{aligned}\tag{20}$$

therefore, it follows

$$\begin{aligned}&e^{-u}\cos u(\mathrm{Fh})(u)(F_c^\alpha f)(u)\\ &=\sqrt{\frac{2}{\pi}}A_\varphi\int_{\mathbb{R}}\int_{\mathbb{R}_+}e^{-u(1+js)}e^{j(v^2+u^2)/2\cot\varphi}h(s)\tilde{f}(s)\cdot\cos u\cos(\csc\varphi\cdot\mathrm{vu})ds\,dv,\end{aligned}\tag{21}$$

from equations (12), (20), and (21), we can obtain

$$\begin{aligned}&2\int_{\mathbb{R}_+}(e^{-u}\cos u(\mathrm{Fh})(u)(F_c^\alpha f)(u))K^{-\varphi}(t,u)\cos(\csc\varphi\cdot\mathrm{tu})du\\ &=4D_\varphi e^{-\mathrm{j}t^2/2\cot\varphi}\int_{\mathbb{R}}\int_{\mathbb{R}_+^2}\cos u\cos(\csc\varphi\cdot\mathrm{vu})\cos(\csc\varphi\cdot\mathrm{tu})e^{-u(1+js)}h(s)\tilde{f}(v)ds\,dv\,du\\ &=D_\varphi e^{-\mathrm{j}t^2/2\cot\varphi}\int_{\mathbb{R}}\int_{\mathbb{R}_+^2}e^{-u(1+\mathrm{j}s)}[\cos(u(1+\csc\varphi(v+t)))+\cos(u(1+\csc\varphi(v-t)))\\ &\quad+\cos(u(1-\csc\varphi(v+t)))+\cos(u(1-\csc\varphi(v-t)))]h(s)\tilde{f}(s)ds\,dv\,du\\ &=D_\varphi e^{-\mathrm{j}t^2/2\cot\varphi}\int_{\mathbb{R}}\int_{\mathbb{R}_+}I(s,v,t)h(s)\tilde{f}(v)ds\,dv.\end{aligned}\tag{22}$$

This completes the proof.

The fractional mixed-weighted convolution is very difficult to implement in the time domain due to the integral operation, as it is evident from Definition 1 and Figure 2. However, thanks to Theorem 4, it can be realized in the FRCT domain (refer to Figure 3). For N points of samples, the computational complexity of the fractional mixed-weighted convolution is given by $O(2N\log_2^N)$.

Remark 5. From Theorem 4, when $\varphi = (2k-1)\pi/2$, $k \in Z$, the fractional mixed-weighted convolution theorem in equation (15) reduces to the corresponding convolution theorem in the Fourier domain.

Remark 6. The fractional mixed-weighted convolution theorem in equation (15) preserves the convolution property for the classical Fourier transform, meaning that the fractional mixed-weighted convolution of two functions is equivalent to multiplying their FT and FRCT. This can be particularly useful in solving convolution integral equations and designing filters.

3.3. Properties of Fractional Mixed-Weighted Convolution. In this subsection, properties of fractional mixed-weighted convolution are given and the corresponding Young's type theorem is also explored as follows.

Theorem 7. *The fractional mixed-weighted convolution for FRCT is not commutative or associative, but is distributive and linear, which satisfies the following equations:*

(1) $(h\overset{\gamma}{*}_{\alpha}(f+g))(t) = (h\overset{\gamma}{*}_{\alpha} f)(t) + (h\overset{\gamma}{*}_{\alpha} g)(t)$,

(2) $((\mu h_1 + \nu h_2)\overset{\gamma}{*}_{\alpha} f)(t) = \mu(h\overset{\gamma}{*}_{\alpha} f)(t) + \nu(h\overset{\gamma}{*}_{\alpha} f)(t)$,

where $h(t), h_1(t)$ *and* $h_2(t) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $f(t)$ *and* $g(t) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, $\mu, \nu \in \mathbb{C}$.

Proof. The distributivity and linearity can be proven by Definition 1 and Theorem 4, therefore, they are omitted here. □

Theorem 8 (Young's type theorem). *Let* $h(t) \in L_p(\mathbb{R}_+, (\sqrt{1+t^2})^{p-1})$, $f(t) \in L_q(\mathbb{R}_+)$, *and* $\omega(t) \in L_r(\mathbb{R}_+)$, *such that* $1/p + 1/q + 1/r = 2$, $p, q, r > 1$, *then we have*

$$\left|\int_{\mathbb{R}_+}\left(h\overset{\gamma}{*}_{\alpha} f\right)t)\cdot\omega(t)\mathrm{d}t\right| \leq 4\pi D_{\varphi}\,\|h\|_{L_p\left(\mathbb{R}_+,\left(\sqrt{1+t^2}\right)^{p-1}\right)}\|f\|_{L_q(\mathbb{R}_+)}\|\omega\|_{L_r(\mathbb{R}_+)}. \tag{23}$$

Proof. Let $p_1, q_1, r_1 > 1$, such that $1/p + 1/p_1 = 1$, $1/q + 1/q_1 = 1$, $1/r + 1/r_1 = 1$, which means $1/p_1 + 1/q_1 + 1/r_1 = 1$. Denote

$$U(s,\nu,t) = \left|f(\nu)^{q/|p_1|}\omega(t)^{r/|p_1|}\frac{I(s,\nu,t)}{\sqrt{1+s^2}}\right|^{1/p_1}, \tag{24}$$

$$V(s,\nu,t) = |h(s)|^{p/q_1}|\omega(t)|^{r/q_1}\left|\sqrt{1+s^2}\right|^{p-1/q_1}|I(s,\nu,t)|^{1/q_1}, \tag{25}$$

$$W(s,\nu,t) = |h(s)|^{p/r_1}|f(\nu)|^{q/r_1}\left|\sqrt{1+s^2}\right|^{p-1/r_1}|I(s,\nu,t)|^{1/r_1}, \tag{26}$$

form equations (24), (25), and (26), we have

$$(U,V,W)(s,\nu,t) = |h(s)\|f(\nu)\|\omega(t)\|I(s,\nu,t)|, \tag{27}$$

due to the following inequality:

$$\begin{aligned}&\int_{\mathbb{R}_+}\frac{\mathrm{d}s}{\left|(1+\mathrm{j}s)^2 + [1\pm(\nu\pm t)\csc\varphi]^2\right|}\\&\leq \int_{\mathbb{R}_+}\frac{ds}{\sqrt{\left(s^2-(1\pm(\nu\pm t)\csc\varphi)^2\right)+1}} \leq \int_{\mathbb{R}_+}\frac{\mathrm{d}s}{(s-(1\pm(\nu\pm t)\csc\varphi))^2+1} \leq \pi,\end{aligned} \tag{28}$$

according to (24) and (28), in the space $L_{p_1}(R_+^3)$, we obtain

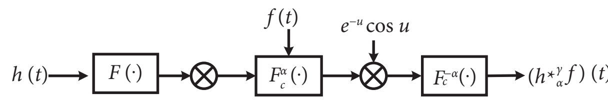

Figure 3: Implementation of the fractional mixed-weighted convolution in FRCT domain.

$$\begin{aligned}\|U\|_{L_{p_1}(\mathbb{R}_+^3)}^{p_1} &= \iiint_{\mathbb{R}_+^3} |f(v)|^q |\omega(t)|^r \left|\frac{I(s,v,t)}{\sqrt{1+s^2}}\right| \mathrm{d}s\mathrm{d}v\mathrm{d}t \\ &\le 4\pi \int_{\mathbb{R}_+} |f(v)|^q \mathrm{d}v \int_{\mathbb{R}_+} |\omega(t)|^r \mathrm{d}t \\ &= 4\pi \|f\|_{L_q(\mathbb{R}_+)}^q \|\omega\|_{L_r(\mathbb{R}_+)}^r.\end{aligned} \tag{29}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}_+} \frac{\mathrm{d}v}{\left|(1+js)^2 + [1 \pm (v \pm t)\csc\varphi]^2\right|} \\ &\le \int_{\mathbb{R}_+} \frac{\mathrm{d}v}{\sqrt{\left(s^2 - (1 \pm (v \pm t)\csc\varphi)^2\right)^2 + 1}} \le \int_{\mathbb{R}_+} \frac{\mathrm{d}v}{(s - (1 \pm (v \pm t)\csc\varphi))^2 + 1} \le \pi,\end{aligned} \tag{30}$$

has the same upper bound as (28), therefore, based on equations (25), (26), and (30), in the space $L_{q_1}(R_+^3), L_{r_1}(R_+^3)$, we have

$$\|V\|_{L_{q_1}(\mathbb{R}_+^3)}^{q_1} = \iiint_{\mathbb{R}_+^3} |h(s)|^p \left(\sqrt{1+s^2}\right)^{p-1} |\omega(t)|^r |I(s,v,t)| \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}t \le 4\pi \|h\|_{L_p\left(\mathbb{R}_+,\left(\sqrt{1+t^2}\right)^{p-1}\right)}^p \|\omega\|_{L_r(\mathbb{R}_+)}^r, \tag{31}$$

and

$$\begin{aligned}\|W\|_{L_{r_1}(\mathbb{R}_+^3)}^{r_1} &= \iiint_{\mathbb{R}_+^3} |h(s)|^p \left|\left(\sqrt{1+s^2}\right)^{p-1}\right| |f(v)|^q |I(s,v,t)| \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}t \\ &\le 4\pi \|h\|_{L_p\left(\mathbb{R}_+,\left(\sqrt{1+t^2}\right)^{p-1}\right)}^p \|f\|_{L_q(\mathbb{R}_+)}^q.\end{aligned} \tag{32}$$

According to equations (29), (31), and (32), we obtain

$$\|U\|_{L_{p_1}(\mathbb{R}_+^3)} \|V\|_{L_{q_1}(\mathbb{R}_+^3)} \|W\|_{L_{r_1}(\mathbb{R}_+^3)} \le 4\pi \|h\|_{L_p\left(\mathbb{R}_+,\left(\sqrt{1+t^2}\right)^{p-1}\right)} \|f\|_{L_q(\mathbb{R}_+)} \|\omega\|_{L_r(\mathbb{R}_+)}, \tag{33}$$

from the Hӧlder's inequality, (12) and (33), we have

$$\begin{aligned}\left|\int_{\mathbb{R}_+} \left(h \overset{\gamma_1}{*}_\alpha f\right) t) \cdot \omega(t) \mathrm{d}t\right| &\le D_\varphi \iiint_{\mathbb{R}_+^3} U(s,v,t) V(s,v,t) W(s,v,t) \mathrm{d}s\, \mathrm{d}v\, \mathrm{d}t \\ &= D_\varphi \|U\|_{L_{p_1}(\mathbb{R}_+^3)} \|V\|_{L_{q_1}(\mathbb{R}_+^2)} \|W\|_{L_{r_1}(\mathbb{R}_+^3)} \\ &\le 4\pi D_\varphi \|h\|_{L_p\left(\mathbb{R}_+,\left(\sqrt{1+t^2}\right)^{p-1}\right)} \|f\|_{L_q(\mathbb{R}_+)} \|\omega\|_{L_r(\mathbb{R}_+)}.\end{aligned} \tag{34}$$

This completes the proof. □

4. Application of Mixed-Weighted Convolution in the Convolution Integral Equation

The convolution integral equation is of great importance in various applications, particularly in solving engineering problems such as optical systems and digital signal processing. These problems can be transformed into the forms of (35) and (43). How to solve the solutions of these equations is one of the meaningful issues of equation theory.

Next, we will use the convolution theorem derived in this paper to study two types of convolution integral equations.

4.1. The First Kind of the Convolution Integral Equation. In this subsection, we shall focus on the following convolution integral equation:

$$\begin{aligned}&\lambda_1 h(t) + e^{-jt^2/2\cot\varphi}\left\{\lambda_2 D_\varphi \int_{\mathbb{R}}\int_{\mathbb{R}_+} e^{jv^2/2\cot\varphi} I(s,v,t)\varphi(s)h(v)\mathrm{d}s\,\mathrm{d}v\right.\\&\left.+\lambda_3 A_\varphi \int_{\mathbb{R}_+} h(s)[\tilde{\psi}(|t-s|)\mathrm{d}s + \tilde{\psi}(|t+s|)]\mathrm{d}s\right\} = g(t),\end{aligned}\tag{35}$$

where $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$, $\varphi, \psi, g \in L_1(\mathbb{R}_+)$ are given, and h is unknown function. After simplification, (35) can be rewritten in the following form:

$$\lambda_1 h(t) + \lambda_2\left(\varphi \overset{\gamma}{*}_\alpha h\right)(t) + \lambda_3\left(h \underset{F_c^\alpha}{*} \psi\right)(t) = g(t),\tag{36}$$

where $(\varphi \overset{\gamma}{*}_\alpha h)(t)$ denotes the fractional mixed-weighted convolution operation in (12), and $(h \underset{F_c^\alpha}{*} \psi)(t)$ denotes convolution operation in [28]. By applying fractional cosine transform to both sides of (36) and utilizing (15) and Theorem 7 (refer to [28]), we can obtain

$$(F_c^\alpha h)(u) = \frac{1}{\lambda_1 + W(u)}(F_c^\alpha g)(u), \quad u > 0,\tag{37}$$

where

$$W(u) = \lambda_2 e^{-u}\cos u(F\varphi)(u) + \lambda_3 e^{-ju^2/2\cot\varphi}(F_c^\alpha\psi)(u).\tag{38}$$

Case 9. When $\lambda_1 \neq 0$ and λ_2, λ_3 are not all zero, from [31], there exists a constant $C > 0$, such that $\lambda_1 + W(u) \neq 0$, for all $u > C$. Hence, $1/(\lambda_1 + W(u))$ is bounded and continuous, and we have $(F_c^\alpha g)(u)/(\lambda_1 + W(u)) \in L_1(\mathbb{R}_+)$. Applying inverse transform of FRCT to equation (34), we can obtain the general solution of equation (32) as follows:

$$h(t) = F_c^{-\alpha}\left[\frac{(F_c^\alpha g)(u)}{\lambda_1 + W(u)}\right](t).\tag{39}$$

Case 10. When $\lambda_1 = 0$ and λ_2, λ_3 are not all zero, for all $u > 0$, such that $W(u) \neq 0$, the general solution of equation (32) is obtained in a similar manner as described as follows:

$$h(t) = F_c^{-\alpha}\left[\frac{(F_c^\alpha g)(u)}{W(u)}\right](t).\tag{40}$$

From the above analysis, we give the main results about the solution of (35).

Theorem 11. *Let $W(u) = \lambda_2 e^{-u}\cos u(F\varphi)(u) + \lambda_3 e^{-ju^2/2\cot\varphi}\cdot(F_c^\alpha\psi)(u)$. Equation (32) has the general solution as follows:*

(1) When $\lambda_1 \neq 0$ and λ_2, λ_3 are not all zero, for all $u > C > 0$. Then, the solution of (35) is given by

$$h(t) = F_c^{-\alpha}\left[\frac{(F_c^\alpha g)(u)}{\lambda_1 + W(u)}\right](t),\tag{41}$$

(2) When $\lambda_1 = 0$ and λ_2, λ_3 are not all zero, for all $u > 0$, then the solution of (35) is given by

$$h(t) = F_c^{-\alpha}\left[\frac{(F_c^\alpha g)(u)}{W(u)}\right](t).\tag{42}$$

4.2. The Second Kind of System of the Convolution Integral Equation. Let $\lambda_1, \lambda_2 \in \mathbb{C}, k_1, k_2, \phi, \psi \in L_1(\mathbb{R}_+)$ be given, h, f be unknown functions, we consider system of convolution integral (43) as follows:

$$\begin{cases} h(t) + \lambda_1 D_\varphi e^{-jt^2/2\cot\varphi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} e^{jv^2/2\cot\varphi} I(s,v,t)\varphi(s)f(v)\mathrm{d}s\,\mathrm{d}v = k_1(t),\\ f(t) + \lambda_2 A_\varphi e^{-jt^2/2\cot\varphi}\int_{\mathbb{R}_+} h(s)[\tilde{\psi}(|t-s|) + \tilde{\psi}(|t+s|)]\mathrm{d}s = k_2(t),\end{cases}\tag{43}$$

where

$$I(s,v,t)=\frac{1+\mathrm{js}}{(1+\mathrm{js})^2+[1+(v+t)\csc\varphi]^2}+\frac{1+\mathrm{js}}{(1+\mathrm{js})^2+[1+(v-t)\csc\varphi]^2}+\frac{1+\mathrm{js}}{(1+\mathrm{js})^2+[1-(v-t)\csc\varphi]^2}+\frac{1+\mathrm{js}}{(1+\mathrm{js})^2+[1-(v+t)\csc\varphi]^2}, \tag{44}$$

and $\widetilde{f}(t)=f(t)e^{jt^2/2\cot\varphi}$, $\widetilde{\psi}(t)=\psi(t)e^{jt^2/2\cot\varphi}$. D_φ and A_φ correspond to (2) and (12), respectively.

Theorem 12. *Let $1-\lambda_1\lambda_2e^{-ju^2/2\cot\varphi}F_c^\alpha(\varphi\overset{\gamma}{*}_\alpha\psi)(u)\neq 0$, for all $u\in\mathbb{R}_+$. Suppose their exists a function $\rho\in L_1(\mathbb{R}_+)$, such that*

$$\left(F_c^\alpha\widetilde{\rho}\right)(u)=\frac{\lambda_1\lambda_2e^{-\mathrm{j}u^2/2\cot\varphi}F_c^\alpha\left(\varphi\overset{\gamma}{*}_\alpha\psi\right)(u)}{1-\lambda_1\lambda_2e^{-\mathrm{j}u^2/2\cot\varphi}F_c^\alpha\left(\varphi\overset{\gamma}{*}_\alpha\psi\right)(u)}, \tag{45}$$

where $\left(F_c^\alpha\widetilde{\rho}\right)(u)=e^{-ju^2/2\cot\varphi}\left(F_c^\alpha\rho\right)(u)$. Then, (43) has the unique solution in $L_1(\mathbb{R}_+)$.

$$h(t)=k_1(t)-\lambda_1\left(\varphi\overset{\gamma}{*}_\alpha k_2\right)(t)+\left(\rho\underset{F_c^\alpha}{*}k_1\right)(t)-\lambda_1\left[\rho\underset{F_c^\alpha}{*}\left(\varphi\overset{\gamma}{*}_\alpha k_2\right)(t).\right. \tag{46}$$

$$f(t)=k_2(t)-\lambda_2\left(\psi\underset{F_c^\alpha}{*}k_1\right)(t)+\left(\rho\underset{F_c^\alpha}{*}k_2\right)(t)-\lambda_2\left[\rho\underset{F_c^\alpha}{*}\left(\psi\underset{F_c^\alpha}{*}k_1\right)(t).\right. \tag{47}$$

Proof. The system of convolution integral equation (37) can be rewritten as follows:

$$\begin{cases} h(t)+\lambda_1\left(\varphi\overset{\gamma}{*}_\alpha f\right)(t)=k_1(t),\\ f(t)+\lambda_2\left(h\underset{F_c^\alpha}{*}\psi\right)(t)=k_2(t), \end{cases} \tag{48}$$

by applying the fractional cosine transform, (15) and Theorem 7 [28] to both sides of (48), we can obtain

$$\begin{aligned} &\left(F_c^\alpha h\right)(u)+\lambda_1e^{-u}\cos u(F\varphi)(u)\left(F_c^\alpha f\right)(u)=\left(F_c^\alpha k_1\right)(u),\\ &\left(F_c^\alpha f\right)(u)+\lambda_2e^{-j\left(u^2/2\right)\cot\varphi}\left(F_c^\alpha h\right)(u)\left(F_c^\alpha\psi\right)(u)=\left(F_c^\alpha k_2\right)(u). \end{aligned} \tag{49}$$

According to Wiener-Levi's Theorem [30] and (45), we can derive

$$\begin{aligned} \left(F_c^\alpha h\right)(u)&=\frac{\left(F_c^\alpha k_1\right)(u)-\lambda_1F_c^\alpha\left(\varphi^\gamma *_\alpha k_2\right)(u)}{1-\lambda_1\lambda_2e^{-\mathrm{j}u^2/2\cot\varphi}F_c^\alpha\left(\varphi^\gamma *_\alpha\psi\right)(u)}\\ &=\left(\left(F_c^\alpha k_1\right)(u)-\lambda_1F_c^\alpha\left(\varphi^\gamma *_\alpha k_2\right)(u)\right)\bullet\left(1+e^{-ju^2/2\cot\varphi}\left(F_c^\alpha\rho\right)(u)\right)\\ &\cdot\left(F_c^\alpha k_1\right)(u)-\lambda_1F_c^\alpha\left(\varphi^\gamma *_\alpha k_2\right)(u)+F_c^\alpha\left(\rho\underset{F_c^\alpha}{*}k_1\right)(u)-\lambda_1F_c^\alpha\left(\rho\underset{F_c^\alpha}{*}\left(\varphi^\gamma *_\alpha k_2\right)\right)(u), \end{aligned} \tag{50}$$

applying inverse transform of the FRCT to (50), we have

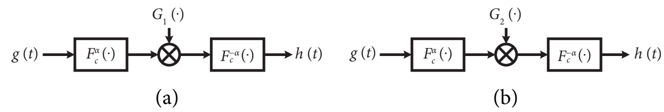

Figure 4: The calculation process for the first kind of convolution integral equation (32), (a) $\lambda_1 \neq 0$. (b) $\lambda_1 = 0$.

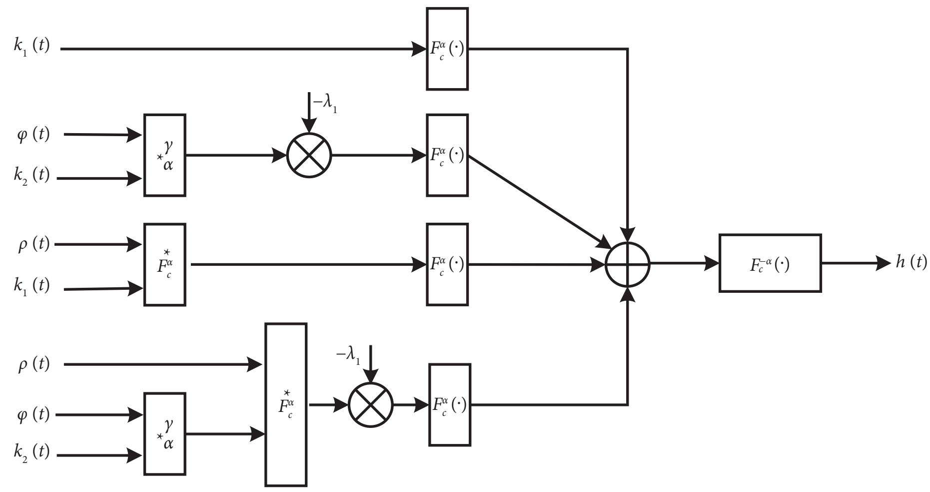

Figure 5: Implementation of solution $h(t)$ of equation (37).

$$h(t) = k_1(t) - \lambda_1\left(\varphi \overset{\gamma}{*}_\alpha k_2\right)(t) + \left(\rho \underset{F_c^\alpha}{*} k_2\right)(t) - \lambda_1\left(\rho \underset{F_c^\alpha}{*}\left(\varphi \overset{\gamma}{*}_\alpha k_2\right)\right)(t). \tag{51}$$

Similarly, we get

$$f(t) = k_2(t) - \lambda_2\left(\psi \overset{\gamma}{*}_\alpha k_1\right)(t) + \left(\rho \underset{F_c^\alpha}{*} k_2\right)(t) - \lambda_2\left(\rho \underset{F_c^\alpha}{*}\left(\psi \overset{\gamma}{*}_\alpha k_1\right)\right)(t). \tag{52}$$

The proof is completed. □

Remark 13. When $1 - \lambda_1\lambda_2 e^{-\mathrm{j}u^2/2\cot\varphi} F_c^\alpha(\varphi \overset{\gamma}{*}_\alpha \psi)(u) = 0$ in Theorem 12, $F_c^\alpha[(\lambda_1(\varphi \overset{\gamma}{*}_\alpha k_2) - k_1)](u) \neq 0$ and $F_c^\alpha[k_2 - \lambda_2(\psi \underset{F_c^\alpha}{*} k_1)](u) \neq 0$, then equation (37) has no solution.

Remark 14. When $1 - \lambda_1\lambda_2 e^{-\mathrm{j}u^2/2\cot\varphi} F_c^\alpha(\varphi \overset{\gamma}{*}_\alpha \psi)(u) = 0$ in Theorem 12, and $F_c^\alpha[\lambda_1(\varphi \overset{\gamma}{*}_\alpha k_2 - k_1)](u) = 0$ or $F_c^\alpha[k_2 - \lambda_2(\psi \underset{F_c^\alpha}{*} k_1)](u) = 0$, then equation (37) has infinitely many solutions.

5. The Complexity Analysis of Solutions to Convolution Integral Equations

The convolution theorem plays an important role in solving convolution integral equations by allowing for the pointwise multiplication of the transformed known function and kernel function, thereby reducing computational complexity.

Now, we provide the computational complexity analysis of the solution to the first kind of convolution integral (35). As shown in Figure 4, the solution to (35) can be realized as follows.

We can see that the major computation for the first kind of convolution integral (35) is mainly focused on calculating $G_1(u)$ and $G_2(u)$ due to the mixed-weighted function, where $G_1(u) = 1/(\lambda_1 + W(u))$ and $G_2(u) = 1/W(u)$. This leads to an increase in calculation. However, by using the classical FFT and considering the relationship between FRCT and FT (refer to (6) and Figure 1), we can calculate the complexity of solution of the first kind of convolution integral (35) is $O(5/2N\log_2^N)$ for all $\lambda_i \in \mathbb{C}$.

Next, let us analyze the computation complexity of the solution achieved in convolution integral (43) in detail. Based on (46) and (47), the solutions $h(t)$ and $f(t)$ of (43) can be implemented in Figures 5 and 6, respectively.

From (46), the solution $h(t)$ can be expressed as the convolution sum, which is difficult to implement in time domain. To simplify calculations, we transform the convolution sum into frequency domain using fractional cosine transform. For a discrete signal of size N, discrete Fourier cosine transform (DFCT) requires a $(1/2N\log_2^N)$ real number multiplications. According to (16), Figure 3, and Theorem 7 (see [28]), we can calculate the complexity of $(\varphi \overset{\gamma}{*}_\alpha k_2)(t)$, $(\rho \underset{F_c^\alpha}{*} k_1)(t)$, and $(\rho \underset{F_c^\alpha}{*} (\varphi \overset{\gamma}{*}_\alpha k_2)(t)$ are

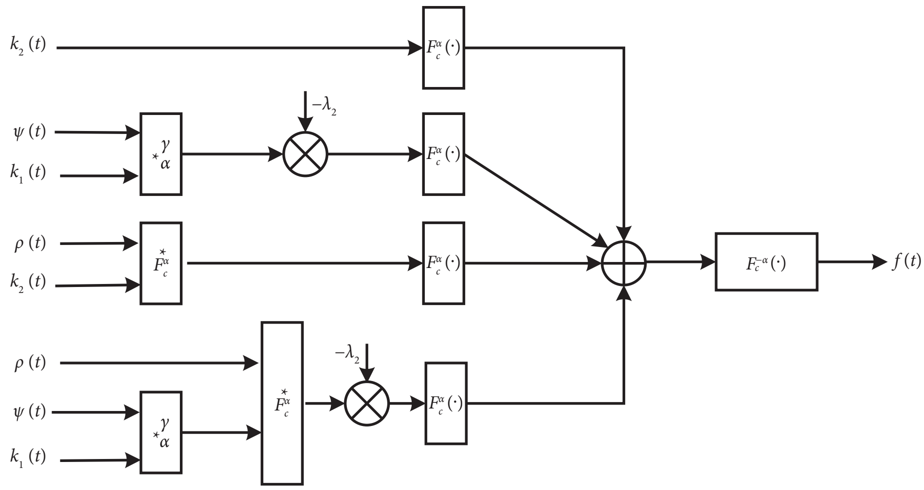

Figure 6: Implementation of solution $f(t)$ of equation (37).

$O(2N\log_2^N)$, $O(3/2N\log_2^N)$, and $O(5/2N\log_2^N)$, respectively. Hence, we obtain the computational complexity of a solution $h(t)$ of (43) via DFCT that is $O(13/2N\log_2^N)$. Similarly, the computational complexity of another solution $f(t)$ of (43) is also $O(13/2N\log_2^N)$.

6. Conclusions

This paper deals with two kinds of convolution integral equations based on the derived fractional convolution theorem. First, fractional mixed-weighted convolution for the fractional cosine transform is proposed. Second, the corresponding convolution theorem is derived, and properties and Young's type theorem for fractional mixed-weighted convolution are studied. Finally, based on the proposed convolution theorem, we discussed two kinds of convolution integral equations and analyzed the computational complexity of the solution of the equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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