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Research Article

A Neural Network Based on a Nonsmooth Equation for a Box Constrained Variational Inequality Problem

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The variational inequality framework holds significant prominence across various domains including economic finance, network transportation, and game theory. In addition, a novel approach utilizing a neural network model is introduced in the current work to address a box constrained variational inequality problem. Initially, the original problem is reformulated into a nonsmooth equation, following which the neural network model is meticulously devised to tackle this reformulated equation. This study comprehensively investigated inherent characteristics and properties of this neural network model. In addition, employing the Lyapunov function method, stability analysis of the neural network model proposed is rigorously demonstrated in the Lyapunov sense. Furthermore, the efficacy of the proposed technique is substantiated through numerical simulations, providing empirical support for its applicability and effectiveness.

1. Introduction

Variational inequalities serve as a comprehensive framework for examining numerous optimization problems and possess significant applications across various domains such as economics, engineering, and transportation, among others, as detailed in the monograph [1] and associated literature. As elucidated in [2], variational inequalities represent a contemporary extension of variational principles, with their historical roots tracing back to seminal works by Euler, Lagrange, and the Bernoulli siblings. The concepts and methodologies inherent in variational inequalities are currently being employed across a spectrum of scientific disciplines, showcasing their efficacy and innovation, as evidenced by scholarly contributions [3–10].

In this article, we study a box constrained variational inequality problem $(\mathrm{BVIP}(l, u, F))$ as follows: finding $x \in X$ and thus.

$$(y - x)^T F(x) \geq 0, \quad \forall y \in X, \tag{1}$$

where $F: R^n \longrightarrow R^n$ is continuously differentiable, $X = \{x \in R^n \mid l \leq x \leq u\}$, $l = (l_1, l_2, \cdots l_n)^T$, $l_i \in R$, $u = (u_1, u_2, \cdots u_n)^T$, $u_i \in R$, and $l_i < u_i$, $i = 1, 2, \cdots n$.

A number of numerical methods including interior point methods [11], Newton methods [12, 13], penalty methods [14], and extragradient methods [15–17], have been proposed for solving variational inequality problems. Numerical methods are predominantly approached from a discrete time standpoint, owing to their practical feasibility for implementation on digital computers.

The utilization of neural networks in addressing variational inequalities, especially within engineering applications demanding real-time solutions, has garnered significant attention in recent years, as discussed extensively in the literature [18–31]. ANNs offer promising prospects due to their potential for efficient hardware implementation and the ability to provide real-time solutions, which may be challenging to achieve with traditional numerical algorithms, particularly for high-dimensional and dense problem settings. Various neural network architectures and methodologies have been proposed to tackle different aspects of variational inequality problems. For instance, in [24], the authors introduce a novel neural network approach for addressing constrained variational inequalities, which demonstrates stability properties under Lyapunov sense. Similarly, in [25] shows a projection neural network for addressing variational inequalities, with proven stability under certain conditions. Moreover, for mixed variational inequalities, the author proposes a proximal projection neural network method in [26], showcasing convergence properties under Lipschitz continuity conditions. These studies highlight the versatility and efficacy of neural networks in addressing diverse variational inequality problem formulations. Despite the advancements, it is acknowledged that most existing neural network approaches focus on general variational inequality problems and may not fully exploit the specific structure of $\text{BVIP}(l, u, F)$ formulations. This presents an opportunity for further research to develop specialized neural network architectures tailored to exploit the unique characteristics of box-constrained variational inequality problems, potentially leading to enhanced efficiency and effectiveness in solving such problems.

The introduction of specialized neural networks tailored for addressing time-varying equations has significantly advanced the field, as evidenced by the Zeroing Neural Network (ZNN) model discussed in [32, 33]. The ZNN model offers exponential convergence towards theoretical solutions of time-varying equations, representing a notable improvement over existing methods. Building upon this foundation, subsequent research efforts have yielded valuable outcomes, as documented in various studies such as [34–37]. One limitation of the classic ZNN is its reliance on infinite time cost for convergence to the theoretical solution. To address this limitation, some new Neural Networks are introduced in [38]. Moreover, they have been shown to be effective in tackling nonconvex Quadratic Programming (QP) problems [39]. However, as computational scales increase, the time required to obtain results becomes prohibitive, necessitating even faster convergence speeds for practical applications. In response to this need, a neural network with varying parameters was developed in [40–42]. This innovation represents a significant advancement in accelerating convergence speeds, addressing the challenges posed by larger computational scales. In addition, in [43], the authors integrate a redefined error monitor function into the neural network design. This integration enhances control over mobile redundant manipulators during tracking tasks, offering superior performance in terms of overshoot, robustness, and convergence speed compared to traditional neural networks, as demonstrated in [44]. These advancements underscore the potential of specialized neural networks in addressing complex dynamic equations and hold promise for future research endeavors in the field.

The paper introduces a novel neural network method aimed at solving (1). Stability analysis of the neural network proposed is shown on the basis of Lyapunov's sense, and the convergence of the solution sequence is guaranteed. Compared to existing studies, the article's main contributions can be summarized as follows:

(1) Utilizing the structure of $\text{BVIP}(l, u, F)$, the paper provides a nonsmooth equation formulation for solving the problem defined by (1). Subsequently, a neural network method is proposed to tackle (1).

(2) In contrast to the method proposed in [6] and the classical neural network method described in [31], the proposed method facilitates faster convergence of the solution trajectory towards the equilibrium point. This improvement is evidenced by the numerical experiments conducted in Section 5.

(3) Unlike neural networks relying on projection functions as discussed in [6], our neural network in this paper operates independently of estimating any parameters, simplifying the computational process.

Overall, these contributions highlight the effectiveness of the neural network put forward in addressing $\text{BVIP}(l, u, F)$, offering advancements over existing approaches and demonstrating promise for future applications.

The structure of the current study is presented: Section 2 gives preliminaries necessary for understanding the subsequent sections. Section 3 introduces a neural network model for a nonsmooth equation. Section 4 establishes the consistency and stability analysis results. Section 5 conducts several numerical tests.

The notations specified below will be utilized consistently throughout this paper. A^T is used to denote the transpose of a matrix A, $\langle x, y\rangle$ to represent the inner product of x and y in vector space, $\|z\|$ denotes the Euclidean norm for any $z \in \mathbb{R}^n$. In addition, for $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, $\nabla\varphi(x)$ signifies the gradient of φ at x. Furthermore, for $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $J\phi(x)$ represents the Jacobian matrix of ϕ evaluated at x. These notations will remain consistent throughout the entirety of this paper to ensure clarity and coherence.

2. A Neutral Network Based on Nonsmooth Equation

A neutral network based on a nonsmooth equation formulation of $\text{BVIP}(l, u, F)$ is proposed in the current section. Some foundational concepts such as P matrix, P_0 function, uniform P function, Clarke subdifferentiation, isolated equilibrium point, Stability in the sense of Lyapunov, exponential stability and asymptotical stability are sourced from [1, 45, 46].

We adopt a nonsmooth equation formulation of $\text{BVIP}(l, u, F)$ from [47]. First, we introduce $\psi\colon R^n \longrightarrow R_+$ defined by

$$\psi(a, b) = \min{}^2\{[-\phi(a, b)]_+, a\}, \tag{2}$$

where

$$\phi(a, b) = \sqrt{a^2 + b^2} - (a + b), \tag{3}$$

with $\phi\colon R^2 \longrightarrow R$ being called the F-B function [48].

Then, $\Psi, \Phi\colon R^n \longrightarrow R^n$ can be defined by

$$\begin{aligned}\Phi_i(x) &= \sqrt{\psi(u_i - x_i, -F_i(x))} = \min\{[-\phi(u_i - x_i, -F_i(x))]_+, u_i - x_i\},\\ \Psi_i(x) &= \sqrt{\psi(x_i - l_i, F_i(x))} = \min\{[-\phi(x_i - l_i, F_i(x))]_+, x_i - l_i\},\end{aligned} \tag{4}$$

where $i = 1, 2, \ldots, n$. We also introduce $G: R^n \longrightarrow R^n$ defined by

$$G_i(x) = \sqrt{\Psi_i(x)^2 + \Phi_i(x)^2} = \begin{cases} \sqrt{(x_i - l_i)^2 + \phi(u_i - x_i, -F_i(x))^2}, & x_i < l_i \text{ and } F_i(x) < 0,\\ -\phi(x_i - l_i, F_i(x)), & l_i \le x_i \le u_i \text{ and } F_i(x) \ge 0,\\ -\phi(u_i - x_i, -F_i(x)), & l_i \le x_i \le u_i \text{ and } F_i(x) < 0,\\ x_i - u_i, & x_i > u_i \text{ and } F_i(x) < 0\\ l_i - x_i, & x_i < l_i \text{ and } F_i(x) \ge 0,\\ \sqrt{(u_i - x_i)^2 + \phi(x_i - l_i, F_i(x))^2}, & x_i > u_i \text{ and } F_i(x) \ge 0,\end{cases} \tag{5}$$

with $i = 1, 2, \ldots, n$.

Consider

$$f(x) = \frac{1}{2}\|G(x)\|^2, \tag{6}$$

where

$$G(x) = \begin{pmatrix} \sqrt{\Phi_1(x)^2 + \Psi_1(x)^2}\\ \vdots\\ \sqrt{\Phi_n(x)^2 + \Psi_n(x)^2}\end{pmatrix}. \tag{7}$$

According to Theorem 2.2 in [47], $f(x)$ is nonnegative, and $f(x) = 0$ is equivalent to that $x \in \mathbb{R}^n$ is a solution of $\mathrm{BVIP}(l, u, F)$. Moreover, if f is continuously differentiable when F is continuously differentiable.

Utilizing the fastest descent method for $\mathrm{BVIP}(l, u, F)$, we now delve into a neural network model of the first order as follows:

$$\frac{d(x(t))}{dt} = -\tau\nabla f(x), x(t_0) = x_0, \tag{8}$$

where t_0 refers to the initial time and τ represents a factor that determines the step size in simulation. If τ is greater than 1, it suggests that a larger step can be utilized during the simulation process. In addition, Figure 1 shows the block diagram framework of (8).

3. Consistency and Stability of (8)

We focus on consistency analysis and stability analysis of the neutral network (8) proposed in this part.

We begin by examining the connection between the equilibrium point of (8) and the solutions to $\mathrm{BVIP}(l, u, F)$.

Theorem 1. *Suppose that x^* represents a solution of $BVIP(l, u, F)$. In such a case, x^* also serves as an equilibrium point of (8). Conversely, if x^* serves as an equilibrium point of (8) and all elements $V \in \partial^c G(x^*)$ are nonsingular, or if l_i and $u_i, i = 1, 2, \ldots, n$ are finite and F satisfies the properties of a P_0 function, in what follows, x^* serves as a solution to $BVIP(l, u, F)$.*

Proof. If x^* is a solution of $\mathrm{BVIP}(l, u, F)$, according to [47], $f(x^*) = 0$, implying $G(x^*) = 0$. Denote V be an element in $\partial^c G(x^*)$, then according to [49], the following can be acquired:

$$\nabla f(x^*) = V^T G(x^*) = 0, \tag{9}$$

which means that x^* is an equilibrium point of (8). Conversely, if $\nabla f(x^*) = 0$ and all $V \in \partial^c G(x^*)$ are nonsingular, then from (9) we have $G(x^*) = 0$, hence $f(x^*) = 0$, indicating that x^* represents a solution of $\mathrm{BVIP}(l, u, F)$. If F is a P_0 function, the conclusion follows directly from Theorem 4.2 in [47]. □

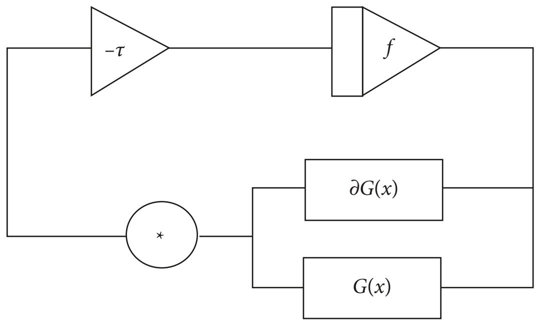

Figure 1: Block diagram of (8).

Next, we study the trajectory of the solution to (8).

Lemma 2. *The function $f(x(t))$ decreases or remains constant as the variable t increases.*

Proof. Since

$$\frac{df((x(t)))}{dt} = \nabla f(x(t))^T \frac{d(x(t))}{dt} = \nabla f(x(t))^T (-\tau \nabla f(x(t))) = -\tau \|\nabla f(x(t))\|^2 \leq 0. \tag{10}$$

Therefore, the function $f(x(t))$ decreases or remains constant as the variable t increases. □

We next define the level set of the starting point x_0 as

$$L_f(x_0) = \{x \in R^n \mid f(x) \leq f(x_0)\}. \tag{11}$$

Theorem 3. *For an arbitrary initial state $x_0 \in R^n$*

(i) There exists exactly one maximal solution $x(t)$, $t \in [t_0, \tau(x_0))$ and $\tau(x_0) > T_0$;

(ii) If X is bounded or F satisfies the uniform P property, then $\tau(x_0) = +\infty$.

Proof. (i) It can be found that $\nabla f(x)$ is continuous, hence by Theorem 2.5 in [50], we know that the maximal solution remains unique. (ii) If otherwise, we have $\tau(x_0) < +\infty$, and by Theorem 2.6 in [50], $\lim_{t \longrightarrow \tau(x_0)} \|x(t)\| = \infty$, let

$$\tau_0 = \inf\left\{s \geq 0 \mid s < \tau(x_0), x(s) \in L_f^c(x_0)\right\} < \infty, \tag{12}$$

where $L_f^c(x_0) = R^n \backslash L_f(x_0)$. By the continuity of f, $L_f(x_0)$ refers to a closed set. If X is bounded or F satisfies the uniform P property, then by Theorem 3.2 in [51], the level set is bounded, so $L_f(x_0)$ is a bounded closed set. Then, we get $x(\tau_0) \in L_f(x_0)$ and $\tau_0 < \tau(x_0)$. Moreover, this means that a $s \in (\tau_0, \tau(x_0))$ exists that

$$f(x(s)) > f(x_0) > f(x(\tau_0)), \tag{13}$$

which, by Lemma 2, contradicts that $f(x(\cdot))$ is non-increasing regarding t. As a result, $\tau(x_0) = +\infty$. □

Inspired by Corollary in 4.3 in [50], the following result can be obtained.

Theorem 4. *Let $x(t) \in [t_0, \tau(x_0))$ be the unique maximum solution of the differential equation model (8), $\tau(x_0) = +\infty$ and $\{x(t)\}$ is bounded, then*

$$\lim_{t \longrightarrow \infty} \nabla f(x(t)) = 0. \tag{14}$$

Furthermore, if x^* denotes the convergence point of trajectory $x(t)$ and all elements $V \in \partial^c G(x^*)$ are nonsingular for, then x^* is a solution of $BVIP(l, u, F)$.

Proof. According to Lemma 2, $f(x(t))$ has a lower bound. And the unconstrained minimization problem (6) corresponding to model (1) is the steepest descending dynamic model. Therefore, according to Corollary 4.3 in [50], the analysis of this model shows that the trajectory of (8) will reach a steady state, and the conclusion can be established.

Furthermore, if x^* is the convergence point of the trajectory $x(t)$, $\lim_{t \longrightarrow \infty} x(t) = x^*$. According to (14), it can be concluded that $\nabla f(x^*) = 0$. Since all $V \in \partial^c G(x^*)$ are nonsingular, the conclusion can be drawn from Theorem 1. □

Remark 5. If l_i and $u_i, i = 1, 2, \ldots, n$ are finite, then nonsingularity of elements in $\partial^c G(x^*)$ in Theorem 4 can be replaced by P_0 property of function the F.

Invoking from [46, 50], we provide some stability results for (8).

Theorem 6. *Let x^* be an isolated equilibrium point of (8), we have for (8), x^* is asymptotically stable.*

Proof. At first, it is demonstrated that $f(x)$ serves as a Lyapunov function over the set Ω_*, a neighborhood of x^*, for equation (8). Using the definition of $f(x)$, it is nonnegative across R^n. Because x^* is isolated, $f(x^*) = 0$ and for any $x \in \Omega_*/\{x^*\}$, $f(x) > 0$. Next, we verify the second condition in the definition of a Lyapunov function. It can be found that:

$$\frac{df(x(t))}{dt} = \nabla_x f(x(t))^T \frac{d(x(t))}{dt} = -\left|\nabla_x f(x(t))^T \nabla_x f(x(t))\right| \leq 0. \tag{15}$$

Thus, the function $f(x)$ acts as a Lyapunov function for (8) over the set Ω_*. As x^* is an isolated equilibrium point, we have $df(x(t))/dt < 0$, $\forall x \in \Omega_*/\{x^*\}$. By Lemma 5.3 in [46], it can follow that x^* is asymptotically stable for (8). □

Theorem 7. *Let x^* be a solution of $BVIP(l, u, F)$ and $\mathcal{J}_x F(x^*)$ is a P matrix, then we have for (8), x^* is exponentially stable.*

Proof. Define x^* be a solution of $\mathrm{BVIP}(l, u, F)$, and later it holds that

$$\nabla f(x^*) = V^T G(x^*) = 0, \tag{16}$$

for $V \in \partial^c G(x^*)$, therefore, x^* is an equilibrium point. Suppose that x^* is not an isolated equilibrium point, later we can select a sequence $\{x_k\}$ which converges to x^* as k tends to infinity and satisfies $x_k \neq x^*$.

$$\nabla f(x_k) = V_k^T G(x_k) = 0, \tag{17}$$

for $V_k \in \partial^c G(x_k)$. Since $\mathcal{J}_x F(x^*)$ is a P matrix, we know from Corollary 5.3 in [47] that when k is large enough, V_k is nonsingular, which, by (17), means that $G(x_k) = 0$. Therefore, x_k refers to a solution of $\mathrm{BVIP}(l, u, F)$ for k large enough. However, by [1], it holds that under condition of $\mathcal{J}_x F(x^*)$ being a P matrix, $\mathrm{BVIP}(l, u, F)$ has at most a solution, which is a contradiction. As a result, x^* stands alone as an equilibrium point.

By Theorem 6, x^* is asymptotically stable. By Corollary 5.3 in [47], $\exists c > 0$ and $\delta > 0$ such that for every $x \in \mathbb{B}(x^*, \delta)$ and every $V \in \partial^c G(x)$, V is invertible and fulfills $\|V^{-1}\| \leq c$. So $\exists \kappa_1 > 0$ and $\kappa_2 > 0$ and thus:

$$\kappa_1 \|v\|^2 \leq v^T V^T V \leq \kappa_2 \|v\|^2, \quad \forall x \in \mathbb{B}(x^*, \delta). \tag{18}$$

By Proposition 2.4 in [47], G is semismooth, which, by Proposition 2.4 in [50], means the following expansion

$$G(x) = G(x^*) + V(x - x^*) + o(\|x - x^*\|), \tag{19}$$

for any $V \in \partial^c G(x)$.

The proof that follows bears resemblance to the proof found in Theorem 5.5 of [46], and we write them out for completeness.

Define δ be sufficiently small such that.

$$\left|o(\|x - x^*\|)\right| \leq \epsilon \|x - x^*\|, \tag{20}$$

for $\forall x \in \mathbb{B}(x^*, \delta)$ and some $0 < \epsilon < \kappa_1$. Next, let

$$\Gamma(t) = \|x(t) - x^*\|^2, \quad t \in [t_0, +\infty). \tag{21}$$

In what follows, we have

$$\begin{aligned}\frac{d\Gamma(t)}{dt} &= 2(x(t) - x^*)^T \frac{dx(t)}{dt} \\ &= -2(x(t) - x^*)^T \nabla_x f(x(t)) \\ &= -2(x(t) - x^*)^T \left(V^T G(x(t))\right),\end{aligned} \tag{22}$$

for every $V \in \partial^c G(x)$. Suppose

$$\bar{\tau} = \inf\{t \in [t_0, +\infty) \mid \|x(t) - x^*\| \geq \delta\}, \tag{23}$$

is the time at which the solution first exits the ball $\mathbb{B}(x^*, \delta)$. Therefore, we can obtain $G(x^*) = 0$, and for $\forall t \in \bar{I} = [t_0, \bar{\tau})$

$$\begin{aligned}\frac{d\Gamma(t)}{dt} &= -2(x(t) - x^*)^T \left(V^T G(x(t))\right) \\ &= -2(x(t) - x^*)^T V^T \left[G(x^*) + V(x(t) - x^*) + o(\|x(t) - x^*\|)\right] \\ &\leq -2(x(t) - x^*)^T V^T V(x(t) - x^*) + \epsilon \|x(t) - x^*\| \\ &\leq (-2\kappa_1 + \epsilon) \|x(t) - x^*\| \\ &= (-2\kappa_1 + \epsilon)\Gamma(t).\end{aligned} \tag{24}$$

By [[52], Corollary 2.1], we have the equivalence of

$$\Gamma(t) \leq e^{(-2\kappa_1 + \epsilon)t} \Gamma(t_0), \quad t \in \bar{I}, \tag{25}$$

and

$$\|x(t) - x^*\| \leq e^{\omega t} \|x(t_0) - x^*\|, \quad t \in \bar{I}, \tag{26}$$

where $\omega = -\kappa_1 + \epsilon/2 < 0$. When $\overline{\tau} < +\infty$, then

$$\delta \leq \limsup_{t \longrightarrow \overline{\tau}} \|x(t) - x^*\| \leq e^{\omega\overline{\tau}} \|x(t_0) - x^*\| < \delta, \tag{27}$$

which refers to a contradiction. Therefore, we have $\overline{\tau} = +\infty$ and the proof is finished. □

4. Numerical Tests

In the current section, multiple instances of box-constrained variational inequalities are provided for validating the developed neural network model. Our simulation is based on MATLAB (2018B) and its ode45 solver. The examples come from [31].

Example 1. Consider (1), where

$$F(x) = \begin{pmatrix} 4x_1 + 2x_2 + 2x_3 + x_4 - 8 \\ 2x_1 + 4x_2 + x_4 - 6 \\ 2x_1 + 2x_3 + 2x_4 - 4 \\ -x_1 - x_2 - 2x_3 + 3 \end{pmatrix}, \tag{28}$$

and $X = [0, 5]^4$. The solution to the variational inequality is $x^* = (4/3, 7/9, 4/9, 2/9)^T$.

Next, the neutral network (8) will be used to calculate the variational inequality (1). Some numerical results are reported. In [6, 31], different neutral networks are proposed to solving BVIP, we compare those numerical test results with ours. In [6, 31], the neutral networks are based on the following differential equations, respectively:

$$\frac{dx}{dt} = -\zeta_1 \frac{x - h_x}{\|x - h_x\|^{\mu}} - \zeta_2 \frac{x - h_x}{\|x - h_x\|^{\nu}}, \tag{29}$$

and

$$\frac{dx}{dt} = -x + h_x, \tag{30}$$

where $h_x = P_X(x - \beta F(x)), \zeta_1 > 0, \zeta_2 > 0, \mu \in (0, 1), \nu < 0$, $P_X(\cdot)$ denotes the projection operator on set X.

Figures 2–4 depict the numerical test outcomes derived from the model (8), (27), and (29) of variational inequality (1). $x_0 = (0, 0, 0, 0)^T$ is selected as the initial point.

We know from Figures 2 and 3 that the trajectories of solutions of neural network based on models (8), (27), and (29) of box constrained variational inequality problem (1) all converge to equilibrium point $x^* = (4/3, 7/9, 4/9, 2/9)^T$. Moreover, compare with neural networks based on models (29) and (30), the trajectories of solutions of the neural network on the basis of model (8) converge to the equilibrium point faster.

Example 2. Consider the BVIP (l, u, F) (1), where

$$F(x) = \begin{pmatrix} x_1^3/100 + e^{x_1 x_2} \\ 2x_2^3/100 + e^{x_2 x_3} \\ \cdots \\ n x_n^3/100 + e^{x_n x_1} \end{pmatrix}, \tag{31}$$

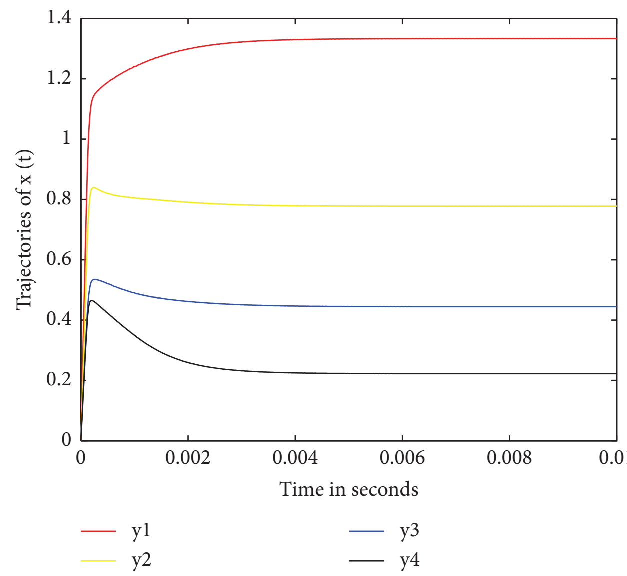

Figure 2: Numerical results for model (8).

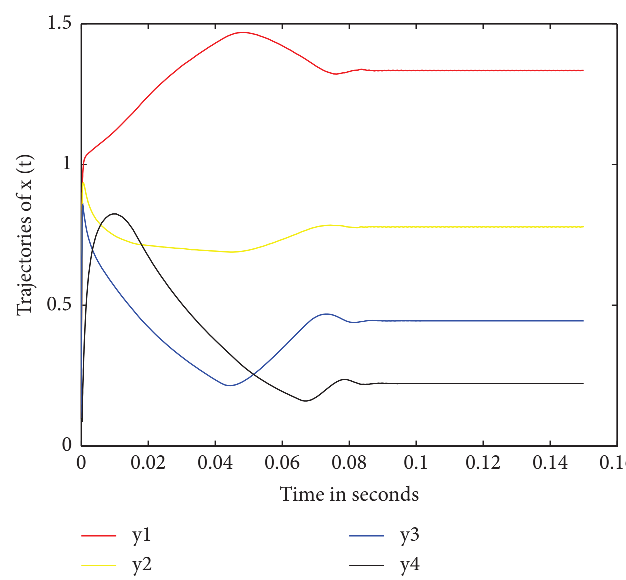

Figure 3: Numerical results for model (29).

and $X = [-1, 6]^n$. Table 1 displays the numerical test results based on the of model (8) with different n, $x_0 = (0, 0, \ldots, 0)^T$ is selected as the initial point, n refers to the dimension of x, k represents the number of iterations, Time(s) is the CPU time and obj is the value of $1/2\|G(x)\|^2$.

Example 3. Consider the BVIP (l, u, F) (1), where $F(x) = Dx + q$ with

$$D = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 & 0 \\ 1 & 4 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -2 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{pmatrix}, \tag{32}$$

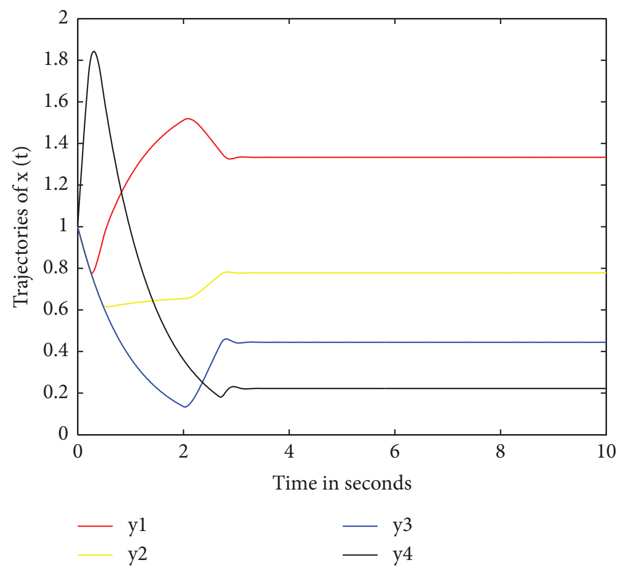

Figure 4: Numerical results for model (30).

Table 1: Numerical findings of Example 2 based on (8).

n	k	Time (s)	obj
100	57	0.998	$2.7978e-005$
300	413	12.084	$3.0695e-004$
500	533	14.922	$1.7702e-004$

Table 2: Numerical findings of Example 3 based on (8).

X	n	Time (s)	obj
$[-5,5]^4$	4	3.076	$3.7083e-006$
$[-5,5]^{10}$	10	6.275	$8.0301e-005$
$[-5,5]^{100}$	100	72.382	$1.9e-003$
$[-5,5]^{1000}$	1000	1536.45	$2.01e-002$

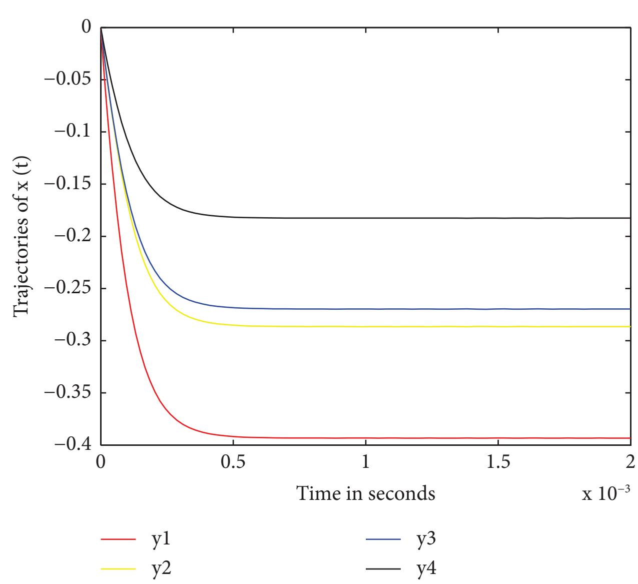

Figure 5: Numerical findings of Example 3 for model (8) with $n=4$.

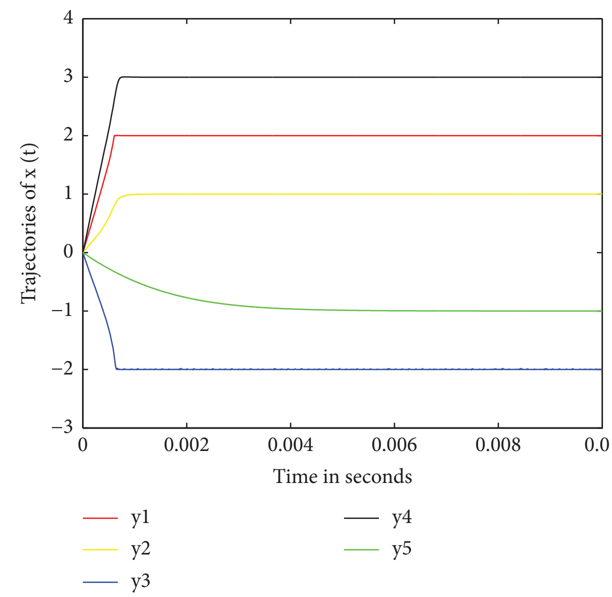

Figure 6: Numerical results of Example 4 for model (8) with $\tau = 1000$.

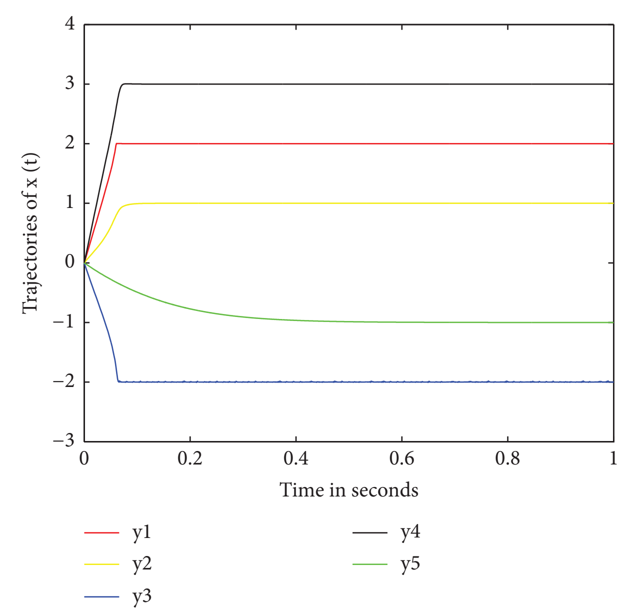

Figure 7: Numerical results of Example 4 for model (8) with $\tau = 10$.

$q = (1,1,1,\dots,1)^T$ and $X = [-5,5]^n$.

Table 2 shows the numerical test results based on the of model (8) with different n, $x_0 = (0,\dots,0)^T$ is selected as the initial point, n is the dimension of x, Time (s) represents the CPU time, and obj is the value of $1/2\|G(x)\|^2$.

Figure 5 shows the numerical test results of Example 3 based on the of model (8) with $n = 4$.

Example 4. Consider the (1), where

$$F(x) = \begin{pmatrix} x_1^3 - 8 \\ x_1 + x_2^3 - 3 \\ x_3^3 + 8 \\ x_4^2 + x_2 - 10 \\ x_5 + 1 \end{pmatrix}, \qquad (33)$$

and $X = \{x\colon a \le x \le b\}$, $a = (-1, 0, -5, 2, -3)^T, b = (4, 6, -1, 7, 0)^T$. In addition, the solution to the variational inequality is $x^* = (2, 1, -2, 3, -1)^T$.

Figures 6 and 7 show the numerical test results based on the of model (8) of variational inequality 1.1 with different τ.

5. Conclusions

To conclude, this study introduces a neural network approach for addressing the box-constrained variational inequality problem. Alongside exploring the existence and convergence of neural network trajectories, we also examine the stability of solutions. These stability results include asymptotic stability and exponential stability. Finally, numerical experiments demonstrate the effectiveness of the neural network method. Of course, like all algorithms, the neural network method put forward in the present study also has drawbacks. For example, due to the involvement of subdifferential estimation, the computing time may be limited. The smoothing method may be able to address this drawback, which may be our future research topic.

Data Availability

The authors confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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