Research Article

Flag-Transitive $2-(v, 5, \lambda)$ Designs Admitting a Two-Dimensional Projective Group

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The focus of this study is to classify flag-transitive 2-designs. We have come to the conclusion that if $\mathcal{D}$ is a nontrivial 2-design having block size 5 and $G$ is a two-dimensional projective special linear group which acts flag-transitively on $\mathcal{D}$ with $q \not\equiv 1 \pmod{4}$, then $\mathcal{D}$ is a 2-(11, 5, 2) design, a 2-(11, 5, 12) design, a 2-(q + 1, 5, 2(q − 1)) design with $q \equiv 3 \pmod{4}$ or a 2-(q + 1, 5, (q − 1)/3) design with $q = 2^f$ (where $f > 2$ is an even).

1. Introduction

In the year of 1987, Davies [1] drew a conclusion that if a $2-(v, k, \lambda)$ design $\mathcal{D}$ has a point-imprimitive flag-transitive automorphism group $G$, then given the value of $\lambda$, there is a superior limit on the block size $k$. We can easily arrive at a conclusion that the existence of the pairs $(\mathcal{D}, G)$ is not infinite only if $\lambda$ is fixed, where $G$ acting on $\mathcal{D}$ is point-imprimitive and flag-transitive. Meanwhile, the hidden meaning is that, as long as we set $k$ fixed, there are only a finite number of such designs. Still further, on the basis of the proof of ([2], Proposition 4.1), one proving that for a nontrivial point-imprimitive and flag-transitive $2-(v, k, \lambda)$ design $\mathcal{D}$, its block size $k \geq 6$. Without a doubt, we have that the flag-transitive automorphism group $G$ for our paper of the $2-(v, 5, \lambda)$ design is definitely point-primitive. The O’Nan–Scott theorem shows that one of the following is doomed to hold for any finite primitive permutation group $G$ (for more details, see [3]):

(a) $G$ is of almost simple type
(b) $G$ is of affine type
(c) $G$ is of simple diagonal type
(d) $G$ is of product type
(e) $G$ is of twisted wreath product type

In order to solve the problem more completely, we narrow down the scope of our problem; here, we only consider the case (a) and $G$ is a two-dimensional projective special linear group $\text{PSL}(2, q)$.

For $q \not\equiv 1 \pmod{4}$, the group $\text{PSL}(2, q)$ acts 3-homogeneously on the one-dimensional projective line. We immediately inferred that if $\mathcal{D}$ is a set consisting of some $k$-subsets of the projective line $\mathcal{P}$, then $(\mathcal{P}, \mathcal{B})$ is a 3-$(q + 1, k, \lambda)$ design for some $\lambda$ in the condition that $\mathcal{B}$ is a union of $G$-orbits with $G = \text{PSL}(2, q)$. Thereafter that is why $\text{PSL}(2, q)$ becomes a coveted treasure when it comes to construct a 3-design. For instance, through the way $\text{PSL}(2, q)$ acts on the 4-element subsets of the one-dimensional projective line, the authors of [4] determined all 3-designs of $\text{PSL}(2, q)$ with block size 4. By using the same processing method, Keranen and Kreher [5] completely solved such designs of block size 5. On the other hand, for $q \equiv 1 \pmod{4}$, Keranen et al. [6] has completely determined all quadruple systems admitting $\text{PSL}(2, q)$ as their automorphism group. Unfortunately, the 3-$(q + 1, 5, \lambda)$ designs with the automorphism group $\text{PSL}(2, q)$ remain uncertain.

Many scholars have turned to another direction of exploration, focusing on 2-designs whose automorphism...
The following lemma is almost involved for each time. Let 
\[ G = \text{Aut}(\mathcal{D}) \] 
be a flag-transitive automorphism group. Furthermore, the most classification of 2-(\nu, k, \lambda) designs, characterized by \( \text{PSL}(2, q) \) and flag-transitive automorphism groups that are the point-primitive and flag-transitive, respectively.

Take any block \( B \in \mathcal{B} \), and thereafter \( G_B \) is the setwise stabilizer. Below we will introduce a classical and commonly used verdict about them.

**Lemma 3.** For a 2-design \( \mathcal{D} = (\mathcal{V}, \mathcal{B}) \), let \( G \leq \text{Aut}(\mathcal{D}) \), and \( \mathcal{F} \) be the flag set of \( \mathcal{D} \). If \( B \in \mathcal{B} \), then the two statements given below are interchangeable:

(i) \( G \) acts flag-transitively on \( \mathcal{F} \)

(ii) \( G \) is a transitive group of \( \mathcal{B} \), and \( G_B \) has a transitive action on \( B \)

The following lemma adds the finishing touch to the proof of our paper. Literature [1] demonstrates the proof in detail and helps us to understand it better.

**Lemma 4.** Provided that \( G \) acts on a 2-design \( \mathcal{D} \) flag-transitively, let \( d \) be a subdegree of \( G \) with \( d \not| 1 \); then \( r \) divides \( Ad \).

From ([4], Theorem 6) and ([5], Theorem 3.3), we can reach the following result which is of most importance through a simple calculation.

**Lemma 5.** Let \( G = \text{PSL}(2, q) \) with \( q \not| 1 \) (mod \( 4 \)), and \( B \) be a 5-element subset of one-dimensional projective line. Then the following two situations are to be true:

(i) If \( q \equiv 3 \) (mod \( 4 \)), then \( |B| \) is one of \( q(q^2 - 1)/2 \) with \( q \geq 3 \), \( q(q^2 - 1)/6 \) with \( q \equiv 7, 19, 31, 43 \) (mod \( 60 \)) or \( q(q^2 - 1)/10 \) with \( q \equiv 11, 19, 31, 59 \) (mod \( 60 \))

(ii) If \( q \equiv 2 \) (mod \( 4 \)), then \( |B| \) is one of \( q(q^2 - 1) \) with \( f \geq 4 \), \( q(q^2 - 1)/4 \) with \( f \geq 3 \) or \( q(q^2 - 1)/60 \) with \( f \) even

The following lemma displays the correlation between 3/2-transitive and 2-transitive which lies in ([11], Theorem 1.2). It is a special case that will arise in our paper discussion.

**Lemma 6.** Let \( G = \text{PSL}(2, q) \) be a finite almost simple group, and \( G \) acts on a set \( \mathcal{P} \) with size \( v \) 3/2-transitively. Then the following conclusions are true:

(i) \( G \) has a 2-transitive action on \( \mathcal{P} \)

(ii) \( v = q(q - 1)/2 \) where \( q \geq 2 \), and each nontrivial subdegree of \( G \) is \( q + 1 \)

**3. Proof of Theorem 1**

From the information provided above, we have known that if \( \mathcal{D} \) is a 2-design with \( k = 5 \) and \( G \leq \text{Aut}(\mathcal{D}) \) is a flag-transitive group, then \( G \) must be a point-primitive group. If
Proposition 7. Suppose that $G \leq \text{Aut}(\mathcal{D})$ is flag-transitive for which $\mathcal{D}$ is a 2-design with block size 5. Then rank $(G) \leq 5$.

Proof. Following Lemma 2, $r = \lambda (v - 1)/k - 1 = \lambda (v - 1)/4$. Assuming $d \neq 1$ is a subdegree of $G$, then combining with Lemma 4, we can easily derive that $\nu - 1 \leq 4d$. Consequently, $d \in \{\nu - 1/4, \nu - 1/2, 3(\nu - 1)/4, \nu - 1\}$. As a result, the possible value of the rank of $G$ is 2, 3, 4, or 5. Moreover,

(i) If rank $(G) = 3$, then $G$ has subdegrees $\{1, \nu - 1/4, 3(\nu - 1)/4\}$ or $\{1, \nu - 1/2, (\nu - 1)/2\}$

(ii) If rank $(G) = 4$, then $G$ has subdegrees $\{1, \nu - 1/4, (\nu - 1)/4, (\nu - 1)/2\}$

(iii) If rank $(G) = 5$, then $G$ has subdegrees $\{1, \nu - 1/4, (\nu - 1)/4, \nu - 1/4, (\nu - 1)/4\}$

As mentioned above, the two-dimensional projective special linear group $PSL(2, q)$ acts doubly transitively on the one-dimensional projective line. It is for this reason that if $B$ is a G-orbit on 5-subsets of the one-dimensional projective line, then $\mathcal{D}$ is destined to be the set of block of a $2-(q + 1, 5, \lambda)$ design for some $\lambda$. At the same time, it can be easily seen that $G$ is destined to be the block-transitive automorphism group of such designs. \square

Proposition 8. Let $\mathcal{D}$ be a nontrivial 2-$(v, 5, \lambda)$ design. Let $G = PSL(2, q)$ be a flag-transitive automorphism group of $\mathcal{D}$ with $q \equiv 1 \pmod{4}$. Then rank $(G) = 2$, and one of the following three statements is to be true:

1. $\mathcal{D}$ has parameters $(v, \lambda) = (11, 2)$ or $(11, 12)$, and $G = PSL(2, 11)$

2. $\mathcal{D}$ has parameters $(v, \lambda) = (q + 1, 2(q - 1))$ with $q \equiv 11, 19, 31$ or 59 (mod 60)

3. $\mathcal{D}$ has parameters $(v, \lambda) = (q + 1, (q - 1)/3)$ with $q = 2^{f}$, where $f > 2$ is an even

Proof. Suppose that rank $(G) = 5$; then $G$ is a 3/2-transitive group by Proposition 7. According to Lemma 2(ii), we get $
u = q(q - 1)/2 = 4(q + 1) + 1$. Thus, $q = 10$, a contradiction.

Assume that rank $(G) = 4$. If $H$ is maximal subgroup of $PSL(2, q)$, then the subdegrees of the representation $PSL(2, q)$ on the cosets of $H$ can be found in $[9, 12]$. Obiously, we know that $G$ has no subdegrees $\{1, \nu - 1/4, (\nu - 1)/4, (\nu - 1)/2\}$.

Suppose that rank $(G) = 3$. For the case $G$ has subdegrees $\{1, \nu - 1/2, (\nu - 1)/2\}$, $G$ is a 3/2-transitive group. Similarly, Lemma 2(ii) yields $\nu = q(q - 1)/2 = 2(q + 1) + 1$. Thus, $q = 6$, a contradiction. Again by $[9, 12]$, we reach that $G$ has no subdegrees $\{1, \nu - 1/4, 3(\nu - 1)/4\}$.

Now we discuss the only remaining circumstance that rank $(G) = 2$; then $\nu = 7, 11$ or $q + 1$. We will now examine these three cases.

Case (1): $G = PSL(2, 7)$ with degree 7.

Here the order of $G$ is 168 and note that its representations are

$$\begin{align*}
(1, 4)(6, 7), \\
(1, 3, 2)(4, 7, 5).
\end{align*}$$

Clearly, $G$ is a doubly transitive group of the point set $\mathcal{D} = \{1, 2, \ldots, 7\}$. Let $B = \{1, 2, 3, 4, 5\}$; then $\mathcal{D} = \langle \mathcal{D}, B^c \rangle$ is a block-transitive 2-design such that its block number $b = \binom{7}{2}$ as $G$ is 2-transitive. Easy calculation shows that the setwise stabilizer $G_B$ has three orbits on $\mathcal{D}$ as follows:

$$\begin{align*}
&\{2\}, \{6, 7\}, \{1, 3, 4, 5\}.
\end{align*}$$

Thus, $\mathcal{D}$ is impossible to admit $PSL(2, 7)$ as its flag-transitive automorphism group by Lemma 3.

Case (2): $G = PSL(2, 11)$ with degree 11.

There are two inequivalent 2-transitive permutation representations of $PSL(2, 11)$ which possess degree 11, namely, $G_1$ and $G_2$. Here we will confine our discussion to the case $G = G_1$, and another case can be discussed similarly. Suppose that $S$ is a Sylow 5-subgroup of $G$; then $S$ partitions the 10 points into two orbits with size 5 and fixes one point, and we denote them by $B_1$ and $B_2$, respectively. Now, there is exactly one conjugacy class of subgroups $G$ isomorphic to alternating group $A_5$ partitioning the 11 points into two orbits of lengths 5 and 6. Let $K$ be the representative of the conjugacy class such that $S \leq K$. So $K$ partitions the 11 points into two orbits of length 5 and 6, respectively. It is safe to assume, without losing generality, that $B_1$ is the K-orbit of length 5. Then $G_{B_1} = K$, where $\mathcal{D} = \langle \mathcal{D}, B_1^c \rangle$ is the flag-transitive 2-$(11, 5, 2)$ symmetric design by Lemma 3 and the doubly transitivity of $G$. Note that $\mathcal{D}_1$ is a Hadamard design of order 3. Finally, $B_2$ is properly contained in the doubly transitive $K$ orbit of length 6; then $G_{B_2} \equiv D_{10}$, and hence $\mathcal{D}_2 = \langle \mathcal{D}, B_2^c \rangle$ is the flag-transitive 2-$(11, 5, 12)$ design again by the doubly transitivity of $G$ and Lemma 3.

Case (3): $G = PSL(2, q)$ with degree $q + 1$.

If $q \equiv 3 \pmod{4}$, from Lemma 5(1), there exist 5-element subsets $E_1$, $E_2$, and $E_3$ such that $|E_1| = q(q^2 - 1)/2$, $|E_2| = q(q^2 - 1)/6$ and $|E_3| = q(q^2 - 1)/10$. Lemma 3 yields that $|G_E|$ can be divided by 5, where $E \in B$ is a block. Thus, we conclude that $E_1$, $E_2$, and $E_3$ cannot be the block set of $\mathcal{D}$ as $|G_E| = 1$ and $|G_{E_2}| = 3$. Let $\mathcal{D}_3 = \langle \mathcal{D}, E_3^c \rangle$; then $\mathcal{D}_3$ is a 2-design with

$$b = \frac{q(q^2 - 1)}{10}$$

and $q \equiv 11, 19, 31$ or 59 (mod 60). \(3\)

Therefore, Lemma 2 yields that $\mathcal{D}_3$ has parameters $(v, \lambda) = (q + 1, 2(q - 1))$. Moreover, $G_{E_3} \equiv Z_5$, acts point-transitively on $E_3$. Then by what we have already shown, $PSL(2, q)$ is a transitive permutation group on the flags.
Now, let $q = 2^j$. Because of the nontriviality of $\mathcal{D}$, we should restrict $f > 2$. Similarly, from Lemma 5(2), there exist 5-element subsets $O_1$, $O_2$, and $O_3$ such that $|O_1^2| = q(q^2 - 1)$, $|O_2^2| = q(q^2 - 1)/4$, and $|O_3^2| = q(q^2 - 1)/60$. By Lemma 3, we infer that $O_1$ or $O_2$ cannot be a base block of a flag-transitive design for $|O_1| \geq 1$ and $|O_2| \geq 4$. Set $\mathcal{D}_4 = (\mathcal{P}, O_3^2)$, and we conclude that $\mathcal{D}_4$ is a 2-design with $b = \frac{q(q^2 - 1)}{60}$ and $f$ is an even. \hfill (4)

Now the result provided in Lemma 2 shows again that the 2-design $\mathcal{D}_4$ has parameters $(v, \lambda) = (q + 1, (q - 1)/3)$. Since $|G_{O_3}| = 60$, we have that $G_{O_3} \cong PSL(2, 4)$ acts point-transitively on $O_3$ by ([13], Lemma 16). Accordingly, $G$ acting on $\mathcal{D}_4$ is flag-transitive [14, 15].

Finally, Theorem 1 can be obtained from Proposition 8. \hfill □

Data Availability
The data used to support the findings of this study are included within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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