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Research Article

Flag-Transitive 2-$(v, 5, \lambda)$ Designs Admitting a Two-Dimensional Projective Group

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The focus of this study is to classify flag-transitive 2-designs. We have come to the conclusion that if $\mathcal{D}$ is a nontrivial 2-design having block size 5 and G is a two-dimensional projective special linear group which acts flag-transitively on $\mathcal{D}$ with $q \not\equiv 1 \pmod 4$, then $\mathcal{D}$ is a 2-(11, 5, 2) design, a 2-(11, 5, 12) design, a 2-$(q+1, 5, 2(q-1))$ design with $q \equiv 3 \pmod 4$ or a 2-$(q+1, 5, (q-1)/3)$ design with $q = 2^f$ (where $f > 2$ is an even).

1. Introduction

In the year of 1987, Davies [1] drew a conclusion that if a 2-(v, k, λ) design $\mathcal{D}$ has a point-imprimitive flag-transitive automorphism group G, then given the value of λ, there is a superior limit on the block size k. We can easily arrive at a conclusion that the existence of the pairs $(\mathcal{D}, G)$ is not infinite only if λ is fixed, where G acting on $\mathcal{D}$ is point-imprimitive and flag-transitive. Meanwhile, the hidden meaning is that, as long as we set k fixed, there are only a finite number of such designs. Still further, on the basis of the proof of ([2], Proposition 4.1), one proving that for a nontrivial point-imprimitive and flag-transitive 2-(v, k, λ) design $\mathcal{D}$, its block size $k \geq 6$. Without a doubt, we have that the flag-transitive automorphism group G for our paper of the 2-$(v, 5, \lambda)$ design is definitely point-primitive. The O'Nan–Scott theorem shows that one of the following is doomed to hold for any finite primitive permutation group G (for more details, see [3]):

(a) G is of almost simple type
(b) G is of affine type
(c) G is of simple diagonal type
(d) G is of product type
(e) G is of twisted wreath product type

In order to solve the problem more completely, we narrow down the scope of our problem; here, we only consider the case (a) and G is a two dimensional projective special linear group $\mathrm{PSL}(2, q)$.

For $q \not\equiv 1 \pmod 4$, the group $\mathrm{PSL}(2, q)$ acts 3-homogeneously on the one-dimensional projective line. We immediately inferred that if $\mathcal{B}$ is a set consisting of some k-subsets of the projective line $\mathcal{P}$, then $(\mathcal{P}, \mathcal{B})$ is a 3-$(q+1, k, \lambda)$ design for some λ in the condition that $\mathcal{B}$ is a union of G-orbits with $G = \mathrm{PSL}(2, q)$. Thereafter that is why $\mathrm{PSL}(2, q)$ becomes a coveted treasure when it comes to construct a 3-design. For instance, through the way $\mathrm{PSL}(2, q)$ acts on the 4-element subsets of the one-dimensional projective line, the authors of [4] determined all 3-designs of $\mathrm{PSL}(2, q)$ with block size 4. By using the same processing method, Keranen and Kreher [5] completely solved such designs of block size 5. On the other hand, for $q \equiv 1 \pmod 4$, Keranen et al. [6] has completely determined all quadruple systems admitting $\mathrm{PSL}(2, q)$ as their automorphism group. Unfortunately, the 3-$(q+1, 5, \lambda)$ designs with the automorphism group $\mathrm{PSL}(2, q)$ remain uncertain.

Many scholars have turned to another direction of exploration, focusing on 2-designs whose automorphism group is the projective group $PSL(2,q)$, which acts transitively on the flags. Here, flags are point-block pairs (α, B) such that $\alpha \in B$. In 1986, Delandtsheer [7] discovered a 2-$(q(q-1)/2, q/2, 1)$ design with $q = 2^f \geq 8$ (it is usually called Witt-Bose-Shrikhandle space $W(q)$) on the way to classify flag-transitive 2-$(v,k,1)$ design. Three savants of literature [8] are absorbed in symmetric designs and give them a complete classification. Zhan and Zhou [9], in 2018, dealt with the situation where $\mathscr{D}$ is a nonsymmetric design with parameters r and λ that are coprime. The most classification of 2-$(v,4,\lambda)$ designs, characterized by $PSL(2,q)$ as a flag-transitive automorphism group, is presented in [10], offering a complete and elegant classification of such designs.

Our paper aims to studying 2-designs with block sizes of 5 and flag-transitive automorphism groups that are the two-dimensional special linear projective group $PSL(2,q)$, where $q \not\equiv 1 \pmod 4$. Obviously, it is of great significance to contribute to fully classify the 2-$(v,5,\lambda)$ designs permitting a flag-transitive automorphism group. Furthermore, the main outcomes of our paper are displayed as follows.

Theorem 1. *Assume that $\mathscr{D}$ is a nontrivial 2-design with block size 5. Let $G = PSL(2,q)$ act flag-transitively on $\mathscr{D}$ with $q \not\equiv 1$ (mod 4). Then one of the following conclusions is proved to be tenable:*

(1) $\mathscr{D}$ is a 2-(11,5,2) design or a 2-(11,5,12) design with $q = 11$

(2) $\mathscr{D}$ is a 2-$(q+1, 5, 2(q-1))$ design with $q \equiv 11, 19, 31$ or 59 (mod 60)

(3) $\mathscr{D}$ is a 2-$(q+1, 5, (q-1/3))$ design with $q = 2^f$ and $f > 2$ is an even integer

In order to have a better understanding for the proving process of Theorem 1 in the third section, we will demonstrate some basic concepts and general principles which will be employed during the process of proving our conclusion.

2. Notation and Preliminaries

Assuming that $\mathscr{B}$ is a set containing b blocks and $\mathscr{P}$ is a set containing v points, the pair $(\mathscr{P}, \mathscr{B})$ points to a 2-design $\mathscr{D}$ that satisfies every block including k points, 2 different points are exactly comprised in λ blocks, and a given point is relative to r blocks. In general, we handle the case $2 < k < v-1$, where $\mathscr{D}$ is called nontrivial. Particularly, a design $\mathscr{D}$ is said to be symmetric if the total number of blocks in $\mathscr{D}$ is equal to the total number of points; otherwise, it is called nonsymmteric. A 2-(v,k,λ) design is often called a finite linear space when $\lambda = 1$. For the study of a 2-design, the following lemma is almost involved for each time.

Lemma 2 (see ([4], 1.2, 1.9)). *The following properties hold for a 2-design:*

(i) $bk = vr$

(ii) $r(k-1) = \lambda(v-1)$

More often than not, we use $\mathrm{Aut}(\mathscr{D})$ to represent the full automorphism group of a design $\mathscr{D}$. That is, $\mathrm{Aut}(\mathscr{D})$ is the group in particular to those composed of all automorphisms of $\mathscr{D}$, where an automorphism of $\mathscr{D}$ refers to a permutation that can permutate not only the point set $\mathscr{P}$ but also the block set $\mathscr{B}$. That is to say, when G is an automorphism group of $\mathscr{D}$, any element of G must belong to $\mathrm{Aut}(\mathscr{D})$, in short, $G \leq \mathrm{Aut}(\mathscr{D})$. If G has a primitive action on point set and a transitive action on flag set, then we say design $\mathscr{D}$ is point-primitive and flag-transitive, respectively.

Take any block $B \in \mathscr{B}$, and thereafter G_B is the setwise stabilizer. Below we will introduce a classical and commonly used verdict about them.

Lemma 3. *For a 2-design $\mathscr{D} = (\mathscr{P}, \mathscr{B})$, let $G \leq Aut(\mathscr{D})$, and $\mathscr{F}$ be the flag set of $\mathscr{D}$. If $B \in \mathscr{B}$, then the two statements given below are interchangeable:*

(i) G acts flag-transitively on $\mathscr{F}$

(ii) G is a transitive group of $\mathscr{B}$, and G_B has a transitive action on B

The following lemma adds the finishing touch to the proof of our paper. Literature [1] demonstrates the proof in detail and helps us to understand it better.

Lemma 4. *Provided that G acts on a 2-design $\mathscr{D}$ flag-transitively, let d be a subdegree of G with $d \neq 1$; then r divides λd.*

From ([4], Theorem 6) and ([5], Theorem 3.3), we can reach the following result which is of most importance through a simple calculation.

Lemma 5. *Let $G = PSL(2,q)$ with $q \not\equiv 1$ (mod 4), and B be a 5-element subset of one-dimensional projective line. Then the following two situations are to be true:*

(1) If $q \equiv 3$ (mod 4), then $|B^G|$ is one of $q(q^2-1)/2$ with $q > 3$, $q(q^2-1)/6$ with $q \equiv 7, 19, 31, 43$ (mod 60) or $q(q^2-1)/10$ with $q \equiv 11, 19, 31, 59$ (mod 60)

(2) If $q = 2^f$, then $|B^G|$ is one of $q(q^2-1)$ with $f \geq 4$, $q(q^2-1)/4$ with $f \geq 3$ or $q(q^2-1)/60$ with f even

The following lemma displays the correlation between 3/2-transitive and 2-transitive which lies in ([11], Theorem 1.2). It is a special case that will arise in our paper discussion.

Lemma 6. *Let $G = PSL(2,q)$ be a finite almost simple group, and G acts on a set $\mathscr{P}$ with size v3/2-transitively. Then the following conclusions are true:*

(i) G has a 2-transitive action on $\mathscr{P}$

(ii) $v = q(q-1)/2$ where $q = 2^f \geq 8$, and each nontrivial subdegree of G is $q+1$

3. Proof of Theorem 1

From the information provided above, we have known that if $\mathscr{D}$ is a 2-design with $k = 5$ and $G \leq \mathrm{Aut}(\mathscr{D})$ is a flag-transitive group, then G must be a point-primitive group. If

G_α is the set keeping a certain point α in $\mathscr{P}$ stationary, then G_α must be one of the maximal subgroups in G. In the following convention, rank (G) is denoted by the rank of G.

Proposition 7. *Suppose that $G \leq Aut(\mathscr{D})$ is flag-transitive for which $\mathscr{D}$ is a 2-design with block size 5. Then rank $(G) \leq 5$.*

Proof. Following Lemma 2, $r = \lambda(v-1)/k - 1 = \lambda(v-1)/4$. Assuming $d \neq 1$ is a subdegree of G, then combining with Lemma 4, we can easily derive that $v - 1 \mid 4d$. Consequently, $d \in \{v - 1/4, v - 1/2, 3(v-1)/4, v-1\}$. As a result, the possible value of the rank of G is 2, 3, 4, or 5. Moreover,

(i) If rank $(G) = 3$, then G has subdegrees $\{1, v - 1/4, 3(v-1)/4\}$ or $\{1, v - 1/2, (v-1)/2\}$
(ii) If rank $(G) = 4$, then G has subdegrees $\{1, v - 1/4, (v-1)/4, (v-1)/2\}$
(iii) If rank $(G) = 5$, then G has subdegrees $\{1, v - 1/4, (v-1)/4, v - 1/4, (v-1)/4\}$

As mentioned above, the two-dimensional projective special linear group $PSL(2,q)$ acts doubly transitively on the one-dimensional projective line. It is for this reason that if $\mathscr{B}$ is a G-orbit on 5-subsets of the one-dimensional projective line, then $\mathscr{B}$ is destined to be the set of block of a 2-$(q+1, 5, \lambda)$ design for some λ. At the same time, it can be easily seen that G is deemed to be the block-transitive automorphism group of such designs. □

Proposition 8. *Let $\mathscr{D}$ be a nontrivial 2-$(v,5,\lambda)$ design. Let $G = PSL(2,q)$ be a flag-transitive automorphism group of $\mathscr{D}$ with $q \not\equiv 1 \pmod 4$. Then rank $(G) = 2$, and one of the following three statements is to be true:*

(1) $\mathscr{D}$ has parameters $(v,\lambda) = (11,2)$ or (11,12), and $G = PSL(2,11)$
(2) $\mathscr{D}$ has parameters $(v,\lambda) = (q+1, 2(q-1))$ with $q \equiv 11, 19, 31$ or 59 (mod 60)
(3) $\mathscr{D}$ has parameters $(v,\lambda) = (q+1, (q-1)/3)$ with $q = 2^f$, where $f > 2$ is an even

Proof. Suppose that rank $(G) = 5$; then G is a 3/2-transitive group by Proposition 7. According to Lemma 2(ii), we get $v = q(q-1)/2 = 4(q+1) + 1$. Thus, $q = 10$, a contradiction.

Assume that rank $(G) = 4$. If H is maximal subgroup of $PSL(2,q)$, then the subdegrees of the representation $PSL(2,q)$ on the cosets of H can be found in [9, 12]. Obviously, we know that G has no subdegrees $\{1, v - 1/4, (v-1)/4, (v-1)/2\}$.

Suppose that rank $(G) = 3$. For the case G has subdegrees $\{1, v - 1/2, (v-1)/2\}$, G is a 3/2-transitive group. Similarly, Lemma 2(ii) yields $v = q(q-1)/2 = 2(q+1) + 1$. Thus, $q = 6$, a contradiction. Again by [9, 12], we reach that G has no subdegrees $\{1, v - 1/4, 3(v-1)/4\}$.

Now we discuss the only remaining circumstance that rank $(G) = 2$; then $v = 7, 11$ or $q + 1$. We will now examine these three cases.

Case (1): $G = PSL(2,7)$ with degree 7.

Here the order of G is 168 and note that its representations are

$$\begin{aligned} &(1,4)(6,7), \\ &(1,3,2)(4,7,5). \end{aligned} \tag{1}$$

Clearly, G is a doubly transitive group of the point set $\mathscr{P} = \{1, 2, \ldots, 7\}$.

Let $B = \{1,2,3,4,5\}$; then $\mathscr{D} = (\mathscr{P}, B^G)$ is a block-transitive 2-design such that its block number $b = \binom{7}{2}$ as G is 2-transitive. Easy calculation shows that the setwise stabilizer G_B has three orbits on $\mathscr{P}$ as follows:

$$\{2\}, \{6,7\}, \{1,3,4,5\}. \tag{2}$$

Thus, $\mathscr{D}$ is impossible to admit $PSL(2,7)$ as its flag-transitive automorphism group by Lemma 3.

Case (2): $G = PSL(2,11)$ with degree 11.

There are two inequivalent 2-transitive permutation representations of $PSL(2,11)$ which possess degree 11, namely, G_1 and G_2. Here we will confine our discussion to the case $G = G_1$, and another case can be discussed similarly. Suppose that S is a Sylow 5-subgroup of G; then S partitions the 10 points into two orbits with size 5 and fixes one point, and we denote them by B_1 and B_2, respectively. Now, there is exactly one conjugacy class of subgroups G isomorphic to alternating group A_5 partitioning the 11 points into two orbits of lengths 5 and 6. Let K be the representative of the conjugacy class such that $S \leq K$. So K partitions the 11 points into two orbits of length 5 and 6, respectively. It is safe to assume, without losing generality, that B_1 is the K-orbit of length 5. Then $G_{B_1} = K$ and $\mathscr{D}_1 = (\mathscr{P}, B_1^G)$ is the flag-transitive 2-$(11,5,2)$ symmetric design by Lemma 3 and the doubly transitivity of G. Note that $\mathscr{D}_1$ is a Hadamard design of order 3. Finally, B_2 is properly contained in the doubly transitive K orbit of length 6; then $G_{B_2} \cong D_{10}$, and hence $\mathscr{D}_2 = (\mathscr{P}, B_2^G)$ is the flag-transitive 2-$(11,5,12)$ design again by the doubly transitivity of G and Lemma 3.

Case (3): $G = PSL(2,q)$ with degree $q+1$.

If $q \equiv 3 \pmod 4$, from Lemma 5(1), there exist 5-element subsets E_1, E_2, and E_3 such that $|E_1^G| = q(q^2-1)/2$, $|E_2^G| = q(q^2-1)/6$ and $|E_3^G| = q(q^2-1)/10$. Lemma 3 yields that $|G_E|$ can be divided by 5, where $E \in \mathscr{B}$ is a block. Thus, we conclude that E_1^G or E_2^G cannot be the block set of $\mathscr{D}$ as $|G_{E_1}| = 1$ and $|G_{E_2}| = 3$. Let $\mathscr{D}_3 = (\mathscr{P}, E_3^G)$; then $\mathscr{D}_3$ is a 2-design with

$$b = \frac{q(q^2-1)}{10} \text{ and } q \equiv 11, 19, 31 \text{ or } 59 \pmod{60}. \tag{3}$$

Therefore, Lemma 2 yields that $\mathscr{D}_3$ has parameters $(v,\lambda) = (q+1, 2(q-1))$. Moreover, $G_{E_3} \cong \mathbb{Z}_5$ acts point-transitively on E_3. Then by what we have already shown, $PSL(2,q)$ is a transitive permutation group on the flags.

Now, let $q = 2^f$. Because of the nontriviality of $\mathscr{D}$, we should restrict $f > 2$. Similarly, from Lemma 5(2), there exist 5-element subsets O_1, O_2, and O_3 such that $|O_1^G| = q(q^2-1)$, $|O_2^G| = q(q^2-1)/4$, and $|O_3^G| = q(q^2-1)/60$. By Lemma 3, we infer that O_1 or O_2 cannot be a base block of a flag-transitive design for $|G_{O_1}| = 1$ and $|G_{O_2}| = 4$. Set $\mathscr{D}_4 = (\mathscr{P}, O_3^G)$, and we conclude that $\mathscr{D}_4$ is a 2-design with

$$b = \frac{q\left(q^2-1\right)}{60} \text{ and } f \text{ is an even.} \tag{4}$$

Now the result provided in Lemma 2 shows again that the 2-design $\mathscr{D}_4$ has parameters $(v, \lambda) = (q+1, (q-1/3))$. Since $|G_{O_3}| = 60$, we have that $G_{O_3} \cong \mathrm{PSL}(2,4)$ acts point-transitively on O_3 by ([13], Lemma 16). Accordingly, G acting on $\mathscr{D}_4$ is flag-transitive [14, 15].

Finally, Theorem 1 can be obtained from Proposition 8. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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