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Research Article

On Implicit Atangana–Baleanu–Caputo Fractional Integro-Differential Equations with Delay and Impulses

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In this paper, we study the existence and uniqueness of solutions for impulsive Atangana-Baleanu-Caputo ($\mathscr{ABC}$) fractional integro-differential equations with boundary conditions. Schaefer's fixed point theorem and Banach contraction principle are used to prove the existence and uniqueness results. An example is presented to illustrate the results.

1. Introduction

To analyze the fractional dynamics of the provided model, we use the Atangana–Baleanu fractional operator in the Caputo sense. Because of their nonlocal characteristics, $\mathscr{AB}$ fractional derivatives are used. Many authors studied the $\mathscr{ABC}$ fractional derivative with applications, see for example [1–11]. The prime reason is the theory of fractional calculus's quick development, which is used extensively in many different fields including biology, mathematics, chemistry, physics, mechanics, medicine, environmental science, control theory, image and signal processing, finance, and others, see reference [12–17].

Numerous phenomena encounter abrupt or sudden changes in their state of motion or rest in real-world issues. Impulsive differential equations are used to model these sudden changes. Regarding ordinary derivatives and integrals, the field that deals with the aforementioned issues has a strong foundation. Researchers have employed fixed point theory and nonlinear analysis techniques to find the results of investigations. The authors have investigated the theory of these differential equations, see reference [18–22]. However, the study of impulsive problems using the theory of fractional calculus has also progressed well. A delay differential equation is a differential equation where the time derivatives at the current time depend on the solution and possibly its derivatives at previous times. These models are used, among other things, in the fields of biology, economics, and mechanics, see [23]. The delay in this differential equation comes from the interval between the beginning of cellular production in the bone marrow and the release of mature cells into the blood. These equations were developed to render models more reasonable because many practices depend on historical data, refer [22, 24, 25]. The fact that these models only consider past states and not past rates is one of their drawbacks.

In [26], Benchohra et al. investigated the existence and stability results for the following fractional differential equations:

$$\begin{aligned} &{}^{c}D^{\varsigma}_{\mathfrak{k}_{\mathfrak{z}}}\left[\mathfrak{y}(\mathfrak{k})-\mathfrak{A}(\mathfrak{k},\mathfrak{y}_{\mathfrak{k}})\right]=\mathfrak{B}\left(\mathfrak{k},\mathfrak{y}_{\mathfrak{k}},{}^{c}D^{\varsigma}_{\mathfrak{k}_{\mathfrak{z}}}\mathfrak{y}(\mathfrak{k})\right), \quad \text{for each } \mathfrak{k}\in\left[\mathfrak{k}_{\mathfrak{z}},\mathfrak{k}_{\mathfrak{z}+1}\right], \mathfrak{z}=0,1,\ldots n, 0<\varsigma\leq 1,\\ &\Delta(\mathfrak{y})|_{\mathfrak{k}=\mathfrak{k}_{\mathfrak{z}}}=\mathfrak{I}_{\mathfrak{z}}\left(\phi_{\mathfrak{k}^{-}_{\mathfrak{z}}}\right), \quad \mathfrak{z}=1,\ldots n,\\ &\mathfrak{y}(\mathfrak{k})=\varphi(\mathfrak{k}), \quad \mathfrak{k}\in[-\mathfrak{r},0], \mathfrak{r}>0, \end{aligned} \tag{1}$$

where ${}^{c}D^{\varsigma}_{\mathfrak{k}_{\mathfrak{z}}}$ is the Caputo fractional derivative, $\mathfrak{B}\colon [0,\mathfrak{T}]\times PC([-\mathfrak{r},0],\mathfrak{R})\times\mathfrak{R}\longrightarrow\mathfrak{R}$, $\mathfrak{A}\colon [0,\mathfrak{T}]\times PC([-\mathfrak{r},0],\mathfrak{R})\longrightarrow\mathfrak{R}$ are given functions with $\mathfrak{A}(0,\varphi)=0$, $\mathfrak{I}_{\mathfrak{z}}\colon PC([-\mathfrak{r},0],\mathfrak{R})\longrightarrow\mathfrak{R}, \varphi\in PC([-\mathfrak{r},0],\mathfrak{R})$, $0=\mathfrak{k}_0<\mathfrak{k}_1<\cdots<\mathfrak{k}_n<\mathfrak{k}_{n+1}=\mathfrak{T}$. $\Delta\mathfrak{y}|_{\mathfrak{k}_{\mathfrak{z}}}=\mathfrak{y}(\mathfrak{k}^{+}_{\mathfrak{z}})-\mathfrak{y}(\mathfrak{k}^{-}_{\mathfrak{z}})$, where $\mathfrak{y}(\mathfrak{k}^{+}_{\mathfrak{z}})=\lim_{h\longrightarrow 0^{+}}\mathfrak{y}(\mathfrak{k}_{\mathfrak{z}}+h)$ and $\mathfrak{y}(\mathfrak{k}^{-}_{\mathfrak{z}})=\lim_{h\longrightarrow 0^{-}}\mathfrak{y}(\mathfrak{k}_{\mathfrak{z}}-h)$ represent the right and left limits of $\mathfrak{y}$ at $\mathfrak{k}=\mathfrak{k}_{\mathfrak{z}}$, respectively.

In [6], Gul et al. examined the existence of the following boundary value problems under the $\mathscr{ABC}$ fractional derivative:

$$\begin{aligned} &{}^{\mathscr{ABC}}_{0}D^{\wp}_{t}[\varkappa(t)-\mathfrak{B}(t,\varkappa(t))]=\mathfrak{W}t,\varkappa(t), \quad 0<\wp\leq 1, t\in[0,\mathscr{T}]=\mathfrak{J}',\\ &\varkappa(0)=\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathfrak{U}(\nu,\varkappa(\nu))d\nu, \end{aligned} \tag{2}$$

where ${}^{\mathscr{ABC}}_{0}D^{\wp}_{t}$ is the $\mathscr{ABC}$ fractional derivative of order $\wp$, $\mathfrak{B},\mathfrak{U},\mathfrak{W}\colon \mathfrak{J}'\times\mathscr{R}\longrightarrow\mathscr{R}$.

In [27], Reunsumrit et al. discussed the existence results for the following problem:

$$\begin{aligned} &{}^{\mathscr{ABC}}_{0}D^{\wp}_{t}[\varkappa(t)-\mathfrak{U}(t,\varkappa(t))]=\mathfrak{B}(t,\varkappa(t),\mathfrak{L}\varkappa(t)), \quad 0<\wp\leq 1, t\in[0,\mathscr{T}]=\mathfrak{J}',\\ &\Delta(\varkappa)|_{t=t_k}=\mathfrak{I}_k(\varkappa(t^{-}_k)),\\ &\varkappa(0)=\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathfrak{S}(\nu,\varkappa(\nu))d\nu, \end{aligned} \tag{3}$$

where ${}^{\mathscr{ABC}}_{0}D^{\wp}_{t}$- is the $\mathscr{ABC}$ fractional derivative of order $\wp$, $\mathfrak{U},\mathfrak{S}\colon \mathfrak{J}'\times\mathscr{R}\longrightarrow\mathscr{R}$ and $\mathfrak{B}, g\colon \mathfrak{J}'\times\mathscr{R}^2\longrightarrow\mathscr{R}$ is a continuous function. Here, $\mathfrak{L}\varkappa(t)=\int_0^t g(t,\tau,\phi(\tau))d\tau$, and $\mathfrak{I}_k\colon \mathscr{R}\longrightarrow\mathscr{R}, k=1,2,\ldots m$. $0=t_0<t_1<t_2<\ldots<t_m=\mathscr{T}$, $\Delta\varkappa|_{t=t_k}=\varkappa(t^{+}_k)-\varkappa(t^{-}_k)$, and $\varkappa(t^{+}_k)=\lim_{h\longrightarrow 0^{+}}\varkappa(t_k+h)$ and $\varkappa(t^{-}_k)=\lim_{h\longrightarrow 0^{-}}\varkappa(t_k+h)$ indicates the right and left hand limits of $\varkappa(t)$ at $t=t_k$.

Motivated by the works, consider the impulsive $\mathscr{ABC}$ fractional integro-differential equations with boundary conditions of the form:

$$\begin{cases} {}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}[\mathfrak{x}(\mathfrak{k})-\mathfrak{P}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}})]=\mathfrak{Q}\left(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}},{}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}\right), \quad \mathfrak{k}\in[0,\mathfrak{T}]=\mathfrak{J}, 0<\varsigma\leq 1, \\ \Delta(\mathfrak{x})|_{\mathfrak{k}=\mathfrak{k}_{\mathfrak{z}}}=\mathfrak{I}_{\mathfrak{z}}\left(\mathfrak{x}_{\mathfrak{k}_{\mathfrak{z}}^{-}}\right), \\ \mathfrak{x}(\mathfrak{k})=\varphi(\mathfrak{k}), \quad \mathfrak{k}\in[-\mathfrak{r},0], \\ \mathfrak{x}(0)=\int_{0}^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{E}(\ell,\mathfrak{x}_{\ell})d\ell, \end{cases} \tag{4}$$

where ${}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}$- is the $\mathscr{ABC}$ fractional derivative of order ς, $\mathfrak{P},\mathfrak{E}\colon \mathfrak{J}\times\grave{\mathfrak{R}}\longrightarrow\grave{\mathfrak{R}}$ and $\mathfrak{Q}\colon \mathfrak{J}\times\grave{\mathfrak{R}}^{2}\longrightarrow\grave{\mathfrak{R}}$ is continuous function. Here, $\mathfrak{I}_{\mathfrak{z}}\colon \grave{\mathfrak{R}}\longrightarrow\grave{\mathfrak{R}}, \mathfrak{z}=1,2,\ldots n.$ $0=\mathfrak{k}_0<\mathfrak{k}_1<\mathfrak{k}_2<\ldots<\mathfrak{k}_n=\mathfrak{T}, \Delta\mathfrak{x}|_{\mathfrak{k}}=\mathfrak{k}_{\mathfrak{z}}=\mathfrak{x}(\mathfrak{k}_{\mathfrak{z}}^{+})-\mathfrak{x}(\mathfrak{k}_{\mathfrak{z}}^{-})$, $\mathfrak{x}(\mathfrak{k}_{\mathfrak{z}}^{-})=\lim_{\mathfrak{r}\longrightarrow 0^{-}}\mathfrak{x}(\mathfrak{k}_{\mathfrak{z}}+\mathfrak{r})$, and $\mathfrak{x}(\mathfrak{k}_{\mathfrak{z}}^{+})=\lim_{\mathfrak{r}\longrightarrow 0^{+}}\mathfrak{x}(\mathfrak{k}_{\mathfrak{z}}+\mathfrak{r})$ represent the left and right hand limits of $\mathfrak{x}(\mathfrak{k})$ at $\mathfrak{k}=\mathfrak{k}_{\mathfrak{z}}$. For any $\mathfrak{k}\in\mathfrak{J}$, we represent $\mathfrak{x}_{\mathfrak{k}}$ by

$$\mathfrak{x}_{\mathfrak{k}}(s)=\mathfrak{x}(\mathfrak{k}+s)\text{ and } -\mathfrak{r}\leq s\leq 0. \tag{5}$$

The contents of this paper are organized as follows. Section 2 provides some fundamental definitions and lemmas. The existence and uniqueness of fractional implicit differential equations are studied in Section 3. In Section 4, the applications are illustrated through an example.

2. Preliminaries

Define

$$\mathfrak{PC}([-\mathfrak{r},0],\grave{\mathfrak{R}})=\left\{\mathfrak{x}\colon [-\mathfrak{r},0]\longrightarrow\grave{\mathfrak{R}}\colon \mathfrak{x}\in\mathfrak{C}\left(\mathfrak{k}_{\mathfrak{z}},\mathfrak{k}_{\mathfrak{z}+1},\grave{\mathfrak{R}}\right), \quad \mathfrak{z}=0,1,\ldots l,\text{ and }\exists\,\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{-}\right)\text{ and }\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{+}\right), \mathfrak{z}=1\ldots .l,\text{ with }\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{-}\right)=\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{+}\right)\right\}, \tag{6}$$

$\mathfrak{PC}([-\mathfrak{r},0],\grave{\mathfrak{R}})$ is a Banach space with the norm

$$\begin{aligned} \|\mathfrak{x}\|_{\mathfrak{PC}} &= \sup_{\mathfrak{k}\in[-\mathfrak{r},0]}|\mathfrak{x}(\mathfrak{k})|, \\ \mathfrak{PC}([0,\mathfrak{T}],\grave{\mathfrak{R}}) &= \left\{\mathfrak{x}\colon [0,\mathfrak{T}]\longrightarrow\grave{\mathfrak{R}}\colon \mathfrak{x}\in\mathfrak{C}\left(\mathfrak{k}_{\mathfrak{z}},\mathfrak{k}_{\mathfrak{z}+1},\grave{\mathfrak{R}}\right), \quad \mathfrak{z}=0,1,\ldots l,\text{ and }\exists\,\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{-}\right)\text{ and }\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{+}\right), \mathfrak{z}=1\ldots .l,\text{ with }\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{-}\right)=\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{+}\right)\right\}, \end{aligned} \tag{7}$$

$\mathfrak{PC}([0,\mathfrak{T}],\grave{\mathfrak{R}})$ is a Banach space with the norm

$$\begin{aligned} \|\mathfrak{x}\|_{\mathfrak{PC}_1} &= \sup_{\mathfrak{k}\in[0,\mathfrak{T}]}|\mathfrak{x}(\mathfrak{k})|, \\ \Xi &= \left\{\mathfrak{x}\colon [-\mathfrak{r},\mathfrak{T}]\longrightarrow\grave{\mathfrak{R}}\colon \mathfrak{x}|_{[-\mathfrak{r},0]}\in\mathfrak{PC}([-\mathfrak{r},0],\grave{\mathfrak{R}})\text{ and }\mathfrak{x}|_{[0,\mathfrak{T}]}\in\mathfrak{PC}([0,\mathfrak{T}],\grave{\mathfrak{R}})\right\}, \end{aligned} \tag{8}$$

Ξ is a Banach space with the norm

$$\|\mathfrak{x}\|_{\Xi}=\sup_{\mathfrak{k}\in[-\mathfrak{r},\mathfrak{T}]}|\mathfrak{x}(\mathfrak{k})|. \tag{9}$$

Definition 1 (see [27]). Let $\varsigma\in\mathfrak{E}^{1}(0,\mathfrak{T})$ with $\varsigma\in[0,1]$, the fractional order $\mathscr{ABC}$ derivative is defined as

$${}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}w(\mathfrak{k})=\frac{\mathfrak{M}(\varsigma)}{1-\varsigma}\int_{0}^{\mathfrak{T}}\frac{dw}{d\ell}\mathscr{E}_{\varsigma}\left[\frac{-\varsigma(\mathfrak{k}-\ell)}{1-\varsigma}\right]d\ell, \tag{10}$$

where $\mathfrak{M}(\varsigma)$ is called normalization function satisfying $\mathfrak{M}(0)=\mathfrak{M}(1)=1$ and $\mathscr{E}_{\varsigma}=\sum_{i=0}^{\infty}\mathfrak{k}^{i\varsigma}/(\varsigma i+1)$ is a Mittag–Leffler function.

Definition 2 (see [27]). The $\mathscr{ABC}$ fractional integral for w is written as

$$ {}_{0}^{\mathscr{ABC}}\mathfrak{I}_{\mathfrak{k}}^{\varsigma}w(\mathfrak{k}) = \frac{1-\varsigma}{\mathfrak{M}(\varsigma)}w(\mathfrak{k}) + \frac{\varsigma}{\mathfrak{M}(\varsigma)}\int_{0}^{\mathfrak{k}}\frac{(\mathfrak{k}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}w(\ell)d\ell, \tag{11} $$

where $\mathfrak{I}^{\varsigma}$ is the Riemann–Liouville fractional integral.

Lemma 3 (see [27]). *Consider the following problem:*

$$ \begin{aligned} {}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}\mathfrak{x}(\mathfrak{k}) &= \mathfrak{z}(\mathfrak{k}),\\ \mathfrak{x}(0) &= \mathfrak{x}_0. \end{aligned} \tag{12} $$

Then, the solution is given by

$$ \mathfrak{x}(0) = \mathfrak{x}_0 + \frac{1-\varsigma}{\mathfrak{M}(\varsigma)}\mathfrak{z}(\mathfrak{k}) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{0}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\mathfrak{z}(\ell)d\ell. \tag{13} $$

Proof. By using Definition 2, we get

$$ \begin{aligned} \mathfrak{x}(\mathfrak{k}) &= \mathfrak{x}_0 + {}_{0}^{\mathscr{ABC}}\mathfrak{I}_{\mathfrak{k}}^{\varsigma}\mathfrak{z}(\mathfrak{k})\\ &= \mathfrak{x}_0 + +\frac{1-\varsigma}{\mathfrak{M}(\varsigma)}\mathfrak{z}(\mathfrak{k}) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{0}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\mathfrak{z}(\ell)d\ell. \end{aligned} \tag{14} $$

□

Theorem 4 (see [26]). *Let* $\mathfrak{Z}$ *be a Banach space, and* $\aleph: \mathfrak{Z} \longrightarrow \mathfrak{Z}$ *is a completely continuous operator. If the set* $E = \{\mathfrak{x} \in \mathfrak{Z}: \mathfrak{x} = \lambda\aleph\mathfrak{x}, \text{for some } \lambda \in (0,1)\}$ *is bounded, then* $\aleph$ *has fixed points.*

Lemma 5 (see [26]). *Let* $v: [0,\mathfrak{T}] \longrightarrow (0,\infty)$ *be a real function and* $w(.)$ *be a nonnegative, locally integrable function on* $[0,\mathfrak{T}]$*, and suppose there are constants* $a > 0$ *and* $0 < b \leq 1$ *such that*

$$ v(\mathfrak{k}) \leq w(\mathfrak{k}) + a\int_{0}^{\mathfrak{k}}(\mathfrak{k}-s)^{-b}v(s)ds. \tag{15} $$

There exists a constant $K = K(b)$ such that

$$ v(\mathfrak{k}) \leq w(\mathfrak{k}) + Ka\int_{0}^{\mathfrak{k}}(\mathfrak{k}-s)^{-b}w(s)ds, \quad \text{for every } \mathfrak{k} \in [0,\mathfrak{T}]. \tag{16} $$

Lemma 6. *Consider the boundary value problem with nonlinear integral boundary conditions if* $\mathfrak{z} \in L(\mathfrak{J})$,

$$ \begin{aligned} {}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}\mathfrak{x}(\mathfrak{k}) &= \mathfrak{z}(\mathfrak{k}), \quad o < \varsigma < 1, \mathfrak{k} \in \mathfrak{J},\\ \mathfrak{x}(0) &= \int_{0}^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}(\ell))d\ell, \end{aligned} \tag{17} $$

then, the solution $\mathfrak{x} \in \mathfrak{A}C(\mathfrak{J})$ *is given by*

$$ \mathfrak{x}(\mathfrak{k}) = \int_{0}^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}(\ell))d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{z}(\mathfrak{k}) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{0}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\mathfrak{z}(\ell)d\ell. \tag{18} $$

Proof. By Lemma 3, we can get the result (18) directly by replacing $\mathfrak{x}_0$ into the boundary condition. □

Lemma 7. *Consider the nonlinear integral boundary value problem*

$$ \begin{aligned} {}_{0}^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}[\mathfrak{x}(\mathfrak{k}) - \mathfrak{P}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}})] &= \mathfrak{p}^{*}(\mathfrak{k}), \quad \mathfrak{k} \in [0,\mathfrak{T}] = \mathfrak{J}, 0 < \varsigma \leq 1,\\ \Delta(\mathfrak{x})|_{\mathfrak{k}=\mathfrak{k}_{\mathfrak{z}}} &= \mathfrak{I}_{\mathfrak{z}}(\mathfrak{x}_{\mathfrak{k}_{\mathfrak{z}}^{-}}),\\ \mathfrak{x}(\mathfrak{k}) &= \varphi(\mathfrak{k}), \quad \mathfrak{k} \in [-\mathfrak{r},0],\\ \mathfrak{x}(0) &= \int_{0}^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}_{\ell})d\ell, \end{aligned} \tag{19} $$

then, the solution of the problem (19) is

$$\mathfrak{x}(\mathfrak{k})=\begin{cases}\varphi(\mathfrak{k}), \quad \mathfrak{k}\in[-\mathfrak{r},0],\\ \mathfrak{P}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}})+\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{E}(\ell,\mathfrak{x}_{\ell})d\ell+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{k})\\ +\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}p^*(\ell)d\ell, \quad \text{if } \mathfrak{k}\in[0,\mathfrak{k}_1],\\ \mathfrak{P}(\mathfrak{k},\mathfrak{x}(\mathfrak{k}))+\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{E}(\ell,\mathfrak{x}_{\ell})d\ell+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{k})\\ +\sum_{i=1}^{\mathfrak{z}}\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{k}_i)+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i}(\mathfrak{k}_i-\ell)^{\varsigma-1}\mathfrak{p}^*(\ell)d\ell\\ +\frac{\varsigma}{\Gamma(\varsigma)\mathfrak{M}(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\mathfrak{p}^*(\ell)d\ell+\sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)), \quad \text{if } \mathfrak{k}\in[\mathfrak{k}_{\mathfrak{z}},\mathfrak{k}_{\mathfrak{z}+1}].\end{cases} \tag{20}$$

Proof. Assume $\mathfrak{k}$ satisfies (19).

If $\mathfrak{k}\in[0,\mathfrak{k}_1]$,

$${}_0^{\mathcal{ABC}}D_{\mathfrak{k}}^{\varsigma}[\mathfrak{x}(\mathfrak{k})-\mathfrak{P}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}})]=\mathfrak{p}^*(\mathfrak{k}). \tag{21}$$

Lemma 6 implies

$$\begin{aligned}\mathfrak{x}(\mathfrak{k})-\mathfrak{P}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}})&=\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{E}(\ell,\mathfrak{x}_{\ell})d\ell+{}_0^{\mathcal{ABC}}\mathfrak{I}_{\mathfrak{k}}^{\varsigma}\mathfrak{p}^*(\mathfrak{k})\\ &=\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{E}(\ell,\mathfrak{x}_{\ell})d\ell+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{k})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}p^*(\ell)d\ell.\end{aligned} \tag{22}$$

If $\mathfrak{k}\in[\mathfrak{k}_1,\mathfrak{k}_2]$, then Lemma 6 implies

$$\begin{aligned}
\mathfrak{x}(\mathfrak{t})-\mathfrak{P}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}}) &= \mathfrak{x}(\mathfrak{t}_1^+)-\mathfrak{P}(\mathfrak{t}_1,\mathfrak{x}_{\mathfrak{t}_1})+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_1}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell \\
&= \Delta\mathfrak{x}|_{\mathfrak{t}=\mathfrak{t}_1}+\mathfrak{x}(\mathfrak{t}_1^-)-\mathfrak{P}(\mathfrak{t}_1,\mathfrak{x}_{\mathfrak{t}_1})+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_1}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell \\
&= \mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-})+\left[\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}_\ell)\mathrm{d}\ell+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t}_1)\right. \\
&\quad \left.+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\ell)^{\varsigma-1}p^*(\ell)d\ell\right]+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_1}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell \\
&= \mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-})+\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}_\ell)\mathrm{d}\ell+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t}_1) \\
&\quad +\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_1}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell.
\end{aligned} \tag{23}$$

If $\mathfrak{t}\in[\mathfrak{t}_1,\mathfrak{t}_2]$, then Lemma 6 implies

$$\begin{aligned}
\mathfrak{x}(\mathfrak{t})-\mathfrak{P}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}}) &= \mathfrak{x}(\mathfrak{t}_2^+)-\mathfrak{P}(\mathfrak{t}_2,\mathfrak{x}_{\mathfrak{t}_2})+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_2}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell \\
&= \Delta\mathfrak{x}|_{\mathfrak{t}=\mathfrak{t}_2}+\mathfrak{x}(\mathfrak{t}_2^-)-\mathfrak{P}(\mathfrak{t}_2,\mathfrak{x}_{\mathfrak{t}_2})+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_2}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell \\
&= \mathfrak{I}_2(\mathfrak{x}_{\mathfrak{t}_2^-})+\left[\int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}_\ell)\mathrm{d}\ell+\mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-})+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t}_2)+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t}_1)\right. \\
&\quad \left.+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell++\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell\right] \\
&\quad +\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t})+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_2}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell \\
&= \int_0^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell,\mathfrak{x}_\ell)d\ell+\left[\mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-})+\mathfrak{I}_2(\mathfrak{x}_{\mathfrak{t}_2^-})\right]+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{t}) \\
&\quad +\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}[\mathfrak{p}^*(\mathfrak{t}_1)+\mathfrak{p}^*(\mathfrak{t}_2)]+\left[\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell\right. \\
&\quad \left.+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell\right]+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{t}_2}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\varsigma-1}p^*(\ell)\mathrm{d}\ell.
\end{aligned} \tag{24}$$

Repeating this process in these ways, the solution $\mathfrak{x}(\mathfrak{t})$, for $\mathfrak{t}\in[\mathfrak{t}_{\mathfrak{z}},\mathfrak{t}_{\mathfrak{z}+1}]$, where $\mathfrak{z}=1,\ldots,n$ can be written as

$$\mathfrak{x}(\mathfrak{k}) = \mathfrak{P}(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{G}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_i)$$
$$+ \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell + \frac{\varsigma}{\Gamma(\varsigma)\mathfrak{M}(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k}-\ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell \tag{25}$$
$$+ \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)).$$

3. Main Results

The following hypotheses are needed to prove the main results.

(A1) For the constants $\mathfrak{K}_{\mathfrak{u}} > 0$, we have for any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{Z}$

$$|\mathfrak{P}(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) - \mathfrak{P}(\mathfrak{k}, \mathfrak{y}(\mathfrak{k}))| \le \mathfrak{K}_{\mathfrak{u}} \|\mathfrak{x}(\mathfrak{k}) - \mathfrak{y}(\mathfrak{k})\|_{\mathfrak{PC}}, \tag{26}$$

(A2) For constants $\mathfrak{K}_{\mathfrak{v}}$, we have for any $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{y}_1, \mathfrak{y}_2 \in \mathscr{Z}$

$$\left|\mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_1(\mathfrak{k}), \mathfrak{x}_2(\mathfrak{k})) - \mathfrak{Q}(\mathfrak{k}, \mathfrak{y}_1(\mathfrak{k}), \mathfrak{y}_2(\mathfrak{k}))\right| \le \mathfrak{K}_{\mathfrak{v}} \left\|\mathfrak{x}_1(\mathfrak{k}) - \mathfrak{y}_1(\mathfrak{k})\right\|_{\mathfrak{PC}} + \mathfrak{L}_{\mathfrak{v}} \left|\mathfrak{x}_2(\mathfrak{k}) - \mathfrak{y}_2(\mathfrak{k})\right|. \tag{27}$$

(A3) For the constants $\mathfrak{K}_{\mathfrak{i}} > 0$, we have for any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{Z}$

$$\left|\mathfrak{I}_{\mathfrak{z}}\mathfrak{x}(\mathfrak{k}) - \mathfrak{I}_k \mathfrak{y}(\mathfrak{k})\right| \le \mathfrak{K}_{\mathfrak{i}} \left\|\mathfrak{x}_1(\mathfrak{k}) - \mathfrak{y}_1(\mathfrak{k})\right\|_{\mathfrak{PC}}. \tag{28}$$

(A4) For the constants $\mathfrak{K}_{\mathfrak{g}} > 0$, we have for any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{Z}$

$$|\mathfrak{G}(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) - \mathfrak{G}(\mathfrak{k}, \mathfrak{y}(\mathfrak{k}))| \le \mathfrak{K}_{\mathfrak{g}} \|\mathfrak{x}(\mathfrak{k}) - \mathfrak{y}(\mathfrak{k})\|_{\mathfrak{PC}}. \tag{29}$$

(A5) There exists $p, q, r \in \mathfrak{C}(\mathfrak{J}, \grave{\mathfrak{R}}_+)$ with $r^* = \sup_{\mathfrak{k}\in\mathfrak{J}} r(\mathfrak{k}) < 1$ such that

$$|\mathfrak{Q}(\mathfrak{k}, \mathfrak{x}, \mathfrak{y})| \le p(\mathfrak{k}) + q(\mathfrak{k})\|\mathfrak{x}\|_{\mathfrak{PC}} + r(\mathfrak{k})|\mathfrak{y}|. \tag{30}$$

For $\mathfrak{k} \in \mathfrak{J}$, $\mathfrak{x} \in \mathfrak{PC}([-\mathfrak{r}, 0], \grave{\mathfrak{R}})$ and $\mathfrak{y} \in \grave{\mathfrak{R}}$.

(A6) There exist constants $N^*, M^* > 0$ such that

$$\left|\mathfrak{I}_{\mathfrak{z}}(\mathfrak{x})\right| \le N^* \|\mathfrak{x}\|_{\mathfrak{PC}} + M^*. \tag{31}$$

For each $\mathfrak{x} \in \mathfrak{PC}([-\mathfrak{r}, 0], \grave{\mathfrak{R}})$, $\mathfrak{z} = 1, \ldots, n$.

(A7) $\mathfrak{P}$ is a completely continuous function, and for each bounded set B_{τ^*} in Ξ, the set $\mathfrak{k} \longrightarrow \mathfrak{P}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}})$: $\mathfrak{x} \in B_{\tau^*}$ is equicontinuous in $\mathfrak{PC}(\mathfrak{J}, \grave{\mathfrak{R}})$ and there exist two constants $d_1 > 0$, $d_2 > 0$ with $nN^* + d_1 < 1$ such that

$$|\mathfrak{P}(\mathfrak{k}, \mathfrak{x})| \le d_1 \|\mathfrak{x}\|_{\mathfrak{PC}} + d_2, \tag{32}$$

$\mathfrak{k} \in \mathfrak{J}$, $\mathfrak{x} \in \mathfrak{PC}([-\mathfrak{r}, 0], \grave{\mathfrak{R}})$.

Theorem 8. *Under hypotheses (A1)–(A4), the considered problem (4) has a unique solution if*

$$\Theta = \left\{ \mathfrak{K}_{\mathfrak{u}} + \frac{\mathfrak{T}^{\varsigma}}{\Gamma(\varsigma+1)} \mathfrak{K}_{\mathfrak{g}} + \left[\frac{1-\varsigma}{\mathfrak{M}(\varsigma)} + \frac{\mathfrak{T}^{\varsigma}}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \right] (n+1) \frac{\mathfrak{K}_{\mathfrak{v}}}{1 - \mathfrak{L}_{\mathfrak{v}}} + n\mathfrak{K}_{\mathfrak{i}} \right\} \le 1. \tag{33}$$

Proof. Consider the operator $\aleph: \mathfrak{Z} \longrightarrow \mathfrak{Z}$ by

$$\aleph\mathfrak{x}(\mathfrak{k}) = \begin{cases} \varphi(\mathfrak{k}); \quad \mathfrak{k} \in [-\mathfrak{r}, 0], \\ \mathfrak{P}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{E}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}), \\ + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_i) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell, \\ + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k} - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)), \quad \mathfrak{k} \in \mathfrak{J}, \end{cases} \tag{34}$$

where $\mathfrak{p}^*(\mathfrak{k}) \in \mathfrak{C}(\mathfrak{J}, \grave{\mathfrak{R}})$ be such that

$$\mathfrak{p}^*(\mathfrak{k}) = \mathfrak{Q}\left(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}, {}_0^{\mathscr{ABC}}D_{\mathfrak{k}}^{\varsigma}\right). \tag{35}$$

If $\mathfrak{x}, \mathfrak{y} \in \mathfrak{Z}$, If $\mathfrak{k} \in [-\mathfrak{r}, 0]$, then

$$\|\aleph(\mathfrak{x}) - \aleph(\mathfrak{y})\| = 0. \tag{36}$$

For any $\mathfrak{k} \in \mathfrak{J}$ and from (33), we have

$$\begin{aligned} \|\aleph(\mathfrak{x}) - \aleph(\mathfrak{y})\|_{\mathfrak{Z}} &= \max_{\mathfrak{k}\in\mathfrak{J}} |\aleph\mathfrak{x}(\mathfrak{k}) - \aleph\mathfrak{y}(\mathfrak{k})| \\ &\le \max_{\mathfrak{k}\in\mathfrak{J}} \Bigg| \mathfrak{P}(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{E}(\ell, \mathfrak{x}(\ell)) d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}) \\ &\quad + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_i) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell \\ &\quad + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k} - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)) \\ &\quad - \Bigg\{ \mathfrak{P}(\mathfrak{k}, \mathfrak{y}(\mathfrak{k})) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{E}(\ell, \mathfrak{y}(\ell)) d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \overline{\mathfrak{p}}^*(\mathfrak{k}) \\ &\quad + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \overline{\mathfrak{p}}^*(\mathfrak{k}_i) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} \overline{\mathfrak{p}}^*(\ell) d\ell \\ &\quad + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k} - \ell)^{\varsigma-1} \overline{\mathfrak{p}}^*(\ell) d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{y}(\mathfrak{k}_i^-)) \Bigg\} \Bigg| \\ &\le \max_{\mathfrak{k}\in\mathfrak{J}} |\mathfrak{P}(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) - \mathfrak{P}(\mathfrak{k}, \mathfrak{y}(\mathfrak{k}))| + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} |\mathfrak{E}(\ell, \mathfrak{x}(\ell)) - \mathfrak{E}(\ell, \mathfrak{y}(\ell))| d\ell \end{aligned}$$

$$+\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\left|\mathfrak{p}^{*}(\mathfrak{k})-\overline{\mathfrak{p}}^{*}(\mathfrak{k})\right|+\sum_{i=1}^{\mathfrak{z}}\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\left|\mathfrak{p}^{*}(\mathfrak{k}_i)-\overline{\mathfrak{p}}^{*}(\mathfrak{k}_i)\right|$$
$$+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i}(\mathfrak{k}_i-\ell)^{\varsigma-1}\left|\mathfrak{p}^{*}(\ell)-\overline{\mathfrak{p}}^{*}(\ell)\right|d\ell \tag{37}$$
$$+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\left|\mathfrak{p}^{*}(\ell)-\overline{\mathfrak{p}}^{*}(\ell)\right|d\ell+\sum_{i=1}^{\mathfrak{z}}\left|\mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^{-}))-\mathfrak{I}_i(\mathfrak{y}(\mathfrak{k}_i^{-}))\right|,$$

where $\mathfrak{p}^{*},\overline{\mathfrak{p}}^{*}\in\mathfrak{C}(\mathfrak{J},\grave{\mathfrak{R}})$ such that

$$\begin{aligned}\mathfrak{p}^{*}(\mathfrak{k})&=\mathfrak{Q}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}},\mathfrak{p}^{*}(\mathfrak{k})),\\ \overline{\mathfrak{p}}^{*}(\mathfrak{k})&=\mathfrak{Q}(\mathfrak{k},\mathfrak{y}_{\mathfrak{k}},\overline{\mathfrak{p}}^{*}(\mathfrak{k})).\end{aligned} \tag{38}$$

By $(A2)$, we have

$$\begin{aligned}\left|\mathfrak{p}^{*}(\mathfrak{k})-\overline{\mathfrak{p}}^{*}(\mathfrak{k})\right|&=\left|\mathfrak{Q}(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}},\mathfrak{p}^{*}(\mathfrak{k}))-\mathfrak{Q}(\mathfrak{k},\mathfrak{y}_{\mathfrak{k}},\overline{\mathfrak{p}}^{*}(\mathfrak{k}))\right|\\ &\le\mathfrak{K}_{\mathfrak{v}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}+\mathfrak{L}_{\mathfrak{v}}\left|\mathfrak{p}^{*}(\mathfrak{k})-\overline{\mathfrak{p}}^{*}(\mathfrak{k})\right|\\ \left|\mathfrak{p}^{*}(\mathfrak{k})-\overline{\mathfrak{p}}^{*}(\mathfrak{k})\right|&\le\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}},\\ \left\|\aleph(\mathfrak{x})-\aleph(\mathfrak{y})\right\|_{\mathfrak{Z}}&\le\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}+\frac{\mathfrak{T}^{\varsigma}}{\Gamma(\varsigma+1)}\mathfrak{K}_{\mathfrak{s}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}+\frac{1-\varsigma}{\mathfrak{M}(\varsigma)}\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}\\ &\quad+n\frac{1-\varsigma}{\mathfrak{M}(\varsigma)}\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}+\frac{\varsigma\mathfrak{T}^{\varsigma}}{\mathfrak{M}(\varsigma)\Gamma(\varsigma+1)}n\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}\\ &\quad+\frac{\varsigma\mathfrak{T}^{\varsigma}}{\mathfrak{M}(\varsigma)\Gamma(\varsigma+1)}\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}+n\mathfrak{K}_{\mathfrak{i}}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}\\ &\le\left\{\mathfrak{K}_{\mathfrak{u}}+\frac{\mathfrak{T}^{\varsigma}}{\Gamma(\varsigma+1)}\mathfrak{K}_{\mathfrak{s}}+\left[\frac{1-\varsigma}{\mathfrak{M}(\varsigma)}+\frac{\mathfrak{T}^{\varsigma}}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\right](n+1)\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}+n\mathfrak{K}_{\mathfrak{i}}\right\}\left\|\mathfrak{x}_{\mathfrak{k}}-\mathfrak{y}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}.\end{aligned} \tag{39}$$

Hence, we obtain

$$\|\aleph(\mathfrak{x}) - \aleph(\mathfrak{y})\|_3 \leq \Theta\|\mathfrak{x} - \mathfrak{y}\|_3. \tag{40}$$

Therefore, $\aleph$ is a contraction and (4) has a unique solution. □

Theorem 9. *Assume the hypotheses (A1)–(A7) hold, then problem (4) has at least one solution.*

Proof. We consider the operator $\aleph_1: \Xi \longrightarrow \Xi$ defined by

$$\aleph_1 = \begin{cases} \varphi(\mathfrak{k}); \quad \mathfrak{k} \in [-\mathfrak{r}, 0], \\ \displaystyle\int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{G}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_i) \\ \displaystyle + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell \\ \displaystyle + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k} - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)), \quad \mathfrak{k} \in \mathfrak{J}. \end{cases} \tag{41}$$

The operator $\aleph$ defined in (33) can be written as

$$\aleph = \mathfrak{P}(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) + \aleph_1, \tag{42}$$

for each $\mathfrak{k} \in \mathfrak{J}$.

We shall use Schaefer's fixed point theorem to prove that $\aleph$ has a fixed point. So, we have to show that $\aleph$ is completely continuous. Since $\mathfrak{P}$ is completely continuous by (A7), we shall show that $\aleph_1$ is completely continuous. □

Step 10. $\aleph_1$ is continuous. Let the sequence $\{\mathfrak{x}_m\}$ such that $\mathfrak{x}_m \longrightarrow \mathfrak{x}$ in Ξ.

If $\mathfrak{k} \in [-\mathfrak{r}, 0]$, then

$$|\aleph_1(\mathfrak{x}) - \aleph_1(\mathfrak{y})| = 0. \tag{43}$$

For $\mathfrak{k} \in \mathfrak{J}$, we have

$$\begin{aligned} |\aleph_1(\mathfrak{x}) - \aleph_1(\mathfrak{y})| \leq & \int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} |\mathfrak{G}(\ell, \mathfrak{x}_{m\ell}) - \mathfrak{G}(\ell, \mathfrak{x}_\ell)| \mathrm{d}\ell + + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} |\mathfrak{p}_m^*(\mathfrak{k}) - \mathfrak{p}^*(\mathfrak{k})| \\ & + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} |\mathfrak{p}_m^*(\mathfrak{k}_i) - \mathfrak{p}^*(\mathfrak{k}_i)| + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} |\mathfrak{p}_m^*(\ell) - \mathfrak{p}^*(\ell)| \mathrm{d}\ell \\ & + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k} - \ell)^{\varsigma-1} |\mathfrak{p}_n^*(\ell) - \mathfrak{p}^*(\ell) d\ell| + \sum_{i=1}^{\mathfrak{z}} |\mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)) - \mathfrak{I}_i(\mathfrak{y}(\mathfrak{k}_i^-))|, \end{aligned} \tag{44}$$

where $\mathfrak{p}^*, \overline{\mathfrak{p}}^* \in \mathfrak{C}(\mathfrak{J}, \grave{\mathfrak{R}})$ such that

$$\begin{aligned}\mathfrak{p}_m^*(\mathfrak{k}) &= \mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{n\mathfrak{k}}, \mathfrak{p}_n^*(\mathfrak{k})),\\ \mathfrak{p}^*(\mathfrak{k}) &= \mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}, \mathfrak{p}^*(\mathfrak{k})).\end{aligned} \tag{45}$$

By $(A2)$, we have

$$\begin{aligned}\left|\mathfrak{p}_m^*(\mathfrak{k}) - \mathfrak{p}^*(\mathfrak{k})\right| &= \left|\mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{m\mathfrak{k}}, \mathfrak{p}_m^*(\mathfrak{k})) - \mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}, \mathfrak{p}^*(\mathfrak{k}))\right|\\ &\leq \mathfrak{K}_{\mathfrak{v}}\left\|\mathfrak{x}_{m\mathfrak{k}} - \mathfrak{x}_{\mathfrak{k}}\right\|_{\mathfrak{PC}} + \mathfrak{L}_{\mathfrak{v}}\left|\mathfrak{p}_m^*(\mathfrak{k}) - \mathfrak{p}^*(\mathfrak{k})\right|,\\ \left|\mathfrak{p}_m^*(\mathfrak{k}) - \mathfrak{p}^*(\mathfrak{k})\right| &\leq \frac{\mathfrak{K}_{\mathfrak{v}}}{1 - \mathfrak{L}_{\mathfrak{v}}}\left\|\mathfrak{x}_{m\mathfrak{k}} - \mathfrak{x}_{\mathfrak{k}}\right\|_{\mathfrak{PC}}.\end{aligned} \tag{46}$$

Since $\mathfrak{x}_m \longrightarrow \mathfrak{x}$, then we get $\mathfrak{p}_m^*(\mathfrak{k}) \longrightarrow \mathfrak{p}^*(\mathfrak{k})$ as $m \longrightarrow \infty$ for each $\mathfrak{k} \in \mathfrak{J}$.

Let $\vartheta > 0$ and for each $\mathfrak{k} \in \mathfrak{J}$, we have $\left|\mathfrak{p}_m^*(\mathfrak{k})\right| \leq \vartheta$ and $\left|\mathfrak{p}^*(\mathfrak{k})\right| \leq \vartheta$.

Then, we have

$$\begin{aligned}(\mathfrak{k} - s)^{\alpha-1}\left|\mathfrak{p}_m^*(s) - \mathfrak{p}^*(\mathfrak{k})\right| &\leq (\mathfrak{k} - s)^{\alpha-1}\left[\left|\mathfrak{p}_m^*(s)\right| + \left|\mathfrak{p}^*(\mathfrak{k})\right|\right]\\ &\leq 2\vartheta(\mathfrak{k} - s)^{\alpha-1},\\ (\mathfrak{k}_k - s)^{\alpha-1}\left|\mathfrak{p}_m^*(s) - \mathfrak{p}^*(\mathfrak{k})\right| &\leq (\mathfrak{k}_k - s)^{\alpha-1}\left[\left|\mathfrak{p}_m^*(s)\right| + \left|\mathfrak{p}^*(\mathfrak{k})\right|\right]\\ &\leq 2\vartheta(\mathfrak{k}_k - s)^{\alpha-1}.\end{aligned} \tag{47}$$

For each $\mathfrak{k} \in \mathfrak{J}$, the functions $s \longrightarrow 2\vartheta(\mathfrak{k} - s)^{\alpha-1}$ and $s \longrightarrow 2\vartheta(\mathfrak{k}_k - s)^{\alpha-1}$ are integrable on $[0, \mathfrak{k}]$, and then the Lebesgue Dominated Convergence Theorem and (44) imply that

$$\left|\aleph_1(\mathfrak{x}_m)(\mathfrak{k}) - \aleph_1(\mathfrak{x})(\mathfrak{k})\right| \longrightarrow \text{as}\, m \longrightarrow \infty. \tag{48}$$

Consequently, $\aleph_1$ is continuous.

Step 11. $\aleph_1$ maps bounded sets into bounded sets in Ξ. Indeed, it is enough to show that for any $\tau^* > 0$, there exists a positive constant $\wp$ such that for each $\mathfrak{x} \in B_{\tau^*} = \{\mathfrak{x} \in \Xi\colon \|\mathfrak{x}\|_{\Xi} \leq \tau^*\}$, we have $\left\|\aleph_1(\mathfrak{x})\right\|_{\Xi} \leq \wp$.

For each $\mathfrak{k} \in \mathfrak{J}$, we have

$$\begin{aligned}\aleph_1 = &\int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}(\ell, \mathfrak{x}_\ell)d\ell + \frac{(1 - \varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{k}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1 - \varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^*(\mathfrak{k}_i)\\ &+ \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i}(\mathfrak{k}_i - \ell)^{\varsigma-1}\mathfrak{p}^*(\ell)d\ell + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k} - \ell)^{\varsigma-1}\mathfrak{p}^*(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)),\end{aligned} \tag{49}$$

where $\mathfrak{p}^* \in \mathfrak{C}(\mathfrak{J}, \grave{\mathfrak{R}})$ such that

$$\mathfrak{p}^*(\mathfrak{k}) = \mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}, \mathfrak{p}^*(\mathfrak{k})). \tag{50}$$

By $(A5)$ and for each $\mathfrak{k} \in \mathfrak{J}$, we have

$$\begin{aligned} |\mathfrak{p}^*(\mathfrak{k})| &= |\mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}, \mathfrak{p}^*(\mathfrak{k}))| \\ &\le p(\mathfrak{k}) + q(\mathfrak{k})\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}} + r(\mathfrak{k})|\mathfrak{p}^*(\mathfrak{k})| \\ &\le p(\mathfrak{k}) + q(\mathfrak{k})\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}_1} + r(\mathfrak{k})|\mathfrak{p}^*(\mathfrak{k})| \\ &\le p(\mathfrak{k}) + q(\mathfrak{k})\tau^* + r(\mathfrak{k})|\mathfrak{p}^*(\mathfrak{k})| \\ &\le p^* + q^*\tau^* + r^*|\mathfrak{p}^*(\mathfrak{k})|, \end{aligned} \tag{51}$$

where $p^* = \sup_{\mathfrak{k}\in\mathfrak{J}} p(\mathfrak{k})$ and $q^* = \sup_{\mathfrak{k}\in\mathfrak{J}} q(\mathfrak{k})$. Then,

$$|\mathfrak{p}^*(\mathfrak{k})| \le \frac{p^* + q^*\tau^*}{1 - r^*} := M. \tag{52}$$

Thus, (49) implies

$$\begin{aligned} |\aleph_1(\mathfrak{x})(\mathfrak{k})| &\le \Bigg| \int_0^{\mathfrak{T}} \frac{(\mathfrak{T} - \ell)^{\varsigma-1}}{\Gamma(\varsigma)} \mathfrak{G}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_i) \\ &\quad + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_i - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}} (\mathfrak{k} - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{k}_i^-)) \Bigg| \\ &\le \frac{\mathfrak{T}^\varsigma}{\Gamma(\varsigma+1)} (e_1\|\mathfrak{x}\| + e_2) + M\frac{1-\varsigma}{\mathfrak{M}(\varsigma)} + Mn\frac{1-\varsigma}{\mathfrak{M}(\varsigma)} + \frac{M\mathfrak{T}^\varsigma n}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} + \frac{M\mathfrak{T}^\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} + \sum_{\mathfrak{z}=1}^{n} \left(N^* \left\|\mathfrak{x}_{\mathfrak{k}_k^-}\right\| + M^* \right) \\ &\le \frac{\mathfrak{T}^\varsigma}{\Gamma(\varsigma+1)} (e_1\vartheta^* + e_2) + M\frac{(n+1)(1-\varsigma)}{\mathfrak{M}(\varsigma)} + \frac{M\mathfrak{T}^\varsigma(n+1)}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} + n(N^*\vartheta^* + M^*) := R. \end{aligned} \tag{53}$$

And if $\mathfrak{k} \in [-\mathfrak{r}, 0]$, then

$$|\aleph_1(\mathfrak{x})(\mathfrak{k})| \le \|\varphi\|_{\mathfrak{PC}}, \tag{54}$$

thus

$$\|\aleph_1(\mathfrak{x})\|_{\Xi} \le \max\left\{R, \|\varphi\|_{\mathfrak{PC}}\right\} := \wp. \tag{55}$$

Step 12. $\aleph_1$ maps bounded sets into equicontinuous sets of Ξ.

Let $\mathfrak{k}_{\mathfrak{z}-1}, \mathfrak{k}_{\mathfrak{z}} \in (0, \mathfrak{T}), \mathfrak{k}_{\mathfrak{z}-1} < \mathfrak{k}_{\mathfrak{z}}, B_{\tau^*}$ be a bounded set of Ξ as in Step 11, and let $\mathfrak{x} \in B_{\tau^*}$. Then,

$$\begin{aligned} &\left|\aleph_1(\mathfrak{x})(\mathfrak{k}_{\mathfrak{z}}) - \aleph_1(\mathfrak{x})(\mathfrak{k}_{\mathfrak{z}-1})\right| \\ &= \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_{\mathfrak{z}}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)} \mathfrak{p}^*(\mathfrak{k}_{\mathfrak{z}}) + \frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i} (\mathfrak{k}_{\mathfrak{z}} - \ell)^{\varsigma-1} \mathfrak{p}^*(\ell) d\ell \end{aligned}$$

$$
\begin{aligned}
&+\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}\left(\mathfrak{k}_{\mathfrak{z}}-\ell\right)^{\varsigma-1}\mathfrak{p}^{*}(\ell)d\ell+\sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_{i}\left(\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{-}\right)\right)\\
&-\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^{*}\left(\mathfrak{k}_{\mathfrak{z}-1}\right)-\sum_{i=1}^{\mathfrak{z}}\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^{*}\left(\mathfrak{k}_{\mathfrak{z}-1}\right)-\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_{i}}\left(\mathfrak{k}_{\mathfrak{z}-1}-\ell\right)^{\varsigma-1}\mathfrak{p}^{*}(\ell)d\ell\\
&-\frac{\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}\left(\mathfrak{k}_{\mathfrak{z}-1}-\ell\right)^{\varsigma-1}\mathfrak{p}^{*}(\ell)d\ell-\sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_{i}\left(\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}-1}^{-}\right)\right)\Bigg| \qquad (56)\\
&\leq\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\left|\mathfrak{p}^{*}\left(\mathfrak{k}_{\mathfrak{z}}\right)-\mathfrak{p}^{*}\left(\mathfrak{k}_{\mathfrak{z}-1}\right)\right|+\sum_{i=1}^{\mathfrak{z}}\frac{(1-\varsigma)}{\mathfrak{M}(\varsigma)}\left|\mathfrak{p}^{*}\left(\mathfrak{k}_{\mathfrak{z}}\right)-\mathfrak{p}^{*}\left(\mathfrak{k}_{\mathfrak{z}-1}\right)\right|+\frac{(n+1)}{\mathfrak{M}((\varsigma)\Gamma(\varsigma))}\left(\mathfrak{k}_{\mathfrak{z}}^{\varsigma}-\mathfrak{k}_{\mathfrak{z}-1}^{\varsigma}\right)\\
&+\sum_{i=1}^{\mathfrak{z}}\left|\mathfrak{I}_{i}\left(\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}}^{-}\right)\right)-\mathfrak{I}_{i}\left(\mathfrak{x}\left(\mathfrak{k}_{\mathfrak{z}-1}^{-}\right)\right)\right|.
\end{aligned}
$$

As $\mathfrak{k}_{\mathfrak{z}}\longrightarrow\mathfrak{k}_{\mathfrak{z}-1}$, the right hand side of the above inequality tents to 0. Hence, $\aleph_1$ is completely continuous.

Step 13. A priori bounds. To prove that the set

$$
E=\left\{\mathfrak{x}\in\Xi:\ \mathfrak{x}=\lambda\aleph_{1}(\mathfrak{x})\quad\text{for some}\,\lambda\in(0,1)\right\}, \qquad (57)
$$

is bounded. Let $\mathfrak{x}\in E$. Then, $\mathfrak{x}=\lambda\aleph_1(\mathfrak{x})$ for some $\lambda\in(0,1)$. Thus, for each $\mathfrak{k}\in\mathfrak{J}$, we have

$$
\begin{aligned}
\mathfrak{x}&=\lambda\left|\mathfrak{P}\left(\mathfrak{k},\mathfrak{x}_{\mathfrak{k}}\right)\right|+\lambda\int_{0}^{\mathfrak{T}}\frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\mathfrak{G}\left(\ell,\mathfrak{x}_{\ell}\right)d\ell+\frac{\lambda(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^{*}(\mathfrak{k})+\sum_{i=1}^{\mathfrak{z}}\frac{\lambda(1-\varsigma)}{\mathfrak{M}(\varsigma)}\mathfrak{p}^{*}\left(\mathfrak{k}_{i}\right)\\
&+\frac{\lambda\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_{i}}\left(\mathfrak{k}_{i}-\ell\right)^{\varsigma-1}\mathfrak{p}^{*}(\ell)d\ell+\frac{\lambda\varsigma}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\mathfrak{p}^{*}(\ell)d\ell+\lambda\sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_{i}\left(\mathfrak{x}\left(\mathfrak{k}_{i}^{-}\right)\right).
\end{aligned} \qquad (58)
$$

And for each $\mathfrak{k}\in\mathfrak{J}$ and by $(A5)$, we have

$$
\begin{aligned}
|\mathfrak{p}^*(\mathfrak{k})| &= |\mathfrak{Q}(\mathfrak{k}, \mathfrak{x}_{\mathfrak{k}}, \mathfrak{p}^*(\mathfrak{k}))| \\
&\le p(\mathfrak{k}) + q(\mathfrak{k})\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}} + r(\mathfrak{k})|\mathfrak{p}^*(\mathfrak{k})| \\
&\le p(\mathfrak{k}) + q(\mathfrak{k})\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}} + r(\mathfrak{k})|\mathfrak{p}^*(\mathfrak{k})| \\
&\le p^* + q^*\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}} + r^*|\mathfrak{p}^*(\mathfrak{k})| \\
|\mathfrak{p}^*(\mathfrak{k})| &\le \frac{1}{1-r^*}\left(p^* + q^*\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}}\right)
\end{aligned} \tag{59}
$$

For each $\mathfrak{k} \in \mathfrak{J}$ and by (58), (A6) and (A7), and we have

$$
\begin{aligned}
|\mathfrak{x}| &\le d_1\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}} + d_2 + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\left(e_1\|\mathfrak{x}_{\mathfrak{k}}\|_{\mathfrak{PC}} + e_2\right)d\ell + \frac{1-\varsigma}{(1-r^*)(\mathfrak{M}(\varsigma))}\left(p^* + q^*\|\mathfrak{x}\|_{\mathfrak{PC}}\right) \\
&+ \frac{n(1-\varsigma)}{(1-r^*)(\mathfrak{M}(\varsigma))}\left(p^* + q^*\|\mathfrak{x}\|_{\mathfrak{PC}}\right) + \frac{\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i}(\mathfrak{k}_i-\ell)^{\varsigma-1}\left(p^* + q^*\|\mathfrak{x}\|_{\mathfrak{PC}}\right)d\ell \\
&+ \frac{\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}\left(p^* + q^*\|\mathfrak{x}\|_{\mathfrak{PC}}\right)d\ell + n\left(N^*\left\|\mathfrak{x}_{\mathfrak{k}_{\mathfrak{z}}^-}\right\|_{\mathfrak{PC}} + M^*\right).
\end{aligned} \tag{60}
$$

Define v by

$$
v(\mathfrak{k}) = \sup\{|\mathfrak{x}(s)|: \quad s \in [-\mathfrak{r}, \mathfrak{k}]\}, \mathfrak{k} \in [0, \mathfrak{T}]. \tag{61}
$$

Then, there exists $\mathfrak{k}^* \in [-\mathfrak{r}, \mathfrak{T}]$ such that $v(\mathfrak{k}) = |\mathfrak{x}(\mathfrak{k}^*)|$. If $\mathfrak{k} \in [0, \mathfrak{T}]$, then by the previous inequality, we have for $\mathfrak{k} \in \mathfrak{J}$

$$
\begin{aligned}
v(\mathfrak{k}) &\le d_1 v(\mathfrak{k}) + d_2 + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}(e_1 v(\ell) + e_2)\mathrm{d}\ell + \frac{(n+1)(1-\varsigma)}{(1-r^*)(\mathfrak{M}(\varsigma))}(p^* + q^* v(\mathfrak{k})) \\
&+ \frac{\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i}(\mathfrak{k}-\ell)^{\varsigma-1}(p^* + q^* v(\ell))\mathrm{d}\ell \\
&+ \frac{\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}(p^* + q^* v(\ell))\mathrm{d}\ell + n(N^* v(\mathfrak{k}) + M^*) \\
&\le \left(d_1 + \frac{(n+1)(1-\varsigma)}{(1-r^*)\mathfrak{M}(\varsigma)}q^* + nN^*\right)v(\mathfrak{k}) + \left(d_2 + \frac{(n+1)(1-\varsigma)}{(1-r^*)\mathfrak{M}(\varsigma)}p^* + nM^*\right) + \frac{np^*}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)(1-r^*)}\mathfrak{T}^{\varsigma}
\end{aligned}
$$

$$+\frac{\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{k}_{i-1}}^{\mathfrak{k}_i}(\mathfrak{k}_i-\ell)^{\varsigma-1}q^*v(\ell)d\ell+\frac{p^*}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)(1-r^*)}\mathfrak{T}^{\varsigma}$$
$$+\frac{\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\int_{\mathfrak{k}_{\mathfrak{z}}}^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}q^*v(\ell)\mathrm{d}\ell$$
$$\leq\frac{1}{(1-(d_1+(n+1)(1-\varsigma)/(1-r^*)\mathfrak{M}(\varsigma)q^*+nN^*))}\left(d_2+\frac{(n+1)(1-\varsigma)}{(1-r^*)\mathfrak{M}(\varsigma)}p^*+nM^*+\frac{(n+1)p^*}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)(1-r^*)}\mathfrak{T}^{\varsigma}\right)$$
$$+\frac{1}{(1-(d_1+nN^*+(n+1)(1-\varsigma)/(1-r^*)\mathfrak{M}(\varsigma)q^*))}\frac{q^*(n+1)\varsigma}{\mathfrak{M}(1-r^*)(\varsigma)\Gamma(\varsigma)}\int_0^{\mathfrak{k}}(\mathfrak{k}-\ell)^{\varsigma-1}v(\ell)\mathrm{d}\ell. \tag{62}$$

Applying Lemma 5, we get

$$v(\mathfrak{k})\leq\frac{1}{(1-(d_1+(n+1)(1-\varsigma)/(1-r^*)\mathfrak{M}(\varsigma)q^*+nN^*))}\left[d_2+\frac{(n+1)(1-\varsigma)}{(1-r^*)\mathfrak{M}(\varsigma)}p^*+nM^*+\frac{(n+1)p^*}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)(1-r^*)}\mathfrak{T}^{\varsigma}\right]$$
$$\times\left[1+\frac{\delta(n+1)\mathfrak{T}^{\varsigma}q^*\varsigma}{(1-r^*)\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\right], \tag{63}$$

where $\delta=\delta\varsigma$ is a constant. If $\mathfrak{k}^*\in[-\mathfrak{r},0]$, then $v(\mathfrak{k})=\|\phi\|_{\mathfrak{PC}}$, thus for any $\mathfrak{k}\in\mathfrak{J}$, $\|\mathfrak{x}\|_{\Xi}\leq v(\mathfrak{k})$, we get

$$\|\mathfrak{x}\|_{\Xi}\leq\max\left\{\|\ell\|_{\mathfrak{PC}},A\right\}. \tag{64}$$

Hence, the set E is bounded. By Theorem 4, the fixed point of $\aleph$ is a solution of problem (4).

4. Example

Consider the following problem:

$$\begin{cases}{}_0^{\mathscr{ABC}}D_{\mathfrak{k}}^{1/2}\left[\mathfrak{x}(\mathfrak{k})-\dfrac{\tan^{-1}|\mathfrak{x}(\mathfrak{k})|}{35}\right]=\dfrac{\mathfrak{k}^3+\sin|\mathfrak{x}(\mathfrak{k})|}{45}-\dfrac{e^{-\mathfrak{k}}}{11+e^{\mathfrak{k}}}\dfrac{\left|{}_0^{\mathscr{ABC}}D_{\mathfrak{k}}^{1/2}\mathfrak{x}(\mathfrak{k})\right|}{1+\left|{}_0^{\mathscr{ABC}}D_{\mathfrak{k}}^{1/2}\mathfrak{x}(\mathfrak{k})\right|}, \mathfrak{k}\in[0,1],\\ \Delta\mathfrak{x}(\mathfrak{k})=\dfrac{\mathfrak{x}(1/2^-)}{10+\mathfrak{x}(1/2^-)},\\ \mathfrak{x}(\mathfrak{k})=\varphi(\mathfrak{k}),\quad \mathfrak{k}\in[-\mathfrak{r},0],\mathfrak{r}>0\\ \mathfrak{x}(0)=\displaystyle\int_0^1\dfrac{(1-\ell)^{\varsigma-1}}{\Gamma(\varsigma)}\dfrac{1}{25}\exp(-\mathfrak{x}(\ell))d\ell,\end{cases} \tag{65}$$

where

$$\mathfrak{P}(\mathfrak{k},\mathfrak{x}(\mathfrak{k}))=\frac{\tan^{-1}|\mathfrak{x}(\mathfrak{k})|}{35},\mathfrak{Q}(\mathfrak{k},\mathfrak{x},\mathfrak{y})=\frac{\mathfrak{k}^3+\sin|\mathfrak{x}(\mathfrak{k})|}{45}-\frac{e^{-\mathfrak{k}}}{11+e^{\mathfrak{k}}}\frac{|\mathfrak{y}|}{1+|\mathfrak{y}|},\mathfrak{G}(\mathfrak{k},\mathfrak{x}(\mathfrak{k}))=\frac{1}{25}\exp(-\mathfrak{x}(\mathfrak{k})). \tag{66}$$

As $\mathfrak{T}=1$ and $\varsigma=1/2$, let $\mathfrak{x},\mathfrak{y}\in\mathfrak{Z}$

$$\begin{aligned}
|\mathfrak{P}(\mathfrak{k},\mathfrak{x}(\mathfrak{k}))-\mathfrak{P}(\mathfrak{k},\mathfrak{y}(\mathfrak{k}))| &= \left|\frac{\tan^{-1}|\mathfrak{x}(\mathfrak{k})|}{35}-\frac{\tan^{-1}|\mathfrak{y}(\mathfrak{k})|}{35}\right| \\
&\le \frac{1}{35}|\mathfrak{x}(\mathfrak{k})-\mathfrak{y}(\mathfrak{k})|, \\
\left|\mathfrak{Q}(\mathfrak{k},\mathfrak{x},\mathfrak{y})-\mathfrak{Q}(\mathfrak{k},\overline{\mathfrak{x}},\overline{\mathfrak{y}})\right| &= \left|\frac{\mathfrak{k}^3+\sin|\mathfrak{x}(\mathfrak{k})|}{45}-\frac{\mathfrak{k}^3+\sin|\mathfrak{y}(\mathfrak{k})|}{45}\right| + \frac{e^{-\mathfrak{k}}}{11+e^{\mathfrak{k}}}\left|\frac{|\overline{\mathfrak{x}}|}{1+|\overline{\mathfrak{x}}|}\frac{|\overline{\mathfrak{y}}|}{1+|\overline{\mathfrak{y}}|}\right| \\
&\le \frac{19}{180}|\mathfrak{x}(\mathfrak{k})-\mathfrak{y}(\mathfrak{k})| + \frac{19}{180}\left|\overline{\mathfrak{x}}(\mathfrak{k})-\overline{\mathfrak{y}}(\mathfrak{k})\right| \\
\left|\mathfrak{I}_{\mathfrak{z}}\mathfrak{x}(\mathfrak{k})-\mathfrak{I}_{k}\mathfrak{y}(\mathfrak{k})\right| &= \left|\frac{\mathfrak{x}}{10+\mathfrak{x}}-\frac{\mathfrak{y}}{10+\mathfrak{x}}=\frac{10|\mathfrak{x}-\mathfrak{y}|}{(10+\mathfrak{x})(10+\mathfrak{y})}\right| \le \frac{1}{10}|\mathfrak{x}-\mathfrak{y}| \\
&\text{and} \\
|\mathfrak{S}(\mathfrak{k},\mathfrak{x}(\mathfrak{k}))-\mathfrak{S}(\mathfrak{k},\mathfrak{y}(\mathfrak{k}))) &= \left|\frac{1}{25}\exp(-\mathfrak{x}(\mathfrak{k}))-\frac{1}{25}\exp(-\mathfrak{y}(\mathfrak{k}))\right| \le \frac{1}{25}|\mathfrak{x}(\mathfrak{k})-\mathfrak{y}(\mathfrak{k})|.
\end{aligned} \tag{67}$$

Thus, we have $\mathfrak{K}_{\mathfrak{u}}=1/35, \mathfrak{K}_{\mathfrak{v}}=\mathfrak{L}_{\mathfrak{v}}=19/180, \mathfrak{K}_{\mathfrak{g}}=1/25$ and choose $n=1, \mathfrak{T}=1, \mathfrak{K}_{\mathfrak{i}}=1/10$.

Now, examine the conditions of the theorems (40) and attain

$$\Theta=\left\{\mathfrak{K}_{\mathfrak{u}}+\frac{\mathfrak{T}^{\varsigma}}{\Gamma(\varsigma+1)}\mathfrak{K}_{\mathfrak{g}}+\left[\frac{1-\varsigma}{\mathfrak{M}(\varsigma)}+\frac{T^{\varsigma}}{\mathfrak{M}(\varsigma)\Gamma(\varsigma)}\right](n+1)\frac{\mathfrak{K}_{\mathfrak{v}}}{1-\mathfrak{L}_{\mathfrak{v}}}+n\mathfrak{K}_{\mathfrak{i}}\right\}=0.29745<1. \tag{68}$$

Therefore, problem (65) has a unique solution.

5. Concluding Remarks

This work has successfully investigated the existence and uniqueness results for the fractional implicit differential equation and integral boundary conditions. These types of problems have numerous applications in mathematical modeling of human diseases and dynamical problems. Based on the Banach fixed point theorem and Schaefer's fixed point theorem, we have established the adequate results for at least one solution. The derived results have been justified by proving a suitable problem. In future, we extend our work with numerical results [28].

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Panjaiyan Karthikeyann, Sadhasivam Poornima, Kulandhaivel Karthikeyan, Chanon Promsakon, and Thanin Sitthiwirattham contributed to the study conception and design. Material preparation, data collection, and analysis were performed by Panjaiyan Karthikeyann, Kulandhaivel Karthikeyan, Chanon Promsakon, and Thanin Sitthiwirattham. The first draft of the manuscript was written by Sadhasivam Poornima, Chanon Promsakon. All authors commented on previous versions of the manuscript. All the authors read and approved the final manuscript. Panjaiyan Karthikeyann, Sadhasivam Poornima, Kulandhaivel Karthikeyan, Chanon Promsakon, and Thanin Sitthiwirattham confirm that all authors meet the ICMJE criteria.

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