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Research Article

Solving a Fractional Differential Equation via the Bipolar Parametric Metric Space

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In this paper, we propose the notion of the bipolar parametric metric space and prove fixed point theorems. The proved results generalize and extend some of the well-known results in the literature. An example and application to support our result is presented.

1. Introduction

Fixed point theory plays a vital role in applications of many fields of mathematics. Discovering FPs (fixed points) of generalized contraction maps has become an exciting field of study in the FP theory. Many researchers have recently released articles on FP theorems and applications in a variety of ways. One of the most recent topics in the FP theory is the presence of FPs in contraction maps in BPMSs (bipolar metric spaces), which can be thought of as generalizations of the Banach contraction principle. In 2016, Mutlu and Gurdal [1] have developed the concepts of BPMS, and they investigated certain basic FP and coupled FP results for covariant and contravariant maps under contractive conditions; see [1, 2]. In BPMSs, a lot of significant work has been done (see [3–9]). In 2021, Gaba et al. [10] proved FP theorems on BPMS. Mani et al. [11] developed the concept and proved coupled fixed point theorems in $\mathscr{C}^\star$ algebra-valued bipolar metric spaces (see [12–14]).

The notion of the parametric metric space was introduced in 2014. Rao et al. [15] presented parametric S-metric spaces and proved common FP theorems. In 2016, Krishnakumar and Nagaral [16] extended the Banach fixed point theorem to continuous mappings on complete parametric b-metric spaces. Tas and Ozgur [17] introduced parametric N_b-metric spaces, obtained some FP results, and proved a fixed-circle theorem on a parametric N_b-metric space as an application. Younis and Bahuguna [18] initiated the concept of controlled graphical metric type spaces, with integrate-controlled metric type spaces, extended b-metric type spaces, and graphical type spaces; also, finding a nonlinear model of a rocket's ascending motion as an application. In 2023, Younis et al. [19] developed FP theorems in graphical spaces to show a solution to fourth-order two-point boundary value problem expressing elastic beam deformations. Smarandache et al. [20] demonstrated the quadruple neutrosophic theory and its applications. Ahmad et al. [21] demonstrated FP solutions in graphical bipolar b-metric spaces, applying covariant and contravariant mapping contractions. In this paper, we present the notion of BPPMS (bipolar parametric metric space) and prove FP theorems on BPPMS.

2. Preliminaries

In this section, we present some basic definitions. Mutlu and Gurdal [1] proposed bipolar metric spaces and proved fixed point theorems.

Definition 1 (see [1]). Let $\mathscr{E}$ and Λ be nonempty sets and $\aleph$: $\mathscr{E}\times\Lambda \longrightarrow \mathbb{R}^{+}$ be a function s.t. (such that)

(a) If $\aleph(\sigma,\eta)=0$, then $\sigma=\eta$, for all $(\sigma,\eta)\in\mathscr{E}\times\Lambda$.
(b) If $\sigma=\eta$, then $\aleph(\sigma,\eta)=0$, for all $(\sigma,\eta)\in\mathscr{E}\times\Lambda$
(c) $\aleph(\sigma,\eta)=\aleph(\eta,\sigma)$, for all $(\sigma,\eta)\in\mathscr{E}\cap\Lambda$
(d) $\aleph(\sigma,\eta)\leq\aleph(\sigma,\omega)+\aleph(\alpha,\omega)+\aleph(\alpha,\eta)$, for all $\sigma,\alpha\in\mathscr{E}$ and $\omega,\eta\in\Lambda$.

The triplet $(\mathscr{E},\Lambda,\aleph)$ is called a BPMS.

Now, we introduce the notion of BPPMSs.

Definition 2. Let $\mathscr{E}$ and Λ be nonempty sets and $\aleph$: $\mathscr{E}\times\Lambda\times(0,\infty)\longrightarrow\mathbb{R}^{+}$ be a function s.t.

(a) If $\aleph(\sigma,\eta,c)=0$ for all $c>0$, then $\sigma=\eta$, for all $(\sigma,\eta)\in\mathscr{E}\times\Lambda$.
(b) If $\sigma=\eta$, then $\aleph(\sigma,\eta,c)=0$, for all $c>0$ and $(\sigma,\eta)\in\mathscr{E}\times\Lambda$
(c) $\aleph(\sigma,\eta,c)=\aleph(\eta,\sigma,c)$, for all $c>0$ and $(\sigma,\eta)\in\mathscr{E}\cap\Lambda$
(d) $\aleph(\sigma,\eta,c)\leq\aleph(\sigma,\omega,c)+\aleph(\alpha,\omega,c)+\aleph(\alpha,\eta,c)$, for all $c>0$, $\sigma,\alpha\in\mathscr{E}$, and $\omega,\eta\in\Lambda$.

The triplet $(\mathscr{E},\Lambda,\aleph)$ is called a BPPMS.

We introduce the notions of covariant mapping, contravariant mapping, convergent sequence, Cauchy sequence, and continuous and contraction mapping as follows.

Definition 3

(A1) Let $(\mathscr{E},\Lambda,\aleph)$ be a BPPMS. Then, the points of the sets $\mathscr{E}$, Λ, and $\mathscr{E}\cap\Lambda$ are named as left, right, and central points, respectively, and any sequence, that is consisted of only left (or right, or central) point is called a left (or right, or central) sequence on $(\mathscr{E},\Lambda,\aleph)$.

(A2) Let $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ and $(\mathscr{E}_2,\Lambda_2,\aleph_2)$ be BPPMSs and Ω: $\mathscr{E}_1\cup\Lambda_1\longrightarrow\mathscr{E}_2\cup\Lambda_2$ be a function. If $\Omega(\mathscr{E}_1)\subseteq\mathscr{E}_2$ and $\Omega(\Lambda_1)\subseteq\Lambda_2$, then Ω is called a covariant map, or a map from $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ to $(\mathscr{E}_2,\Lambda_2,\aleph_2)$, and this is written as $\Omega:(\mathscr{E}_1,\Lambda_1,\aleph_1)\rightrightarrows(\mathscr{E}_2,\Lambda_2,\aleph_2)$. If $\Omega:(\mathscr{E}_1,\Lambda_1,\aleph_1)\rightrightarrows(\Lambda_2,\mathscr{E}_2,\overline{\aleph}_2)$ is a map, then Ω is called a contravariant map from $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ to $(\mathscr{E}_2,\Lambda_2,\aleph_2)$ and this is denoted as $\Omega:(\mathscr{E}_1,\Lambda_1,\aleph_1)\rightleftarrows(\mathscr{E}_2,\Lambda_2,\aleph_2)$.

Definition 4. Let $(\mathscr{E},\Lambda,\aleph)$ be a BPPMS. A left sequence $\{\sigma_a\}$ converges to a right point η if and only if for every $\wp>0$, there exists an $a_0\in\mathbb{N}$ s.t. $\aleph(\sigma_a,\eta,c)<\wp$ for all $a\geq a_0$ and $c>0$. Similarly, a right seqence $\{\eta_a\}$ converges to a left point σ if and only if, for every $\wp>0$ we can find an $a_0\in\mathbb{N}$, satisfying whenever $a\geq a_0, c>0, \aleph(\sigma,\eta_a,c)<\wp$.

Definition 5. Let $(\mathscr{E},\Lambda,\aleph)$ be a BPPMS.

(i) A sequence $(\{\sigma_n\},\{\eta_n\})$ on the set $\mathscr{E}\times\Lambda$ is called a bisequence on $(\mathscr{E},\Lambda,\aleph)$.
(ii) If both $\{\sigma_n\}$ and $\{\eta_n\}$ are convergent, then the bisequence (σ_n,η_n) is called convergent. If $\{\sigma_n\}$ and $\{\eta_n\}$ both converge to a same point $u\in\mathscr{E}\cap\Lambda$, then this bisequence is called biconvergent.
(iii) A bisequence $(\{\sigma_n\},\{\eta_n\})$ on $(\mathscr{E},\Lambda,\aleph)$ is called Cauchy bisequence, if for each $\wp>0$, we can find a number $a_0\in\mathbb{N}$, satisfying for all positive integers $a,b\geq a_0, c>0, \aleph(\sigma_a,\eta_b,c)<\wp$.

Definition 6. Let $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ and $(\mathscr{E}_2,\Lambda_2,\aleph_2)$ be BPPMSs.

(i) A map $\Omega:(\mathscr{E}_1,\Lambda_1,\aleph_1)\rightrightarrows(\mathscr{E}_2,\Lambda_2,\aleph_2)$ is said to be continuous at a point $\sigma_0\in\mathscr{E}_1$, if for every $\wp>0$, we can find a $\delta>0$ satisfying whenever $\eta\in\Lambda_1, c>0$, and $\aleph_1(\sigma_0,\eta,c)<\delta$, $\aleph_2(\Omega(\sigma_0),\Omega(\eta),c)<\wp$. It is continuous at a point $\eta_0\in\Lambda_1$ if for every $\wp>0$, we can find a $\delta>0$ satisfying whenever $\sigma\in\mathscr{E}_1$, $c>0$, and $\aleph_1(\sigma,\eta_0,c)<\mathscr{E}_1$, $\aleph_2(\Omega(\sigma),\Omega(\eta_0),c)<\wp$. If Ω is continuous at each point $\sigma\in\mathscr{E}_1$ and $\eta\in\Lambda_1$, then it is called continuous.
(ii) A contravariant map $\Omega:(\mathscr{E}_1,\Lambda_1,\aleph_1)\leftrightarrows(\mathscr{E}_2,\Lambda_2,\aleph_2)$ is continuous iff it is continuous as a covariant map Ω: $(\mathscr{E}_1,\Lambda_1,\aleph_1)\rightrightarrows(\mathscr{E}_2,\Lambda_2,\overline{\aleph_2})$.

This definition implies that a contravariant map or a covariant Ω from $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ to $(\mathscr{E}_2,\Lambda_2,\aleph_2)$ is continuous, if and only if $\{\pi_a\}\longrightarrow\varsigma$ on $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ implies $\{\Omega(\pi_a)\}\longrightarrow\Omega(\varsigma)$ on $(\mathscr{E}_2,\Lambda_2,\aleph_2)$.

Definition 7. Let $(\mathscr{E}_1,\Lambda_1,\aleph_1)$ and $(\mathscr{E}_2,\Lambda_2,\aleph_2)$ be BPPMSs and $\lambda>0$. A covariant map Ω: $(\mathscr{E}_1,\Lambda_1,\aleph_1)\rightrightarrows(\mathscr{E}_2,\Lambda_2,\aleph_2)$ s.t.

$$\aleph(\Omega(\sigma),\Omega(\eta),c)\leq\lambda\aleph(\sigma,\eta,c)\ \text{for all}\ c>0,\quad \sigma\in\mathscr{E}_1,\eta\in\Lambda_1 \tag{1}$$

or a contravariant map $\Omega:(\mathscr{E}_1,\Lambda_1,\aleph_1)\leftrightarrows(\mathscr{E}_2,\Lambda_2,\aleph_2)$ s.t.

$$\aleph(\Omega(\sigma),\Omega(\eta),c)\leq\lambda\aleph(\sigma,\eta,c)\ \text{for all}\ c>0,\quad \sigma\in\mathscr{E}_1,\eta\in\Lambda_1 \tag{2}$$

is called Lipschitz continuous. If $\lambda=1$, then this covariant or contravariant map is said to be nonexpansive, and if it is fulfilled for a $\lambda\in(0,1)$, it is called a contraction.

3. Main Results

In this section, we prove FP theorems on BPPMS.

Theorem 8. *Let $(\mathscr{E}, \Lambda, \aleph)$ be a complete BPPMS and given a covariant contraction Ω: $(\mathscr{E}, \Lambda, \aleph) \rightrightarrows (\mathscr{E}, \Lambda, \aleph)$. Then, the function Ω: $\mathscr{E} \cup \Lambda \longrightarrow \mathscr{E} \cup \Lambda$ has a UFP (unique fixed point).*

Proof. Let $\sigma_0 \in \mathscr{E}$ and $\eta_0 \in \Lambda$. For each $a \in \mathbb{N}$, define $\Omega(\sigma_a) = \sigma_{a+1}$ and $\Omega(\eta_a) = \eta_{a+1}$. Then, $(\{\sigma_a\}, \{\eta_a\})$ is a bisequence on $(\mathscr{E}, \Lambda, \aleph)$. Say $\mathscr{M} := \aleph(\sigma_0, \eta_0, c) + \aleph(\sigma_0, \eta_1, c)$ and $\mathscr{K}_a := \lambda^a \mathscr{M}/1 - \lambda$. Then, for all $a, \mathfrak{p} \in \mathbb{N}$,

$$\begin{aligned} \aleph(\sigma_a, \eta_a, c) &= \aleph(\Omega(\sigma_{a-1}), \Omega(\eta_{a-1}), c) \\ &\leq \lambda \aleph(\sigma_{a-1}, \eta_{a-1}, c) \\ &\vdots \\ &\leq \lambda^a \aleph(\sigma_0, \eta_0, c), \end{aligned} \tag{3}$$

and also,

$$\begin{aligned} \aleph(\sigma_a, \eta_{a+1}, c) &= \aleph(\Omega(\sigma_{a-1}), \Omega(\eta_a), c) \\ &\leq \lambda \aleph(\sigma_{a-1}, \eta_a, c) \\ &\vdots \\ &\leq \lambda^a \aleph(\sigma_0, \eta_1, c) \\ \aleph(\sigma_{a+\mathfrak{p}}, \eta_a, c) &\leq \aleph(\sigma_{a+\mathfrak{p}}, \eta_{a+1}, c) + \aleph(\sigma_a, \eta_{a+1}, c) + \aleph(\sigma_a, \eta_a, c) \\ &\leq \aleph(\sigma_{a+\mathfrak{p}}, \eta_{a+1}, c) + \lambda^a \mathscr{M} \\ &\leq \aleph(\sigma_{a+\mathfrak{p}}, \eta_{a+2}, c) + \aleph(\sigma_{a+1}, \eta_{a+2}, c) + \aleph(\sigma_{a+1}, \eta_{a+1}, c) + \lambda^a \mathscr{M} \\ &\leq \aleph(\sigma_{a+\mathfrak{p}}, \eta_{a+2}, c) + (\lambda^{a+1} + \lambda^a)\mathscr{M} \\ &\vdots \\ &\leq \aleph(\sigma_{a+\mathfrak{p}}, \eta_{a+\mathfrak{p}}, c) + (\lambda^{a+\mathfrak{p}-1} + \ldots + \lambda^{a+1} + \lambda^a)\mathscr{M} \\ &\leq (\lambda^{a+\mathfrak{p}} + \ldots + \lambda^{a+1} + \lambda^a)\mathscr{M} \\ &\leq \frac{\lambda^a \mathscr{M}}{1-\lambda} = \mathscr{K}_a, \end{aligned} \tag{4}$$

and similarly, $\aleph(\sigma_a, \eta_{a+\mathfrak{p}}, c) \leq \mathscr{K}_a$. Let $\wp > 0$. Since $\lambda \in (0, 1)$, we can find an $a_0 \in \mathbb{N}$ satisfying $\mathscr{K}_{a_0} = \lambda_0^a/1 - \lambda < \wp/3$. Then,

$$\begin{aligned} \aleph(\sigma_a, \eta_b, c) &\leq \aleph(\sigma_a, \eta_{a_0}, c) + \aleph(\sigma_{a_0}, \eta_{a_0}, c) + \aleph(\sigma_{a_0}, \eta_b, c) \\ &\leq 3K_{a_0} < \wp, \end{aligned} \tag{5}$$

and $(\{\sigma_a\}, \{\eta_a\}))$ is a Cauchy bisequence. Since $(\mathscr{E}, \Lambda, \aleph)$ is complete, $(\{\sigma_a\}, \{\eta_a\})$ converges and thus biconverges to a point $\pi \in \mathscr{E} \cap \Lambda$ and

$$\{\Omega(\eta_a)\} = \{\eta_{a+1}\} \longrightarrow \pi \in \mathscr{E} \cap \Lambda \tag{6}$$

guarantees that $\{\Omega(\eta_a)\}$ has a unique limit. Since Ω is continuous, $\Omega(\eta_a) \longrightarrow \Omega(\pi)$, so $\Omega(\pi) = \pi$. Hence, π is a FP of Ω. If ς is any FP of Ω, then $\Omega(\varsigma) = \varsigma$ implies that $\varsigma \in \mathscr{E} \cap \Lambda$ and we have

$$\aleph(\pi, \varsigma, c) = \aleph(\Omega(\pi), \Omega(\varsigma), c) \leq \aleph(\pi, \varsigma, c), \tag{7}$$

where $0 < \lambda < 1$, which implies $\aleph(\pi, \varsigma, c) = 0$, and so $\pi = \varsigma$. □

Example 1. Let $\mathscr{E} = [0, 1]$ and $\Lambda = \{0\} \cup \mathbb{N} - \{1\}$ be equipped with $\aleph(\sigma, \eta, c) = c|\sigma - \eta|$ for all $\sigma \in \mathscr{E}$, $\eta \in \Lambda$, and $c > 0$. Then, $(\mathscr{E}, \Lambda, \aleph)$ is a complete BPPMS. Define Ω: $\mathscr{E} \cup \Lambda \rightrightarrows \mathscr{E} \cup \Lambda$ given by

$$\Omega(\sigma) = \begin{cases} \dfrac{\sigma}{5}, & \text{if } \sigma \in (0, 1], \\ 0, & \text{if } \sigma \in \{0\} \cup \mathbb{N} - \{1\}, \end{cases} \tag{8}$$

$\forall \sigma \in \mathscr{E} \cup \Lambda$. Let $\sigma \in \mathscr{E}$ and $\eta \in \Lambda$, then

$$\begin{aligned} \aleph(\Omega\sigma, \Omega\eta, c) &= c\left|\frac{\sigma}{5} - 0\right| \\ &\leq \frac{c}{2}|\sigma - \eta|. \end{aligned} \tag{9}$$

Therefore, all the conditions of Theorem 8 are satisfied and Ω has a UFP $\sigma = 0$.

Example 2. Let $\mathscr{E} = \{\mathscr{U}_a(\mathbb{R})$: $\mathscr{U}_a(\mathbb{R})$ be an upper triangular matrices over $\mathbb{R}\}$, $\Lambda = \{\mathscr{L}_a(\mathbb{R}): \mathscr{L}_a(\mathbb{R})$ be an upper triangular matrices over $\mathbb{R}\}$ and the map $\aleph : \mathscr{E} \times \Lambda \longrightarrow \mathbb{R}^+$ defined by

$$\aleph(\mathscr{P}, \mathscr{Q}, c) = c \sum_{i,j=1}^{a} \left|\eta_{ij} - \sigma_{ij}\right|, \tag{10}$$

for all $c > 0$, $\mathscr{P} = (\eta_{ij})_{a\times a} \in \mathscr{E}$ and $\mathscr{Q} = (\sigma_{ij})_{a\times a} \in \Lambda$. Then, $(\mathscr{E}, \Lambda, \aleph)$ is a complete BPPMS. Define $\Omega : \mathscr{E} \cup \Lambda \rightrightarrows \mathscr{E} \cup \Lambda$ given by

$$\Omega(\mathscr{P}) = \left(\frac{\eta_{ij}}{6}\right)_{a\times a}, \tag{11}$$

for all $\mathscr{P} = (\eta_{ij})_{a\times a} \in \mathscr{U}_a(\mathbb{R}) \cup \mathscr{L}_a(\mathbb{R})$. Now,

$$\begin{aligned}
\aleph(\Omega(\mathscr{P}), \Omega(\mathscr{Q}), c) &= \frac{c}{6} \sum_{i,j=1}^{a} \left|\eta_{ij} - \sigma_{ij}\right| \\
&\leq \frac{c}{4} \sum_{i,j=1}^{a} \left|\eta_{ij} - \sigma_{ij}\right| \\
&= \frac{c}{4} \sum_{i,j=1}^{a} \left|\eta_{ij} - \sigma_{ij}\right| \\
&= \lambda \aleph(\mathscr{Q}, \mathscr{P}, c),
\end{aligned} \tag{12}$$

for all $\mathscr{P} = (\eta_{ij})_{a\times a} \in \mathscr{E}$ and $\mathscr{Q} = (\sigma_{ij})_{a\times a} \in \Lambda$. All the axioms of Theorem 8 are verified with $\lambda = 1/4$ and Ω has a unique fixed point $(0_{a\times a}, 0_{a\times a}) \in \mathscr{U}_a(\mathbb{R}) \cup \mathscr{L}_a(\mathbb{R})$ where $0_{a\times a}$ is the null matrix.

Theorem 9. *Let $(\mathscr{E}, \Lambda, \aleph)$ be a complete BPPMS and given a contravariant contraction $\Omega : (\mathscr{E}, \Lambda, \aleph) \leftrightarrows (\mathscr{E}, \Lambda, \aleph)$. Then, the function Ω: $\mathscr{E} \cup \Lambda \longrightarrow \mathscr{E} \cup \Lambda$ has a UFP.*

Proof. Let $\sigma_0 \in \mathscr{E}$. For each $a \in \mathbb{N}$, define $\Omega(\sigma_a) = \eta_a$ and $\Omega(\eta_a) = \sigma_{a+1}$. Then, $(\{\sigma_a\}, \{\eta_a\})$ is a bisequence on $(\mathscr{E}, \Lambda, \aleph)$. Say

$$\mathscr{K}_a = \frac{\lambda^{2a}}{1-\lambda} . \aleph(\sigma_0, \eta_0, c). \tag{13}$$

Then, for all $a, \mathfrak{p} \in \mathbb{Z}^+$,

$$\begin{aligned}
\aleph(\sigma_a, \eta_a, c) &= \aleph(\Omega(\eta_{a-1}), \Omega(\sigma_a), c) \\
&\leq \aleph(\sigma_a, \eta_{a-1}, c) \\
&= \lambda \aleph(\Omega(\eta_{a-1}), \Omega(\sigma_{a-1}), c) \\
&\leq \lambda^2 \aleph(\sigma_{a-1}, \eta_{a-1}, c) \\
&\vdots \\
&\leq \lambda^{2a} \aleph(\sigma_0, \eta_0, c) \\
&= (1-\lambda) K_a \\
&\leq \mathscr{K}_a, \\
\aleph(\sigma_{a+1}, \eta_a, c) &= \aleph(\Omega(\eta_a), \Omega(\sigma_a), c) \\
&\leq \lambda \aleph(\sigma_a, \eta_a, c) \\
&\leq \lambda^{2a+1} \aleph(\sigma_0, \eta_0, c), \\
\aleph(\sigma_{a+\mathfrak{p}}, \eta_a, c) &\leq \aleph(\sigma_{a+\mathfrak{p}}, \eta_{a+1}, c) + \aleph(\sigma_{a+1}, \eta_{a+1}, c) + \aleph(\sigma_{a+1}, \eta_a, c)
\end{aligned}$$

$$
\begin{aligned}
&\leq \aleph\left(\sigma_{a+\mathfrak{p}}, \eta_{a+1}, c\right) + \left(\lambda^{2a+2} + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right)\\
&\leq \aleph\left(\sigma_{a+\mathfrak{p}}, \eta_{a+2}, c\right) + \aleph\left(\sigma_{a+2}, \eta_{a+2}, c\right) + \aleph\left(\sigma_{a+2}, \eta_{a+1}, c\right)\\
&\quad + \left(\lambda^{2a+2} + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right)\\
&\leq \aleph\left(\sigma_{a+\mathfrak{p}}, \eta_{a+2}, c\right) + \left(\lambda^{2a+4} + \lambda^{2a+3} + \lambda^{2a+2} + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right)\\
&\quad\vdots\\
&\leq \aleph\left(\sigma_{a+\mathfrak{p}}, \eta_{a+\mathfrak{p}-1}, c\right) + \left(\lambda^{2a+2\mathfrak{p}-2} + \ldots. + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right)\\
&\leq \left(\lambda^{2a+2a-1} + \lambda^{2a+2\mathfrak{p}-2} + \lambda^{2a+2\mathfrak{p}-3} + \ldots. + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right)\\
&\leq \frac{\lambda^{2a+1}}{1-\lambda}\aleph\left(\sigma_0, \eta_0, c\right)\\
&= \lambda \mathscr{K}_a\\
&< \mathscr{K}_a,\\
\aleph\left(\sigma_a, \eta_{a+\mathfrak{p}}, c\right) &\leq \aleph\left(\sigma_a, \eta_a, c\right) + \aleph\left(\sigma_{a+1}, \eta_a, c\right) + \aleph\left(\sigma_{a+1}, \eta_{a+\mathfrak{p}}, c\right)\\
&\leq \left(\lambda^{2a} + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right) + \aleph\left(\sigma_{a+1}, \eta_{a+\mathfrak{p}}, c\right)\\
&\leq \left(\lambda^{2a} + \lambda^{2a+1}\right)\aleph\left(\sigma_0, \eta_0, c\right) + \aleph\left(\sigma_{a+1}, \eta_{a+1}, c\right) + \aleph\left(\sigma_{a+2}, \eta_{a+1}, c\right)\\
&\quad + \aleph\left(\sigma_{a+2}, \eta_{a+\mathfrak{p}}, c\right)\\
&\leq \left(\lambda^{2a} + \lambda^{2a+1} + \lambda^{2a+2} + \lambda^{2a+3}\right)\aleph\left(\sigma_0, \eta_0, c\right) + \aleph\left(\sigma_{a+2}, \eta_{a+\mathfrak{p}}, c\right)\\
&\quad\vdots\\
&\leq \left(\lambda^{2a} + \lambda^{2a+1} + \ldots + \lambda^{2a+2\mathfrak{p}-1}\right)\aleph\left(\sigma_0 \eta_0, c\right) + \aleph\left(\sigma_{a+\mathfrak{p}}, \eta_{a+\mathfrak{p}}, c\right)\\
&\leq \left(\lambda^{2a} + \lambda^{2a+1} + \ldots + \lambda^{2a+2\mathfrak{p}-1} + \lambda^{2a+2\mathfrak{p}}\right)\aleph\left(\sigma_0, \eta_0, c\right)\\
&\leq \frac{\lambda^{2a}}{1-\lambda}\aleph\left(\sigma_0, \eta_0, c\right)\\
&= \mathscr{K}_a.
\end{aligned}
\tag{14}
$$

Now, since $0 < \lambda < 1$, for any $\wp > 0$, we can find an integer a_0 satisfying

$$
\mathscr{K}_{a_0} = \frac{\lambda^{2n_0+1}}{1-\lambda}\aleph\left(\sigma_0, \eta_0, c\right) < \frac{\wp}{3}. \tag{15}
$$

Hence,

$$
\begin{aligned}
\aleph\left(\sigma_a, \eta_b, c\right) &\leq \aleph\left(\sigma_a, \eta_{a_0}, c\right) + \aleph\left(\sigma_{a_0}, \eta_{a_0}, c\right) + \aleph\left(\sigma_{a_0}, \eta_b, c\right)\\
&\leq 3K_{a_0} < \wp,
\end{aligned}
\tag{16}
$$

and $(\{\sigma_a\}, \{\eta_a\})$ is a Cauchy bisequence. Since $(\mathscr{E}, \Lambda, \aleph)$ is a complete BPPMS, $(\{\sigma_a\}, \{\eta_a\})$ converges, and as a convergent Cauchy bisequence, in particular, it biconverges. Let $\{\sigma_a\} \longrightarrow \pi, \{\eta_a\} \longrightarrow \pi$, where $\pi \in \mathscr{E} \cap \Lambda$. Since the contravariant map Ω is continuous,

$$
\{\sigma_n\} \longrightarrow \pi, \tag{17}
$$

which derives that

$$
\{\eta_a\} = \{\Omega\left(\sigma_a\right)\} \longrightarrow \Omega\left(\pi\right), \tag{18}
$$

and combining this with $\{\eta_a\} \longrightarrow \pi$ gives $\Omega(\pi) = \pi$. Let ς be a FP of Ω, then $\Omega(\varsigma) = \varsigma$ implies $\varsigma \in \mathscr{E} \cap \Lambda$ so that

$$
\begin{aligned}
\aleph\left(\pi, \varsigma, c\right) &= \aleph\left(\Omega\left(\pi\right), \Omega\left(\varsigma\right), c\right)\\
&\leq \lambda \aleph\left(\pi, \varsigma, c\right),
\end{aligned}
\tag{19}
$$

which gives $\aleph(\pi, \varsigma, c) = 0$. Hence, $\pi = \varsigma$. $\square$

Example 3. Let $\mathscr{E} = \{0, 1, 2, 7\}$ and $\Lambda = \{0, (1/4), (1/2), (7/4), 3\}$ be equipped with $\aleph(\sigma, \eta, c) = c|\sigma - \eta|$ for all $\sigma \in \mathscr{E}$, $\eta \in \Lambda$, and $c > 0$. Then, $(\mathscr{E}, \Lambda, \aleph)$ is a complete BPPMS. Define $\Omega: \mathscr{E} \cup \Lambda \rightleftarrows \mathscr{E} \cup \Lambda$ given by

$$\Omega(\sigma) = \begin{cases} \dfrac{\sigma}{4}, & \text{if } \sigma \in \{0, 2, 7\}, \\ 0, & \text{if } \sigma \in \left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{7}{4}, 1, 3\right\}, \end{cases} \tag{20}$$

$\forall \sigma \in \mathscr{E} \cup \Lambda$. Let $\sigma \in \mathscr{E}$ and $\eta \in \Lambda$, then we can easily get

$$\aleph(\Omega\sigma, \Omega\eta, c) \leq \frac{1}{2}\aleph(\sigma, \eta, c). \tag{21}$$

Therefore, conditions of Theorem 9 are satisfied and Ω has a UFP $\sigma = 0$.

Finally, we express a theorem based of Kannan's FP result [22].

Theorem 10. *Let Ω: $(\mathscr{E}, \Lambda, \aleph) \leftrightarrows (\mathscr{E}, \Lambda, \aleph)$, where $(\mathscr{E}, \Lambda, \aleph)$ is a complete BPPMS and let $\alpha \in (0, 1/2)$ satisfies*

$$\aleph(\Omega\eta, \Omega\sigma, c) \leq \alpha(\aleph(\sigma, \Omega\sigma, c) + \aleph(\Omega\eta, \eta, c)), \tag{22}$$

which holds for all $c > 0$, $\sigma \in \mathscr{E}$, and $\eta \in \Lambda$. Then, the function Ω: $\mathscr{E} \cup \Lambda \longrightarrow \mathscr{E} \cup \Lambda$ has a UFP.

Proof. Let $\sigma_0 \in \mathscr{E}$, for each non-negative integer a, we define $\eta_a = \Omega\sigma_a$ and $\sigma_{a+1} = \Omega\eta_a$. Then,

$$\begin{aligned} \aleph(\sigma_a, \eta_a, c) &= \aleph(\Omega\eta_{a-1}, \Omega\sigma_a, c) \\ &\leq \alpha(\aleph(\sigma_a, \Omega\sigma_a, c) + \aleph(\Omega\eta_{a-1}, \eta_{a-1}, c)) \\ &= \alpha(\aleph(\sigma_a, \eta_a, c) + \aleph(\sigma_a, \eta_{a-1}, c)), \end{aligned} \tag{23}$$

for all integers $a \geq 1$. Then,

$$\aleph(\sigma_a, \eta_a, c) \leq \frac{\alpha}{1 - \alpha}\aleph(\sigma_a, \eta_{a-1}, c), \tag{24}$$

and

$$\begin{aligned} \aleph(\sigma_a, \eta_{a-1}, c) &= \aleph(\Omega\eta_{a-1}, \Omega\sigma_{a-1}, c) \\ &\leq \alpha(\aleph(\sigma_{a-1}, \Omega\sigma_{a-1}, c) + \aleph(\Omega\eta_{a-1}, \eta_{a-1}, c)) \\ &= \alpha(\aleph(\sigma_{a-1}, \eta_{a-1}, c) + \aleph(\sigma_a, \eta_{a-1}, c)), \end{aligned} \tag{25}$$

so that

$$\aleph(\sigma_a, \eta_{a-1}, c) \leq \frac{\alpha}{1 - \alpha}\aleph(\sigma_{a-1}, \eta_{a-1}, c). \tag{26}$$

If we say $\lambda := \alpha/1 - \alpha$, then we have $\lambda \in (0, 1)$ since $\alpha \in (0, 1/2)$. Now,

$$\begin{aligned} \aleph(\sigma_a, \eta_a, c) &\leq \lambda^{2a}\aleph(\sigma_0, \eta_0, c), \\ \aleph(\sigma_a, \eta_{a-1}, c) &\leq \lambda^{2a-1}\aleph(\sigma_0, \eta_0, c), \end{aligned} \tag{27}$$

and for all $b, a \in \mathbb{N}$,

$$\begin{aligned} \aleph(\sigma_a, \eta_b, c) &\leq \aleph(\sigma_a, \eta_a, c) + \aleph(\sigma_{a+1}, \eta_a, c) + \aleph(\sigma_{a+1}, \eta_b, c) \\ &\leq \left(\lambda^{2a} + \lambda^{2a+1}\right)\aleph(\sigma_0, \eta_0, c) + \aleph(\sigma_{a+1}, \eta_b, c) \\ &\vdots \\ &\leq \left(\lambda^{2a} + \lambda^{2a+1} + \ldots + \lambda^{2b}\right)\aleph(\sigma_0, \eta_0, c), \end{aligned} \tag{28}$$

if $b > a$, and

$$\begin{aligned} \aleph(\sigma_a, \eta_b, c) &\leq \aleph(\sigma_{b+1}, \eta_b, c) + \aleph(\sigma_{b+1}, \eta_{b+1}, c) + \aleph(\sigma_a, \eta_{b+1}, c) \\ &\leq \left(\lambda^{2b+1} + \lambda^{2b+2}\right)\aleph(\sigma_0, \eta_0, c) + \aleph(\sigma_a, \eta_{b+1}, c) \\ &\vdots \\ &\leq \left(\lambda^{2b+1} + \lambda^{2b+2} + \ldots + \lambda^{2a}\right)\aleph(\sigma_0, \eta_0, c) + \aleph(\sigma_a, \eta_a, c) \\ &< \left(\lambda^{2b+1} + \lambda^{2b+2} + \ldots + \lambda^{2a+1}\right)\aleph(\sigma_0, \eta_0, c), \end{aligned} \tag{29}$$

if $b < a$. Since $\lambda \in (0, 1)$. Therefore, $(\{\sigma_a\}, \{\eta_m\})$ is a Cauchy bisequence. Since $(\mathscr{E}, \Lambda, \aleph)$ is complete. Then, $\{\sigma_a\} \longrightarrow \pi$, $\{\eta_b\} \longrightarrow \pi$ and $\pi \in \mathscr{E} \cup \Lambda$. Since

$$\{\Omega\sigma_a\} = \{\eta_a\} \longrightarrow \pi, \aleph(\Omega\pi, \Omega\sigma_a, c) \longrightarrow \aleph(\Omega\pi, \pi, c). \tag{30}$$

On the other hand,

$$\begin{aligned} \aleph(\Omega\pi, \Omega\sigma_a, c) &\leq \alpha(\aleph(\sigma_a, \Omega\sigma_a, c) + \aleph(\Omega\pi, \pi, c)) \\ &= \alpha(\aleph(\sigma_a, \eta_a, c) + \aleph(\Omega\pi, \pi, c));, \end{aligned} \tag{31}$$

which in turn implies that $\aleph(\Omega\pi, \pi, c) \leq \alpha\aleph(\Omega\pi, \pi, c)$. Hence, $\Omega\pi = \pi$. If ς is any FP of Ω, then $\Omega\varsigma = \varsigma$ implies that ς is in $\mathscr{E} \cap \Lambda$. Then,

$$\begin{aligned} \aleph(\pi, \varsigma, c) = \aleph(\Omega\pi, \Omega\varsigma, c) &\leq \alpha(\aleph(\pi, \Omega\pi, c) + \aleph(\Omega\varsigma, \varsigma, c)) \\ &= \alpha(\aleph(\pi, \pi, c) + \aleph(\nu, \varsigma, c)) = 0. \end{aligned} \tag{32}$$

Consequently, $\pi = \varsigma$.

We conclude by establishing a theorem based on the Reich-type FP theorem [23]. □

Theorem 11. *Let $(\mathscr{E}, \Lambda, \aleph)$ be a complete BPPMS. Consider the mapping $\Omega : (\mathscr{E}, \Lambda, \aleph) \leftrightarrows (\mathscr{E}, \Lambda, \aleph)$ s.t.*

$$\aleph(\Omega\sigma, \Omega\eta, c) \leq \alpha\aleph(\eta, \sigma, c) + \mathfrak{p}\aleph(\eta, \Omega\eta, c) + \nu\aleph(\Omega\sigma, \sigma, c), \tag{33}$$

for all $\eta \in \Lambda$ and $\sigma \in \mathscr{E}$, where $\alpha, \mathfrak{p}, \nu \geq 0$ s.t. $\alpha + \mathfrak{p} + \nu < 1$. Then, the function Ω: $\mathscr{E} \cup \Lambda \longrightarrow \mathscr{E} \cup \Lambda$ has a UFP.

Proof. Let $\eta_0 \in \Lambda$. Define $\sigma_a = \Omega\eta_a$ and $\eta_{a+1} = \Omega\sigma_a$ for all $a \in \mathbb{N}$. Then, we have

$$\begin{aligned}\aleph(\eta_a, \sigma_a, c) &= \aleph(\Omega\sigma_{a-1}, \Omega\eta_a, c)\\ &\leq \alpha\aleph(\eta_a, \sigma_{a-1}, c) + \mathfrak{p}\aleph(\eta_a, \Omega\eta_a, c) + \nu\aleph(\Omega\sigma_{a-1}, \sigma_{a-1}, c)\\ &= (\alpha + \nu)\aleph(\eta_a, \sigma_{a-1}, c) + \mathfrak{p}\aleph(\eta_a, \sigma_a, c),\end{aligned} \tag{34}$$

for all integers $a \geq 1$. Now,

$$\begin{aligned}\aleph(\eta_a, \sigma_a, c) &\leq \left(\frac{\alpha + \nu}{1 - \mathfrak{p}}\right)\aleph(\eta_a, \sigma_{a-1}, c), \text{ and}\\ \aleph(\eta_a, \sigma_{a-1}, c) &= \aleph(\Omega\sigma_{a-1}, \Omega\eta_{a-1}, c)\\ &\leq \alpha\aleph(\eta_{a-1}, \sigma_{a-1}, c) + \mathfrak{p}\aleph(\eta_{a-1}, \Omega\eta_{a-1}, c) + \nu\aleph(\Omega\sigma_{a-1}, \sigma_{a-1}, c)\\ &= (\alpha + \mathfrak{p})\aleph(\eta_{a-1}, \sigma_{a-1}, c) + \nu\aleph(\eta_a, \sigma_{a-1}, c),\end{aligned} \tag{35}$$

so

$$\aleph(\eta_a, \sigma_{a-1}, c) \leq \left(\frac{\alpha + \mathfrak{p}}{1 - \nu}\right)\aleph(\eta_{a-1}, \sigma_{a-1}, c). \tag{36}$$

If we say $\rho \coloneqq \alpha + \mathfrak{p}/1 - \nu$ and $g \coloneqq \alpha + \nu/1 - \mathfrak{p}$, then we have $\rho, g \in (0, 1)$. Now,

$$\begin{aligned}\aleph(\eta_a, \sigma_a, c) &\leq g^{2a}\aleph(\eta_0, \sigma_0, c),\\ \aleph(\eta_a, \sigma_{a-1}, c) &\leq \rho^{2a-1}\aleph(\eta_0, \sigma_0, c).\end{aligned} \tag{37}$$

For all natural numbers $a < b$, we have

$$\begin{aligned}\aleph(\eta_a, \sigma_b, c) &\leq \aleph(\eta_a, \sigma_a, c) + \aleph(\eta_{a+1}, \sigma_a, c) + \aleph(\eta_{a+1}, \sigma_b, c)\\ &\leq \aleph(\eta_a, \sigma_a, c) + \aleph(\eta_{a+1}, \sigma_a) + \aleph(\eta_{a+1}, \sigma_{a+1}, c) + \aleph(\eta_{a+2}, \sigma_{a+1}, c)\\ &\quad + \cdots + \aleph(\eta_{b-1}, \sigma_{b-1}, c) + \aleph(\eta_b, \sigma_{b-1}, c) + \aleph(\eta_b, \sigma_b, c)\\ &\leq g^{2a}\aleph(\eta_0, \sigma_0, c) + \rho^{2a+1}\aleph(\eta_0, \sigma_0, c) + g^{2a+2}\aleph(\eta_0, \sigma_0, c) + \rho^{2a+3}\aleph(\eta_0, \sigma_0, c)\\ &\quad + \cdots + g^{2b-2}\aleph(\eta_0, \sigma_0, c) + \rho^{2b-1}\aleph(\eta_0, \sigma_0, c) + g^{2b}\aleph(\eta_0, \sigma_0, c)\\ &= \left(g^{2a} + g^{2a+2} + \cdots + g^{2b}\right)\aleph(\eta_0, \sigma_0, c) + \left(\rho^{2a+1} + \rho^{2a+3} + \cdot + \rho^{2b-1}\right)\aleph(\eta_0, \sigma_0, c)\\ &\leq g^{2a}\left(\frac{1}{1 - g^2}\right)\aleph(\eta_0, \sigma_0, c) + \rho^{2a+1}\left(\frac{1}{1 - \rho^2}\right)\aleph(\eta_0, \sigma_0, c).\end{aligned} \tag{38}$$

For all natural numbers $b < a$, we have

$$\begin{aligned}
\aleph(\eta_a,\sigma_b,c) &\leq \aleph(\eta_{b+1},\sigma_b,c)+\aleph(\eta_{b+1},\sigma_{b+1},c)+\aleph(\eta_a,\sigma_{b+1},c)\\
&\leq \aleph(\eta_{b+1},\sigma_b,c)+\aleph(\eta_{b+1},\sigma_{b+1},c)+\aleph(\eta_{b+2},\sigma_{b+1},c)+\aleph(\eta_{b+2},\sigma_{b+2},c)\\
&\quad+\cdots+\aleph(\eta_{a-1},\sigma_{a-1},c)+\aleph(\eta_a,\sigma_{a-1},c)+\aleph(\eta_a,\sigma_a,c)\\
&\leq \rho^{2b+1}\aleph(\eta_0,\sigma_0,c)+g^{2b+2}\aleph(\eta_0,\sigma_0,c)+\rho^{2b+3}\aleph(\eta_0,\sigma_0,c)+g^{2a+4}\aleph(\eta_0,\sigma_0,c)\\
&\quad+\cdots+g^{2a-2}\aleph(\eta_0,\sigma_0,c)+\rho^{2a-1}\aleph(\eta_0,\sigma_0,c)+g^{2a}\aleph(\eta_0,\sigma_0,c)\\
&=\left(g^{2b+2}+g^{2a+4}+\cdots+g^{2a}\right)\aleph(\eta_0,\sigma_0,c)+\left(\rho^{2b+1}+\rho^{2b+3}+\cdot+\rho^{2a-1}\right)\aleph(\eta_0,\sigma_0,c)\\
&\leq g^{2b+2}\left(\frac{1}{1-g^2}\right)\aleph(\eta_0,\sigma_0,c)+\rho^{2b+1}\left(\frac{1}{1-\rho^2}\right)\aleph(\eta_0,\sigma_0,c).
\end{aligned} \tag{39}$$

Therefore, $(\{\eta_a\},\{\sigma_m\})$ is a Cauchy bisequence. Since $(\mathscr{E},\Lambda,\aleph)$ is complete BPPMS, $\{\eta_a\}\longrightarrow \mathfrak{e}$, $\{\sigma_b\}\longrightarrow \mathfrak{e}$, where $\mathfrak{e}\in\mathscr{E}\cup\Lambda$. Since

$$\{\Omega\eta_a\}=\{\sigma_a\}\longrightarrow \mathfrak{e}, \aleph(\Omega\mathfrak{e},\Omega\eta_a,c)\longrightarrow \aleph(\Omega\mathfrak{e},\mathfrak{e},c). \tag{40}$$

On the other hand,

$$\begin{aligned}
\aleph(\Omega\mathfrak{e},\Omega\eta_a,c) &\leq \alpha\aleph(\eta_a,\mathfrak{e},c)+\mathfrak{p}\aleph(\eta_a,\Omega\eta_a,c)+\nu\aleph(\Omega\mathfrak{e},\mathfrak{e},c)\\
&=\alpha\aleph(\eta_a,\mathfrak{e},c)+\mathfrak{p}\aleph(\eta_a,\sigma_a,c)+\nu\aleph(\Omega\mathfrak{e},\mathfrak{e},c).
\end{aligned} \tag{41}$$

Therefore, $\aleph(\Omega\mathfrak{e},\mathfrak{e},c)\leq \nu\aleph(\Omega\mathfrak{e},\mathfrak{e},c)$. Hence, $\Omega\mathfrak{e}=\mathfrak{e}$. If v is any FP of Ω, then $\Omega v=v$, implies that $v\in\mathscr{E}\cap\Lambda$. Then,

$$\begin{aligned}
\aleph(\mathfrak{e},v,c)=\aleph(\Omega\mathfrak{e},\Omega v,c) &\leq \alpha\aleph(v,\mathfrak{e},c)+\mathfrak{p}\aleph(\mathfrak{e},\Omega\mathfrak{e},c)+\nu\aleph(\Omega v,v,c)\\
&=\alpha\aleph(v,\mathfrak{e},c)\\
&<\aleph(\mathfrak{e},v,c).
\end{aligned} \tag{42}$$

Consequently, $\mathfrak{e}=v$. □

Example 4. Let $\mathscr{E}=[0,1]$ and $\Lambda=[1,2]$ be equipped with $\aleph(\sigma,\eta,c)=c|\sigma-\eta|$ for all $\sigma\in\mathscr{E}$, $\eta\in\Lambda$, and $c>0$. Then, $(\mathscr{E},\Lambda,\aleph)$ is a complete BPPMS. Define Ω: $\mathscr{E}\cup\Lambda\rightleftarrows\mathscr{E}\cup\Lambda$ given by

$$\Omega(\sigma)=\frac{(\sqrt{2}+1)-\sigma}{\sqrt{2}}, \tag{43}$$

$\forall\,\sigma\in\mathscr{E}\cup\Lambda$. Let $\sigma\in\mathscr{E}$ and $\eta\in\Lambda$, then

$$\begin{aligned}
\aleph(\Omega\sigma,\Omega\eta,c) &=\frac{c}{\sqrt{2}}|\sigma-\eta|\\
&=\frac{c}{\sqrt{2}}|\eta-\sigma|\leq\frac{1}{2}\aleph(\eta,\sigma,c)\\
&=\alpha\aleph(\eta,\sigma,c).
\end{aligned} \tag{44}$$

All the axioms of Theorem 11 are verified with $\alpha=1/2$, $\mathfrak{p}=\nu=0$, and Ω has a unique fixed point $\sigma=1$.

4. Application to Fractional Differential Equations

We recall many important definitions from the fractional calculus theory. For a function $\eta\in\complement[0,1]$, the Reimann–Liouville fractional derivative of the order $\delta>0$ is given by

$$\frac{1}{\Gamma(a-\delta)}\frac{d^a}{\mathrm{d}a^a}\int_0^a\frac{\eta(x)dx}{(a-x)^{\delta-a+1}}=\mathscr{D}^\delta\eta(a), \tag{45}$$

provided that the right hand side is pointwise defined on $[0,1]$, where $[\delta]$ is the integer part of the number δ, Γ is the Euler gamma function.

Consider the following fractional differential equation:

$$\begin{aligned}
&{}^x\mathscr{D}^\sigma\eta(a)+g(a,\eta(a))=0,\quad 1\leq a\leq 0,\ 2\leq\sigma>1;\\
&\eta(0)=\eta(1)=0,
\end{aligned} \tag{46}$$

where g: $[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function and ${}^x\mathscr{D}^\sigma$ represents the Caputo fractional derivative of order σ and it is defined by

$$^{x}\mathscr{D}^{\sigma} = \frac{1}{\Gamma(a-\sigma)}\int_0^{\zeta}\frac{\eta^a(x)\mathrm{d}x}{(a-x)^{\sigma-a+1}}. \tag{47}$$

Let $\mathscr{E} = (\mathbb{C}[0,1],(0,\infty]) = \{\mathfrak{g}\colon [0,1] \longrightarrow (0,\infty]$ be a continuous function$\}$. Let $\Lambda = (\mathbb{C}[0,1],(-\infty,0]) = \{\mathfrak{g}\colon [0,1] \longrightarrow (-\infty,0]$ be a continuous function$\}$. Define $\aleph\colon \mathscr{E}\times\Lambda \longrightarrow \mathbb{R}^+$ is given by

$$\aleph(\eta,\eta',c) = c\sup_{a\in[0,1]}\left|\eta(a)-\eta'(a)\right|, \tag{48}$$

for all $(\eta,\eta')\in\mathscr{E}\times\Lambda$ and $c>0$. Obviously, axioms (a), (b), and (c) are satisfied. Now, we prove the axiom (d). For this,

$$\begin{aligned} c\left|\eta(a)-\eta'(a)\right| &= c\left|\eta(a)-\omega(a)-\sigma(a)+\omega(a)+\sigma(a)-\eta'(a)\right| \\ &\le c|\eta(a)-\omega(a)| + c|\sigma(a)-\omega(a)| + c\left|\sigma(a)-\eta'(a)\right|. \end{aligned} \tag{49}$$

Taking the supremum on both sides, we get

$$\aleph(\eta,\eta',c) \le \aleph(\eta,\omega,c) + \aleph(\sigma,\omega,c) + \aleph(\sigma,\eta',c), \tag{50}$$

for all $c>0$, $\sigma,\eta\in\mathscr{E}$, and $\omega,\eta'\in\Lambda$. Then, $(\mathscr{E},\Lambda,\aleph)$ is a complete BPPMS.

Theorem 12. *Assume the nonlinear fractional differential equation (46). Suppose that the following conditions are satisfied:*

(i) There exists $a\in[0,1]$, $\lambda\in(0,1)$, and $(\eta,\eta')\in\mathscr{E}\times\Lambda$ s.t. $|g(a,\eta)-g(a,\eta')| \le \lambda|\eta(a)-\eta'(a)|$;

(ii) $\sup_{a\in[0,1]}\int_0^1|\mathscr{G}(a,x)|\,\mathrm{d}\mathfrak{q} \le 1$.

Then, equation (46) has a unique solution in $\mathscr{E}\cup\Lambda$.

Proof. The given equation (46) is equivalent to the succeeding integral equation

$$\eta(a) = \int_0^1\mathscr{G}(a,x)g(\mathfrak{q},\eta(x))\mathrm{d}x, \tag{51}$$

where

$$\mathscr{G}(a,x) = \begin{cases} \dfrac{[a(1-x)]^{\sigma-1}-(a-x)^{\sigma-1}}{\Gamma(\sigma)}, & 0\le x\le a\le 1, \\[2ex] \dfrac{[a(1-x)]^{\sigma-1}}{\Gamma(\sigma)}, & 0\le a\le x\le 1. \end{cases} \tag{52}$$

Define the covariant mapping $\Omega\colon \mathscr{E}\cup\Lambda \longrightarrow \mathscr{E}\cup\Lambda$ defined by

$$\Omega\eta(a) = \int_0^1\mathscr{G}(a,x)g(\mathfrak{q},\eta(x))\mathrm{d}x. \tag{53}$$

Now,

$$\begin{aligned} c\left|\Omega\eta(a)-\Omega\eta'(a)\right| &= c\left|\int_0^1\mathscr{G}(a,x)g(\mathfrak{q},\eta(x))dx - \int_o^1\mathscr{G}(a,x)g(\mathfrak{q},\eta'(x))dx\right| \\ &\le c\int_0^1|\mathscr{G}(a,x)|dx\cdot\int_0^1\left|g(\mathfrak{q},\eta(x))-g(\mathfrak{q},\eta'(x))\right|dx \\ &\le \lambda c\left|\eta(a)-\eta'(a)\right|. \end{aligned} \tag{54}$$

Taking the supremum on both sides, we get

$$\aleph(\Omega\eta,\Omega\eta',c) \le \lambda\aleph(\eta,\eta',c). \tag{55}$$

Hence, all the hypothesis of Theorem 8 are satisfied and consequently, equation (46) has a unique solution. □

5. Conclusion

The idea of BPPMS was introduced in this article and FP theorems were demonstrated. An illustrative example is provided that show the validity of the hypothesis and the degree of usefulness of our findings.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally in writing this paper. Furthermore, this manuscript were read and approved by all the authors.

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