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Research Article

Some Properties of S-Semiannihilator Small Submodules and S-Small Submodules with respect to a Submodule

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Let R be a commutative ring with nonzero identity, $S \subseteq R$ be a multiplicatively closed subset of R, and M be a unital R-module. In this article, we introduce the concepts of S-semiannihilator small submodules and S-T-small submodules as generalizations of S-small submodules. We investigate some basic properties of them and give some characterizations of such submodules, especially for (finitely generated faithful) multiplication modules.

1. Introduction

Throughout this paper, R is a commutative ring with nonzero identity and M denotes a unital R-module. Also, $S \subseteq R$ is a multiplicatively closed subset of R. We use the notations " $\subseteq$ " and " $\leq$ " to denote inclusion and submodules, respectively. As usual, the rings of natural numbers, integers, and integer modulo n will be denoted by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_n$, respectively. A module M over a ring R (not necessarily commutative) is called prime if for every nonzero submodule N of M, $\operatorname{Ann}(N) = \operatorname{Ann}(M)$. An R-module M is called faithful if $\operatorname{Ann}(M) = 0$. An R-module M is called a multiplication module, if for any submodule N of M, $N = IM$ for some ideal I of R, and in this case, $N = (N\colon M)M$ (see [1]). A submodule of an R-module M is called small (superfluous) which is denoted by $N \ll M$, if for any submodule X of M, $N + X = M$, which implies that $X = M$. It is clear that the zero submodule of every nonzero module is small. More details about small submodules can be found in [2, 3]. In [4], the author introduced the concept of a semiannihilator small submodule of a module over a commutative ring R such that N is called semiannihilator small (sa-small for short), denoted by $N \ll^{sa} M$, if for every submodule X of M with $N + X = M$ implies that $\operatorname{Ann}(X) \ll R$. An ideal I of R is an sa-small ideal of R if it is an sa-small submodule of R as an R-module. Let T be an arbitrary submodule of M. In [5], a submodule N is called a T-small submodule of M provided for each submodule X of M, $T \subseteq N + X$, which implies that $T \subseteq X$.

A nonempty subset S of R is called a multiplicatively closed subset of R if $(i)\, 0 \notin S$, $(ii)\, 1 \in S$, and $(iii)\, ss' \in S$ for all $s, s' \in S$, see [6]. Let M be an R-module and S be a multiplicatively closed subset of R. Then, M is called an S-multiplication module if for each submodule N of M, there exist $s \in S$ and an ideal I of R such that $sN \subseteq IM \subseteq N$ [7]. The concept of S-Noetherian rings has been introduced and investigated by Anderson et al. [8]. Farshadifar introduced and studied in brief the notions of S-secondary submodules and S-copure submodules [9, 10]. Şengelen Sevim et al. in [11] described the concept of S-prime submodules. After, the generalizations of S-prime submodules have been studied in [12, 13]. Recently, the concept of S-small submodules have been studied in [14]. Here, we introduce and study the notions of S-semiannihilator small submodules and S-T-small submodules as generalizations of T-small submodules. In Sections 2 and 3, various properties of such submodules are considered.

2. S-Semiannihilator Small Submodules

In this section, we define the concept of S-semiannihilator small submodules of an R-module and we get some characterizations of them.

We begin with the following definition.

Definition 1. Let M be an R-module.

(1) We say that a submodule N of M is an S-small submodule of M which is denoted by $N \ll_S M$ if there exists $s \in S$ such that whenever $N + X = M$ for some submodule X of M, it implies that $sM \subseteq X$. We say that an R-module M is an S-hollow module if every submodule of M is an S-small submodule of M.

(2) An ideal I of R is called S-small if it is an S-small submodule of R as an R-module. A ring R is an S-hollow ring if it is an S-hollow R-module.

Remark 2. The following results follow from the definition:

(1) Clearly, if $S \cap \mathrm{Ann}(N) \neq \varnothing$, then $N \ll_S M$. Particularly, if M is an R-module with $S \cap \mathrm{Ann}(M) \neq \varnothing$, then M is S-hollow. Moreover, in this case, M is an S-multiplication R-module because suppose that $sM = 0$ for some $s \in S$. It is sufficient to take $I = (s)$; then, $0 = sN \subseteq IM \subseteq N$ for every submodule N of M, as needed.

(2) Let M be an R-module and $N <^{\oplus} M$. Then, there exists a proper submodule X of M such that $M = N \oplus X$. If $N \ll_S M$, then there exists $s \in S$ such that $sM = sN + sX \subseteq X$. This implies that $sN = 0$ and so $S \cap \mathrm{Ann}(N) \neq \varnothing$.

(3) It is clear that every small submodule is also S-small. In particular, the zero submodule is an S-small submodule of M. The following example shows that converse is not necessarily true in general. Clearly, if $S \subseteq U(R)$ and N is an S-small submodule of M, then N is small.

(4) If $N \ll_S M$, then for every submodule X of M with $N + X = M$, $S \cap (X:_R N) \neq \varnothing$, since there exists $s \in S$ such that $sN \subseteq sM = sN + sX \subseteq X$, as needed.

Example 1

(1) Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and take $S = \mathbb{Z} - \{0\}$. Then, $N = \mathbb{Z} \oplus 0$ is not an S-small submodule of M because $N + X = M$ for $X = 0 \oplus \mathbb{Z}$ but $sM \nsubseteq X$ for all $s \in S$.

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_6$ and the submodule $N = \langle \overline{2} \rangle$. Take the multiplicatively closed subset $S = \{3^n | n \in \mathbb{N} \cup \{0\}\}$. Then, N is an S-small submodule of M. Because we have $N + \langle \overline{3} \rangle = \mathbb{Z}_6$ and $N + \mathbb{Z}_6 = \mathbb{Z}_6$, let $s = 3$. Then, $s\mathbb{Z}_6 \subseteq \langle \overline{3} \rangle$ and $s\mathbb{Z}_6 \subseteq \mathbb{Z}_6$. But N is not a small submodule of M because $N + \langle \overline{3} \rangle = \mathbb{Z}_6$ and $\langle \overline{3} \rangle \neq \mathbb{Z}_6$. In general, let p, q be distinct prime numbers and consider the $\mathbb{Z}$-module $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$. Then, the submodule $N = \langle \overline{p} \rangle \cong 0 \oplus \mathbb{Z}_q$ is an S-small submodule of M such that $S = \{q^n | n \in \mathbb{N} \cup \{0\}\}$. Moreover, the submodule $K = \langle \overline{q} \rangle \cong \mathbb{Z}_p \oplus 0$ is an S'-small submodule of M such that $S' = \{p^n | n \in \mathbb{N} \cup \{0\}\}$.

Example 2. Consider $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ as a $\mathbb{Z}$-module such that p is a prime number. It is clear that every proper submodule of M is prime, and for any submodule N of M, $(N:_{\mathbb{Z}} M) = p\mathbb{Z}$. Also, M is a prime $\mathbb{Z}$-module, and it is an S-hollow module such that $S = \{p^n | n \in \mathbb{N} \cup \{0\}\}$.

Proposition 3. *Let M be an R-module. Then, the following statements are true:*

(1) If $N \leq K \leq M$ and $K \ll_S M$, then $N \ll_S M$.

(2) Let $\{N_i\}_{i \in \Lambda}$ be a nonempty set of S-small submodules of M. Then, $\bigcap_{i \in \Lambda} N_i$ is an S-small submodule.

Proof. The proofs are straightforward.

We recall that a submodule N of an R-module M is a semiannihilator small (briefly, sa-small) submodule if whenever $N + X = M$ for some submodule X of M, implying that $\mathrm{Ann}(X) \ll R$. □

Definition 4

(1) A submodule N of M is called an S-semiannihilator small (briefly, S-sa-small) submodule of M which is denoted by $N \ll_S^{sa} M$ if there exists $s \in S$ such that whenever $N + X = M$ for some submodule X of M, it implies that $s\mathrm{Ann}(X) \ll R$.

(2) An ideal I of R is called S-semiannihilator small (briefly, S-sa-small) ideal if it is an S-semiannihilator small submodule of R as an R-module.

Example 3. Consider the $\mathbb{Z}_6$-module $M = \mathbb{Z}_6$ and the submodule $N = \langle \overline{3} \rangle$. Take the multiplicatively closed subset $S = \{\overline{1}, \overline{2}, \overline{4}\}$. Then, N is an S-sa-small submodule of M. Because we have $N + \langle \overline{2} \rangle = \mathbb{Z}_6$ and $N + \mathbb{Z}_6 = \mathbb{Z}_6$, let $s = \overline{2}$. Then, $s\mathrm{Ann}_{\mathbb{Z}_6}(\langle \overline{2} \rangle) = s\langle \overline{3} \rangle \ll \mathbb{Z}_6$ and $n_{\mathbb{Z}_6}(\mathbb{Z}_6) \ll \mathbb{Z}_6$. But N is not an sa-small submodule of M because $N + \langle \overline{2} \rangle = \mathbb{Z}_6$, but $\mathrm{Ann}_{\mathbb{Z}_6}(\langle \overline{2} \rangle) = \langle \overline{3} \rangle \ll \mathbb{Z}_6$.

Lemma 5. *Let M be an R-module and S, T be two multiplicatively closed subsets of R with $S \subseteq T$. If N is an S-sa-small submodule of M, then N is a T-sa-small submodule of M.*

Proof. The proof is straightforward.

Let S be a multiplicatively closed subset of R. The saturation of S is the set $S^* = \{x \in R | xy \in S, \text{for some } y \in R\}$. It is clear that S^* is a multiplicatively closed subset of R and that $S \subseteq S^*$. □

Proposition 6. *Let N be a submodule of an R-module M. Then, N is an S-sa-small submodule of M if and only if N is an S^*-sa-small submodule of M.*

Proof. Let N be an S-sa-small submodule of M. Then, by Lemma 5, N is an S^*-sa-small submodule of M. Conversely, let N be an S^*-sa-small submodule of M. Suppose $N + X = M$ for some submodule X of M. Then, there exists $s^* \in S^*$ such that $s^* \mathrm{Ann}(M) \ll R$. Then, there exists $r \in R$ such that

$s = s^* r \in S$. We have $s\mathrm{Ann}(M) = s^* r\mathrm{Ann}(M) \subseteq s^*\mathrm{Ann}(M) \ll R$ and so $s\mathrm{Ann}(M) \ll R$. □

Proposition 7. *Let M be an R-module and N, K be submodules of M. Then, the following assertions hold:*

(i) If $K \subseteq N$ and $N \ll_S^{sa} M$, then $K \ll_S^{sa} M$

(ii) If $K \ll_S^{sa} N$, then $K \ll_S^{sa} M$

Proof.

(i) It is clear.

(ii) Let $K + X = M$ for some submodule X of M. Then, $(K + X) \cap N = N$, so $K + (X \cap N) = N$ (modular law). Hence, there exists $s \in S$ such that $s\mathrm{Ann}(X \cap N) \ll R$. Therefore, $sAnn(X) \ll R$. □

Proposition 8. *Let M be an R-module and I be an ideal of R. Then, the following assertions hold:*

(i) If $IM \ll_S^{sa} M$, then $I \ll_S^{sa} R$

(ii) If M is a finitely generated faithful multiplication module and $I \ll_S^{sa} R$, then $IM \ll_S^{sa} M$

Proof

(i) Let $I + J = R$ for some ideal J of R. Then, $IM + JM = M$. Hence, there exists $s \in S$ such that $s\mathrm{Ann}(\mathrm{JM}) \ll R$, since $s\mathrm{Ann}(J) \subseteq s\mathrm{Ann}(\mathrm{JM})$, so $s\mathrm{Ann}(J) \ll R$.

(ii) Let $\mathrm{IM} + X = M$ for some submodule X of M. We have $X = JM$ for some ideal J of R. Thus, $\mathrm{IM} + \mathrm{JM} = (I + J)M = M$. By Nakayama's lemma, there exists $a \in I + J$ such that $(1 - a)M = 0$. Since M is faithful, $1 - a = 0$ and so $a = 1$. Thus, $I + J = R$. Since $I \ll_S^{sa} R$, there exists $s \in S$ such that $s\mathrm{Ann}(J) = s\mathrm{Ann}(\mathrm{JM}) \ll R$, as needed. □

Theorem 9. *Let M and M' be R-modules and $f\colon M \longrightarrow M'$ be an R-epimorphism. If $K \ll_S^{sa} M'$, then $f^{-1}(K) \ll_S^{sa} M$.*

Proof. Let $f^{-1}(K) + L = M$ for some submodule L of M. Since f is an epimorphism, we have $f(f^{-1}(K) + L) = f(M) = M'$. Hence, $K + f(L) = M'$, so there exists $s \in S$ such that $sAnn(f(L)) \ll R$. Thus, $sAnn(L) \ll R$ since $s\mathrm{Ann}(L) \subseteq s\mathrm{Ann}(f(L))$. Therefore, $f^{-1}(K) \ll_S^{sa} M$.

By the following example, we show that if $f\colon M \longrightarrow M'$ is an epimorphism, then the image of an S-sa-small submodule of M need not be S-sa-small in M'. □

Example 4. Consider the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}_6$, the multiplicatively closed subset $S = \{2^n | n \in \mathbb{N} \cup \{0\}\}$, and the natural epimorphism $\pi\colon \mathbb{Z} \longrightarrow \mathbb{Z}_6$. Then, $\{0\}$ is an S-sa-small submodule of $\mathbb{Z}$, but $\pi(0) = \overline{0}$ is not an S-sa-small submodule of $\mathbb{Z}_6$.

The following example shows that the sum of S-sa-small submodules of an R-module M need not be an S-sa-small submodule of M.

Example 5. Consider the $\mathbb{Z}$-module $\mathbb{Z}$ and the multiplicatively closed subset $S = \mathbb{Z} - 3\mathbb{Z}$. The submodules $2\mathbb{Z}$ and $3\mathbb{Z}$ are the S-sa-small submodules of $\mathbb{Z}$. But $3\mathbb{Z} + 2\mathbb{Z}$ is not an S-sa-small submodule of $\mathbb{Z}$.

Proposition 10. *Let M_1 and M_2 be R-modules. If $N_1 \ll_S^{sa} M_1$ and $N_2 \ll_S^{sa} M_2$, then $N_1 \oplus N_2 \ll_S^{sa} M_1 \oplus M_2$.*

Proof. Let $\pi_i\colon M_1 \oplus M_2 \longrightarrow M_i$ for $i = 1, 2$ be the projection maps. Since $N_1 \ll_S^{sa} M_1$ and $N_2 \ll_S^{sa} M_2$, by Theorem 9, $N_1 \oplus M_2 = \pi_1^{-1}(N_1) \ll_S^{sa} M_1 \oplus M_2$ and $M_1 \oplus N_2 = \pi_2^{-1}(N_2) \ll_S^{sa} M_1 \oplus M_2$. Hence, $(N_1 \oplus M_2) \cap (M_1 \oplus N_2) = N_1 \oplus N_2 \ll_S^{sa} M_1 \oplus M_2$ by Proposition 7. □

Definition 11. An R-module M is called an S-semi-annihilator hollow (briefly, S-sa-hollow) module if every proper submodule of M is an S-sa-small submodule of M.

Example 6

(1) Consider the $\mathbb{Z}$-module $\mathbb{Z}$ and the multiplicatively closed subset $S = \{2^n | n \in \mathbb{N} \cup \{0\}\}$. Then, $\mathbb{Z}$ is an S-sa-hollow module, but it is not an S-hollow module. Because $3\mathbb{Z} + 5\mathbb{Z} = \mathbb{Z}$, but for any $s \in S$, $s\mathbb{Z} \nsubseteq 5\mathbb{Z}$.

(2) Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$ and the multiplicatively closed subset $S = \mathbb{Z} - 5\mathbb{Z}$. Then, $\mathbb{Z}_6$ is not an S-sa-hollow module. Because $\langle \overline{2} \rangle + \langle \overline{3} \rangle = \mathbb{Z}_6$, but for any $s \in S$, $s\mathrm{Ann}_{\mathbb{Z}}(\langle \overline{3} \rangle) = s(2\mathbb{Z}) \ll \mathbb{Z}$.

Proposition 12. *Let M, M' be R-modules and $f\colon M \longrightarrow M'$ be an epimorphism. If M' is an S-sa-hollow module, then M is an S-sa-hollow module.*

Proof. Let N be a submodule of M. Then, $f(N)$ is a submodule of M'. Since M' is an S-sa-hollow module, then $f(N) \ll_S^{sa} M'$. Thus, $f^{-1}(f(N)) \ll_S^{sa} M$, and since $N \subseteq f^{-1}(f(N))$, by Proposition 7, $N \ll_S^{sa} M$. Therefore, M is an S-sa-hollow module. □

Example 7. (a) We consider the $\mathbb{Z}$ and $\mathbb{Z}_4$ as $\mathbb{Z}$-modules, the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$, and the natural epimorphism $\pi\colon \mathbb{Z} \longrightarrow \mathbb{Z}_4$. Then, $\mathbb{Z}$ is an S-sa-hollow $\mathbb{Z}$-module, and $\{0\} \ll_S^{sa} \mathbb{Z}$; $\pi(\{0\}) = \overline{0}$ is not an S-sa-small submodule of $\mathbb{Z}_4$. Because $\overline{0} + \mathbb{Z}_4 = \mathbb{Z}_4$, but for any $s \in S$, $s\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_4) = s(4\mathbb{Z}) \ll \mathbb{Z}$.

Corollary 13. *Let M be an R-module and N be a submodule of M. If M/N is an S-sa-hollow module, then M is an S-sa-hollow module.*

Proof. Apply Proposition 12. □

Theorem 14. *Let M be an R-module and N be a submodule of M. Assume that F is a faithfully flat R-module. Then, if*

$F\otimes N$ is an S-sa-small submodule of R-module $F\otimes M$, then N is an S-sa-small submodule of M.

Proof. Let $F\otimes N \ll_S^{sa} F\otimes M$ and $N+K=M$ for some submodule K of M. Then,

$$F\otimes (N+K) = F\otimes N + F\otimes K = F\otimes M. \quad (1)$$

By assumption, there exists $s\in S$ such that $s\mathrm{Ann}(F\otimes K)\ll R$. Since F is a faithfully flat R-module, $\mathrm{Ann}(F\otimes K)=\mathrm{Ann}(K)$. Thus, $s\mathrm{Ann}(K)\ll R$, so N is an S-sa-small submodule of M. □

3. S-Small Submodules with respect to a Submodule

Let T be an arbitrary submodule of an R-module M and $S\subseteq R$ be a multiplicatively closed subset of R. In this section, we introduce and study another generalization of S-small and T-small submodules, namely, S-T-small submodules.

Definition 15

(1) Let M be an R-module and T be an arbitrary submodule of M. A submodule N of M is called an S-T-small submodule of M which is denoted by $N\ll_{S-T} M$ if there exists $s\in S$ such that whenever $T\subseteq N+X$ for some submodule X of M, it implies that $sT\subseteq X$.

We say that M is an S-T-hollow module if every submodule of M is S-T-small in M.

(2) Let J be an ideal of R. An ideal I of R is called S-J-ideal of R if there exists $s\in S$ such that whenever $J\subseteq I+K$ for some ideal K of R, then $sJ\subseteq K$. R is an S-T-hollow ring if it is an S-T-hollow as an R-module.

Observation 16. Let M be an R-module.

(1) Take $T=M$; then, $N\ll_{S-T} M$ if and only if $N\ll_S M$. If $T=0$, then every submodule of M is S-T-small in M.

(2) Clearly, every S-T-small submodule is a T-small submodule, but the following example shows that converse is not necessarily true.

(3) If $T\subseteq N$ and $N\ll_{S-T} M$, then there exists $s\in S$ such that $s\in \mathrm{Ann}(T)$, so $S\cap \mathrm{Ann}(T)\neq\varnothing$. Equivalently, if for some submodule T of M, $S\cap\mathrm{Ann}(T)=\varnothing$, then either $T\nsubseteq N$ or $N\ll_{S-T} M$.

Example 8. Consider the $\mathbb{Z}$-module $M=\mathbb{Z}$, the multiplicatively closed subset $S=\mathbb{Z}-\{0\}$, and the submodule $T=4\mathbb{Z}$. Then, the submodule $N=2\mathbb{Z}$ is not a T-small submodule of M because $4\mathbb{Z}\subseteq 2\mathbb{Z}+8\mathbb{Z}$, but $4\mathbb{Z}\nsubseteq 8\mathbb{Z}$. While, $N=2\mathbb{Z}$ is an $S-T$-small submodule of M. Let m be an integer such that $4\mathbb{Z}\subseteq 2\mathbb{Z}+m\mathbb{Z}$. Set $s=l\mathrm{cm}[4,m]$. Thus, $s4\mathbb{Z}\subseteq m\mathbb{Z}$.

Proposition 17. *Let M be an R-module, $L\leq T\leq M$, and $K\leq M$. Then, the following assertions hold:*

(i) If $K\ll_{S-T} M$, then $K\cap T\ll_S M$

(ii) $L\ll_{S-T} M$ if and only if $L\ll_S T$

Proof

(i) Let $(K\cap T)+X=M$ for some submodule X of M. We show that $sM\subseteq X$ for some $s\in S$. We have $T\subseteq (K\cap T)+X\subseteq K+X$. Since $K\ll_{S-T} M$, there exists $s\in S$ such that $sT\subseteq X$. Thus, $sM=s(K\cap T)+sX\subseteq sT+sX\subseteq sT+X=X$, so $K\cap T\ll_S M$.

(ii) Suppose that $L\ll_{S-T} M$ and $L+X=T$ for some submodule X of T. Since $T\subseteq L+X$ and $L\ll_{S-T} M$, so there exists $s\in S$ such that $sT\subseteq X$. Thus, $L\ll_S T$. Conversely, let $L\ll_S T$ and $T\subseteq L+X$ for some $X\leq M$. Then, $T=(L+X)\cap T=L+(X\cap T)$, and hence, $sT\subseteq X\cap T$ for some $s\in S$ because $L\ll_S T$. Thus, $sT\subseteq X$ for some $s\in S$, so $L\ll_{S-T} M$. □

Proposition 18. *Let M be an R-module with submodules $N\leq K\leq M$ and $T\leq K$. Then, $N\ll_{S-T} M$ if and only if $N\ll_{S-T} K$.*

Proof

($\Rightarrow$) It is clear.

($\Leftarrow$) Let $T\subseteq N+X$ for some submodule X of M. Then, $T\subseteq (N+X)\cap K = N+(X\cap K)$. Since $N\ll_{S-T} K$, there exists $s\in S$ such that $sT\subseteq X\cap K\subseteq X$. □

Theorem 19. *Let M be an R-module with submodules $K\leq N\leq M$ and $K\leq T$. If $N\ll_{S-T} M$, then $K\ll_{S-T} M$ and $N/K\ll_{S-T/K} M/K$.*

Proof. Let $T\subseteq K+X$ for some submodule X of M. Thus, $T\subseteq N+X$, so there exists $s\in S$ such that $sT\subseteq X$. Therefore, $K\ll_{S-T} M$. Let $T/K\subseteq N/K+X/K$ for some submodule X/K of M/K. Hence, $T\subseteq N+X$, so there exists $s\in S$ such that $sT\subseteq X$ since $N\ll_{S-T} M$. Thus, $s(T/K)=(sT+K)/K\subseteq X/K$. □

Definition 20. Let M, M' be R-modules and $0\neq T\leq M$. An R-epimorphism $f\colon M\longrightarrow M'$ is called $S-T$-small in case $\mathrm{Ker}(f)$ is an S-T-small submodule of M.

Proposition 21. *Let M be an R-module and K and $0\neq T$ be submodules of M. The following statements are equivalent:*

(i) $K\ll_{S-T} M$

(ii) The natural map $\pi\colon M\longrightarrow M/K$ is $S-T$-small

(iii) For every R-module N and R-homomorphism $g\colon N\longrightarrow M$, $T\subseteq K+Im(g)$ implies that $sT\subseteq Im(g)$ for some $s\in S$

Proof.

$(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (i)$. Let $T \subseteq K + X$ for some submodule X of M. Let $i\colon X \longrightarrow M$ be the inclusion map. Then, $T \subseteq K + X \subseteq K + \mathrm{Im}(i)$, and by (iii), there exists $s \in S$ such that $sT \subseteq \mathrm{Im}(i) = X$. □

Proposition 22. *Let M, M' be R-modules and $f\colon M \longrightarrow M'$ be an R-homomorphism. If K and T are submodules of M such that $K \ll_{S-T} M$, then $f(K) \ll_{S-f(T)} M'$. In particular, if $K \ll_{S-T} L \leq M$, then $K \ll_{S-T} M$.*

Proof. Let $f(T) \subseteq f(K) + X'$ for some submodule X' of M'. It is easy to see that $T \subseteq K + f^{-1}(X')$. Thus, there exists $s \in S$ such that $sT \subseteq f^{-1}(X')$. Hence, $f(sT) \subseteq f(f^{-1}(X')) \subseteq X'$, so $sf(T) \subseteq X'$. Therefore, $f(K) \ll_{S-f(T)} M'$. □

Corollary 23. *Let M, M' be R-modules and $f\colon M \longrightarrow M'$ be an R-monomorphism. If K and T are submodules of M, then $K \ll_{S-T} M$ if and only if $f(K) \ll_{S-f(T)} M'$.*

Proof

$(\Rightarrow)$ By Proposition 22.

$(\Leftarrow)$ Let $T \subseteq K + X$ for some submodule X of M. Then, $f(T) \subseteq f(K) + f(X)$. Thus, there exists $s \in S$ such that $sf(T) \subseteq f(X)$, so $f(sT) \subseteq f(X)$. Hence, $sT \subseteq f^{-1}(f(X)) \subseteq X$ since f is a monomorphism. Therefore, K is an $S-T$-small submodule of M. □

Example 9. Consider the $\mathbb{Z}$-homomorphism $f\colon \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{20}$ with $f(\overline{x}) = \overline{2x}$ and the multiplicatively closed subset $S = \{3^n\colon n \in \mathbb{N} \cup \{0\}\}$. Then, the submodule $\langle\overline{10}\rangle$ is an S-small submodule of $\mathbb{Z}_{20}$, but $f^{-1}(\langle\overline{10}\rangle) = \langle\overline{5}\rangle$ is not S-small in $\mathbb{Z}_{10}$. Because $\langle\overline{5}\rangle + \langle\overline{2}\rangle = \mathbb{Z}_{10}$, $s\mathbb{Z}_{10} \nsubseteq \langle\overline{2}\rangle$ for every $s \in S$.

Theorem 23. *Let M be a Noetherian R-module and N be a submodule of M. Then, N is an S-T-small submodule of M if and only if $S^{-1}N$ is an $S^{-1}T$-small submodule of $S^{-1}M$.*

Proof

$(\Rightarrow)$ Let N be an S-T-small submodule of M and $S^{-1}T \subseteq S^{-1}N + S^{-1}X$ for some submodule $S^{-1}X$ of $S^{-1}M$. Then, we have $T \subseteq N + X$ since M is a Noetherian R-module. Hence, there exists $s \in S$ such that $sT \subseteq X$. Let $a/t \in S^{-1}T$. Then, $a/t = b/t'$ for some $b \in T$ and $t' \in S$, so $a/t = \mathrm{sb}/\mathrm{st}' \in S^{-1}X$. Therefore, $S^{-1}T \subseteq S^{-1}X$.

$(\Leftarrow)$ It is obvious. □

Proposition 24. *Let M_1, M_2 be R-modules and $T_1 \leq M_1$ and $T_2 \leq M_2$. If N_1 is an S-T_1-small submodule of M_1 and N_2 is an S-T_2-small submodule of M_2, then $N_1 \oplus N_2$ is an S-$(T_1 \oplus T_2)$-small submodule of $M_1 \oplus M_2$.*

Proof. Let $T_1 \oplus T_2 \subseteq N_1 \oplus N_2 + X_1 \oplus X_2$ for some submodule $X_1 \oplus X_2$ of $M_1 \oplus M_2$. Then, $T_1 \subseteq N_1 + X_1$ and $T_2 \subseteq N_2 + X_2$. Thus, there exist $s_1, s_2 \in S$ such that $s_1T_1 \subseteq X_1$ and $s_2T_2 \subseteq X_2$. Set $s = s_1s_2$. Hence, $s(T_1 \oplus T_2) \subseteq X_1 \oplus X_2$. □

Theorem 25. *Let M be a finitely generated faithful multiplication R-module and $T \leq M$. Then, N is an S-T-small submodule of M if and only if $(N\colon_R M)$ is an S-$(T\colon M)$-small ideal of R.*

Proof. Let $N \ll_{S-T} M$ and $(T\colon M) \subseteq (N\colon M) + J$ for some ideal J of R. Then, $(T\colon M)M \subseteq ((N\colon M) + J)M = (N\colon M)M + JM$, and since M is a multiplication module, we have $T \subseteq N + JM$. Hence, there exists $s \in S$ such that $sT \subseteq JM$, so $s(T\colon M)M \subseteq JM$. Since M is a cancellation module, $s(T\colon M) \subseteq J$. Therefore, $(N\colon_R M)$ is an S-$(T\colon M)$-small ideal of R. Conversely, let $(N\colon_R M) \ll_{S-(T\colon M)} R$ and $T \subseteq N + X$ for some $X \leq M$. Thus, $(T\colon M)M \subseteq (N\colon M)M + (X\colon M)M = ((N\colon M) + (X\colon M))M$ since M is a multiplication module, so $(T\colon M) \subseteq (N\colon M) + (X\colon M)$ since M is a cancellation module. By assumption, there exists $s \in S$ such that $s(T\colon M) \subseteq (X\colon M)$, and hence, $sT \subseteq X$. This implies that N is an S-T-small submodule of M. □

Theorem 26. *Let M be an R-module and $N, T \leq M$. Assume that F is a faithfully flat R-module. If $F \otimes N$ is an S-$F \otimes T$-small submodule of $F \otimes M$, then N is an S-T-small submodule of M.*

Proof. Let $F \otimes N \ll_{S-F\otimes T} F \otimes M$ and $T \subseteq N + K$ for some submodule K of M. Then,

$$F \otimes T \subseteq F \otimes (N + K) = F \otimes N + F \otimes K. \quad (2)$$

Hence, there exists $s \in S$ such that $F \otimes sT = s(F \otimes T) \subseteq F \otimes K$. Thus, $0 \longrightarrow F \otimes sT \longrightarrow F \otimes K$ is exact, so $0 \longrightarrow sT \longrightarrow K$ is exact since F is faithfully flat. Thus, $sT \subseteq K$ and $N \ll_{S-T} M$, as needed. □

4. Conclusions

In this article, we introduced the concepts of S-semiannihilator small submodules and S-T-small submodules as generalizations of S-small submodules. We showed the concepts of annihilator small submodules and T-small submodules are different from the concept of S-small submodules. Several properties, examples, and characterizations of such submodules have been investigated. Moreover, we investigated the properties and the behavior of these structures under homomorphisms, Cartesian product, and localizations.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Barnard, "Multiplication modules," *Journal of Algebra*, vol. 71, no. 1, pp. 174–178, 1981.

[2] F. W. Anderson and K. R. Fuller, *Rings And Categories of Modules*, Springer Verlag Berlin Heidelberg, New York, NY, USA, 1992.

[3] T. Y. Lam, *Lectures On Modules and Rings*, Springer-Verlag, New York, NY, USA, 1999.

[4] S. M. Yaseen, "Semiannihilator small submodules," *International Journal of Scientific Research*, vol. 25, pp. 955–958, 2016.

[5] R. Beyranvand and F. Moradi, "Small submodules with respect to an arbitrary submodules," *Journal Algebra and Related Topics*, vol. 3, no. 2, pp. 43–45, 2015.

[6] F. Wang and H. Kim, *Foundations Of Commutative Rings And Their Modules*, Springer, Singapore, 2016.

[7] D. D. Anderson, T. Arabaci, Ü. Tekir, and S. Koç, "On S-multiplication modules," *Communications in Algebra*, vol. 48, no. 8, pp. 398–3407, 2020.

[8] D. D. Anderson and T. Dumitrescu, "S-noetherian rings," *Communications in Algebra*, vol. 30, no. 9, pp. 4407–4416, 2002.

[9] F. Farshadifar, "S-secondary submodules of a moduleS-secondary submodules of a module," *Communications in Algebra*, vol. 49, no. 4, pp. 1394–1404, 2021.

[10] F. Farshadifar, "S-copure submodules of a moduleS-copure submodules of a module," *Miskolc Mathematical Notes*, vol. 24, pp. 153–163, 2023.

[11] E. Şengelen Sevi̇m, T. Arabaci, Ü. Teki̇r, and S. Koç, "On S-prime submodulesS-prime submodules," *Turkish Journal of Mathematics*, vol. 43, no. 2, pp. 1036–1046, 2019.

[12] F. Farzalipour and P. Ghiasvand, "On S-1-absorbing prime submodules," *Journal of Algebra and Its Applications*, vol. 14, Article ID 2250115, 2022.

[13] G. Ulucak, Ü. Tekir, and S. Koç, "On S-2-absorbing submodules and vn-regular modulesS-2-absorbing submodules and vn-regular modules," *Analele Universitatii Ovidius Constanta- Seria Matematica*, vol. 28, no. 2, pp. 239–257, 2020.

[14] S. Rajaee, "S-small and S-essential submodules," *Journal Algebra and Related Topics*, vol. 10, no. 1, pp. 1–10, 2022.

[15] A. S. Mijbass, "Cancellation modules," M. Sc. Thesis, Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq, 1992.