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Research Article

On the Exterior Degree of a Finite-Dimensional Lie Algebra

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In this paper, we define the exterior degree for a finite-dimensional Lie algebra over the field $\mathbb{F}_q$ and give upper and lower bounds. Also, we give some relations between this concept and commutativity degree, capability, and Schur multiplier.

1. Introduction

One of the interesting concepts in algebra is Lie algebras. Some classical results correspond to a p-group with a specific Lie algebra. It means that we may obtain some results for Lie algebras similar to them in groups. It is worth to note that they are not analogous in general and exact distinctions should be done. In the last decades, the relation between probability theory and group theory has been considered. The probability that two elements of a group commute is called the commutativity degree of a group. Similarly, the commutativity degree of a Lie algebra L can be defined as follows:

$$d(L) = \frac{\left|\{(x,y) \in L \times L \mid [x,y] = 0\}\right|}{|L|^2}. \quad (1)$$

In this paper, we define the concept of the exterior degree of Lie algebras by following the same line in [1] for groups. Furthermore, we obtain some bounds for this notion.

In the following, we recall some concepts and terminologies which will be used in the rest.

From [2], the nonabelian tensor square of L denoted by $L \otimes L$ is defined as the Lie algebra generated by symbols $l \otimes k$ subject to the relations:

$$\begin{aligned} &c(l \otimes k) = cl \otimes k = l \otimes ck, \ (l + l') \otimes k = l \otimes k + l' \otimes k, \\ &l \otimes (k + k') = l \otimes k + l \otimes k', [l, l'] \otimes k = l \otimes [l', k] - l' \otimes [l, k], \\ &l \otimes [k, k'] = [k', l] \otimes k - [k, l] \otimes k', [l \otimes k, l' \otimes k'] = -[k, l] \otimes [l', k'], \end{aligned} \quad (2)$$

where $c \in \mathbb{F}$ and $l, l', k, k' \in L$.

Define $L \otimes L$ to be the subalgebra of $L \otimes L$ generated by all elements of the form $l \otimes l$, where $l \in L$. Now, the exterior square of L, $L \wedge L$, defined to be the quotient $L \otimes L/LL$ and the coset $l \otimes l' + LL$ is denoted by $l \wedge l'$ for all $l, l' \in L$. It is an easy matter to see that

$$\kappa' \colon L \wedge L \mapsto L^2, \quad (3)$$

given by $l \wedge l' \mapsto [l, l']$ is an epimorphism. The kernel of κ' is isomorphic to the well-known Schur multiplier of L, which is denoted by $\mathscr{M}(L)$. Also, $\mathscr{M}(L)$ can be considered on the second member of a maximal defining pair (see [3, 4] for more details).

The set $C_L^{\wedge}(x) = \{y \in L \mid y \wedge x = 0_{L\wedge L}\}$ is called the exterior centralizer and the exterior center of L is the intersection of all exterior centralizers of elements in L and denoted by $Z^{\wedge}(L)$.

We may assume that the reader is familiar to the notion of abelian and Heisenberg Lie algebras. An abelian Lie algebra of dimension n and a Heisenberg Lie algebra of dimension $2m+1$ are denoted by $A(n)$ and $H(m)$, respectively.

2. The Exterior Degree of $A(n)$ and $H(m)$

In this section, the exterior degree of a Lie algebra is defined, and we compute it for abelian and Heisenberg Lie algebras. First of all, we need to remind the following fact.

The presentation of Heisenberg Lie algebras is $H(m) = \langle x_i, y_i, z \mid [x_i, y_i] = z, 1 \le i \le m\rangle$, where $m \ge 1$ obtained in ([1], Example 3).

From now on, all Lie algebras are considered to be finite-dimensional over the finite field $\mathbb{F}_q$.

Definition 1. Let L be a Lie algebra. Then, the exterior degree of L is defined by

$$d^{\wedge}(L) = \frac{\left|\{(x, y) \in L \times L \mid x \wedge y = 0\}\right|}{|L|^2}. \tag{4}$$

We are going to give two equivalent formulas for $d^{\wedge}(L)$ to simplify computations. To do this, we need the following notations and terminologies. Assume that $B^{\wedge}(L) = \{(x, y) \in L \times L \mid x \wedge y = 0\}$. It is easy to see that $\left|B^{\wedge}(L)\right| = \sum_{x\in L} \sum \left|C_L^{\wedge}(x)\right|$. So, $d^{\wedge}(L) = 1/|L|^2 \sum_{x\in L} |C_L^{\wedge}(x)|$.

For each $x \in L$, the map $\varphi_x\colon L \longrightarrow L \wedge L$ defined by $y \mapsto x \wedge y$ is linear. Now, it is clear that $\ker \varphi_x = C_L^{\wedge}(x)$.

In the following lemma, we rewrite $d^{\wedge}(L)$ by the above notations.

Lemma 2. *Let L be a Lie algebra. Then, $d^{\wedge}(L) = 1/|L| \sum_{x\in L} 1/\left|Im\varphi_x\right|$.*

Proof. By rank-nullity theorem, we have $\dim L = \dim \mathrm{Im}\varphi_x + \dim \ker\varphi_x$, but $\ker\varphi_x = C_L^{\wedge}(x)$. So, $\dim L = \dim \mathrm{Im}\varphi_x + \dim C_L^{\wedge}(x)$. Hence,

$$|L| = q^{\dim L} = q^{\dim \mathrm{Im}\varphi_x + \dim C_L^{\wedge}(x)} = \left|\mathrm{Im}\varphi_x\right| \left|C_L^{\wedge}(x)\right|. \tag{5}$$

Considering $d^{\wedge}(L) = 1/|L|^2 \sum_{x\in L} |C_L^{\wedge}(x)|$ and (1), the result follows.

The concept of capability is a useful tool in computing the exterior degree of some Lie algebras. Recall that a Lie algebra L is capable provided that $L \cong H/Z(H)$ for some Lie algebra H. The epicentre of a Lie algebra L, which is denoted $Z^*(L)$, is introduced in ([5], Definition 1.4). The importance of $Z^*(L)$ is due to the fact that L is capable if and only if $Z^*(L) = 0$. In ([6], Lemma 3.1), it is showed that $Z^*(L) = Z^{\wedge}(L)$. So, L is capable if and only if $Z^{\wedge}(L) = 0$. It is well known ([6], Theorems 3.3 and 3.4) that the only capable Lie algebras in the class of abelian and Heisenberg Lie algebras are $A(n)$, where $n \ge 2$ and $H(1)$.

In the following proposition, we give the exterior degree of all abelian algebras. □

Proposition 3. *The exterior degree of abelian Lie algebras can be computed as follows:*

(i) $d^{\wedge}(A(1)) = 1$,

(ii) $d^{\wedge}(A(n)) = q^n + q^{n-1} - 1/q^{2n-1}$, *where* $n \ge 2$.

Proof. (i) It is obvious that $d^{\wedge}(A(1)) = 1$. (ii) Let $A(n) = \langle x_1, \ldots, x_n\rangle$ be a basis of $A(n)$. Then, $A(n) \wedge A(n) \cong \mathscr{M}(A(n))$ by ([6], Corollary 2.5), hence $\dim A(n) \wedge A(n) = n(n-1)/2$ by ([7], Lemma 23). So, $A(n) \wedge A(n) = \langle x_i \wedge x_j \mid 1 \le i < j \le n\rangle$. Assume that $a \in A(n)$. Thus, $a = \alpha_1 x_1 + \ldots + \alpha_n x_n$ such that $\alpha_i \in \mathbb{F}_q$ and $1 \le i \le n$. If $\alpha_1 \ne 0$, then $a \wedge x_j \ne 0$ for all $2 \le j \le n$. We show that $\{a \wedge x_i | 2 \le i \le n\}$ is linearly independent. Let

$$\gamma_2(a \wedge x_2) + \gamma_3(a \wedge x_3) + \gamma_4(a \wedge x_2) + \ldots + \gamma_n(a \wedge x_2) = 0, \tag{6}$$

for all $2 \le \gamma_k \le n$. Then,

$$\begin{gathered}\gamma_2\left(\left(\alpha_1 x_1 \wedge x_2\right) + \left(\alpha_2 x_2 \wedge x_2\right) + \ldots + \left(\alpha_n x_n \wedge x_2\right)\right) + \\ \gamma_3\left(\left(\alpha_1 x_1 \wedge x_3\right) + \left(\alpha_2 x_2 \wedge x_3\right) + \ldots + \left(\alpha_n x_n \wedge x_3\right)\right) + \ldots + \\ \gamma_{n-1}\left(\left(\alpha_1 x_1 \wedge x_{n-1}\right) + \left(\alpha_2 x_2 \wedge x_{n-1}\right) + \ldots + \left(\alpha_n x_n \wedge x_{n-1}\right)\right) + \\ \gamma_n\left(\left(\alpha_1 x_1 \wedge x_n\right) + \left(\alpha_2 x_2 \wedge x_n\right) + \ldots + \left(\alpha_n x_n \wedge x_n\right)\right) = 0,\end{gathered} \tag{7}$$

or equality

$$\gamma_2 \alpha_1 \left(x_1 \wedge x_2\right) + \gamma_3 \alpha_1 \left(x_1 \wedge x_3\right) + \ldots + \gamma_n \alpha_1 \left(x_1 \wedge x_n\right) + \sum_{1\le i<j\le n} \left(\gamma_j \alpha_i - \gamma_i \alpha_j\right)\left(x_i \wedge x_j\right) = 0, \tag{8}$$

Since $\{x_i \wedge x_j \mid 1 \le i < j \le n\}$ is a basis of $A(n) \wedge A(n)$, then

$$\gamma_2\alpha_1 = 0, \gamma_3\alpha_1 = 0, \ldots, \gamma_n\alpha_1 = 0, \gamma_j\alpha_i - \gamma_i\alpha_j = 0 \text{ for all } 2 \le i < j \le n. \quad (9)$$

On the other hand, $\alpha_1 \ne 0$, thus $\gamma_2 = \gamma_3 = \ldots = \gamma_n = 0$. Therefore, $\{a \wedge x_i | 2 \le i \le n\}$ is linearly independent when $\alpha_1 \ne 0$. Hence, $a \wedge x_1 \in \langle a \wedge x_j \mid 2 \le j \le n\rangle$ and $\dim \mathrm{Im}\varphi_a = n-1$. By a similar way, one can see $\dim\mathrm{Im}\varphi_a = n-1$ for all nonzero elements a in $A(n)$. On the other hand, $A(n)$ is capable for all $n \ge 2$ by ([6], Theorem 3.3) and consequently $Z^\wedge(A(n)) = 0$. Therefore, by Lemma 2,

$$\begin{aligned} d^\wedge(A(n)) &= \frac{1}{|A(n)|}\sum_{a\in A(n)}\frac{1}{|\mathrm{Im}\varphi_a|} = \frac{1}{|A(n)|} + \frac{1}{|A(n)|}\sum_{a\notin Z^\wedge(A(n))}\frac{1}{|\mathrm{Im}\varphi_a|} \\ &= \frac{1}{|A(n)|} + \frac{1}{|A(n)|}\frac{|A(n)|-1}{q^{n-1}} = \frac{q^n + q^{n-1} - 1}{q^{2n-1}}. \end{aligned} \quad (10)$$

In order to compute the exterior degree of Heisenberg Lie algebras, the following proposition is useful. □

Proposition 4. *Let L be a noncapable Lie algebra and $N \subseteq Z^\wedge(L)$. Then, $d^\wedge(L) = d^\wedge(L/N)$.*

Proof. We know that $L \wedge L \cong L/N \wedge L/N$ if and only if $N \subseteq Z^\wedge(L)$ by ([6], Corollary 2.3). Hence, $|N|^2|B^\wedge(L/N)| = |B^\wedge(L)|$, which implies that $d^\wedge(L) = d^\wedge(L/N)$.

Now, we are ready to compute the exterior degree of all Heisenberg Lie algebras. □

Proposition 5. *The exterior degree of Heisenberg Lie algebras is as follows.*

(i) $d^\wedge(H(1)) = q^3 + q^2 - 1/q^5$,

(ii) $d^\wedge(H(m)) = q^{2m} + q^{2m-1} - 1/q^{4m-1}$, where $m \ge 2$.

Proof. (i) We have $H(1) = \langle x, y \mid [x,y] = z\rangle$. Since $H(1) \wedge H(1) \cong A(3)$ by ([6], Lemma 3.2), we have $L \wedge L = \langle x\wedge y, x \wedge z, y \wedge z\rangle \cong A(3)$. We know that L is capable by ([6], Theorem 3.4), so $Z^\wedge(L) = 0$. Assume that $a \in L$ and $a \ne 0$, thus $a = \alpha_1 x + \alpha_2 y + \alpha_3 z$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_q$ and

$$\begin{aligned} a \wedge x &= \alpha_2(y \wedge x) + \alpha_3(z \wedge x), \\ a \wedge y &= \alpha_1(x \wedge y) + \alpha_3(z \wedge y), \\ a \wedge z &= \alpha_1(x \wedge z) + \alpha_2(y \wedge z). \end{aligned} \quad (11)$$

One can see that $\{a \wedge x, a \wedge y, a \wedge z\}$ is linearly dependent. So, $1 \le \dim \ \mathrm{Im}\varphi_a \le 2$. If $\alpha_1 \ne 0$, then $a \wedge y$ and $a \wedge z$ are linearly independent. Thus, $\dim\mathrm{Im}\varphi_a = 2$. Similarly, $\dim\mathrm{Im}\varphi_a = 2$ for other nonzero elements a in L. Hence, by Lemma 2,

$$\begin{aligned} d^\wedge(L) &= \frac{1}{|L|}\sum_{a\in L}\frac{1}{|\mathrm{Im}\varphi_a|} = \frac{1}{|L|} + \frac{1}{|L|}\sum_{a\notin Z^\wedge(L)}\frac{1}{|\mathrm{Im}\varphi_a|} = \frac{1}{|L|} + \frac{1}{|L|}\frac{|L|-1}{q^2} \\ &= \frac{q^3 + q^2 - 1}{q^5}. \end{aligned} \quad (12)$$

(ii) We know that L is noncapable by ([6], Theorem 3.4), and so $Z^\wedge(L) \ne 0$. Since $Z^\wedge(L) \le Z(L)$ and $\dim Z(L) = 1$, we have $Z^\wedge(L) = Z(L) = \langle z\rangle$. On the other hand, $L/Z^\wedge(L) \cong A(2m)$, where $m \ge 2$. Thus,

$$d^\wedge(L) = d^\wedge(L/Z^\wedge(L)) = \frac{q^{2m} + q^{2m-1} - 1}{q^{4m-1}}, \quad (13)$$

by Propositions 3 and 4.

At the end of his section, we compute the commutativity degree of some Lie algebras that play an important role in the rest of the paper. □

Example 1

(i) Let $L = \langle x, y \mid [x,y] = x\rangle$ is a Lie algebra over the field. Then, $Z(L) = 0$ and so $|Z(L)| = 1$. Consider

$ad_x\colon L \longrightarrow L$ such that $y \mapsto [x, y]$. If $x \neq 0$, then $\dim\ker ad_x = \dim C_L(x) = 1$ and so $|C_L(x)| = q$.

$$d(L) = \frac{|\{(x, y) \in L \times L \mid [x, y] = 0\}|}{|L|^2} = \frac{1}{|L|^2}\sum_{x\in L}|C_L(x)| = \frac{1}{|L|} + \frac{1}{|L|^2}(|L| - |Z(L)|)q = \frac{q^2 + q - 1}{q^3}. \tag{14}$$

(ii) Let m be a positive integer and $L = H(m) = \langle x_i, y_i, z \mid [x_i, y_i] = z, 1 \leq i \leq m\rangle$ be a Lie algebra over the field $\mathbb{F}_q$. By a similar method in part (i), one can see that $d(L) = q^{2m} + q - 1/q^{2m+1}$.

3. Upper and Lower Bounds for $d^\wedge(L)$

In this section, we give some bounds for the exterior degree of Lie algebras. In addition, the structure of Lie algebras with $d^\wedge(L) = q^2 + q - 1/q^3$ is obtained.

We start with the following theorem, which gives lower and upper bounds for $d^\wedge(L)$.

Theorem 6. *Let L be an arbitrary Lie algebra. Then,*

$$\frac{d(L)}{|\mathcal{M}(L)|} + \frac{|Z^\wedge(L)|}{|L|}\left(1 - \frac{1}{|\mathcal{M}(L)|}\right) \leq d^\wedge(L) \leq d(L) - \left(\frac{q-1}{q}\right)\left(\frac{|Z(L)| - |Z^\wedge(L)|}{|L|}\right). \tag{15}$$

Proof. Since the map $f_x\colon C_L(x) \longrightarrow \mathcal{M}(L)$ with rule $y \mapsto x \wedge y$ is a linear map and $\ker f_x = C_L^\wedge(x)$, we have $|C_L^\wedge(x)|/|C_L(x)| \geq 1/|\mathcal{M}(L)|$. Moreover, $|C_L^\wedge(x)| = |C_L(x)| = |L|$ for all $x \in Z^\wedge(L)$. Hence,

$$\begin{aligned} d^\wedge(L) &= \frac{|Z^\wedge(L)|}{|L|} + \frac{1}{|L|^2}\sum_{x\in L/Z^\wedge(L)}|C_L^\wedge(x)| \geq \frac{|Z^\wedge(L)|}{|L|} + \frac{|L|d(L) - |Z^\wedge(L)|}{|\mathcal{M}(L)|\|L|} \\ &= \frac{d(L)}{|\mathcal{M}(L)|} + \frac{|Z^\wedge(L)|}{|L|}\left(1 - \frac{1}{|\mathcal{M}(L)|}\right). \end{aligned} \tag{16}$$

Since $C_L^\wedge(x) \leq C_L(x)$, $|\mathrm{Im}\varphi_x| \geq q$ for all $x \notin Z^\wedge(L)$ and $|C_L^\wedge(x)| = |L|/|\mathrm{Im}\varphi_x|$, we have

$$\begin{aligned} d^\wedge(L) &= \frac{|Z^\wedge(L)|}{|L|} + \frac{1}{|L|^2}\sum_{x\in Z(L)/Z^\wedge(L)}|C_L^\wedge(x)| + \frac{1}{|L|^2}\sum_{x\in L/Z(L)}|C_L^\wedge(x)| \\ &= \frac{|Z^\wedge(L)|}{|L|} + \frac{1}{|L|}\sum_{x\in Z(L)/Z^\wedge(L)}\frac{1}{|\mathrm{Im}\varphi_x|} + \frac{1}{|L|^2}\sum_{x\in L/Z(L)}|C_L^\wedge(x)| \\ &\leq \frac{|Z^\wedge(L)|}{|L|} + \frac{|Z(L)| - |Z^\wedge(L)|}{q|L|} + \frac{1}{|L|^2}\sum_{x\in L/Z(L)}|C_L(x)| \\ &= d(L) - \left(\frac{q-1}{q}\right)\frac{|Z(L)| - |Z^\wedge(L)|}{|L|}. \end{aligned} \tag{17}$$

By Theorem 6, it is obvious that $d^\wedge(L) \leq d(L)$ for every Lie algebra L. If we have the equality $d^\wedge(L) = d(L)$, then L is called unidegree. Lie algebras with the trivial Schur multiplier are a known example of unidegree Lie algebra. Also, a Lie algebra is called unicentral if $Z^\wedge(L) = Z(L)$. One can easily see that every unidegree Lie algebra is unicentral, by Theorem 6, but the converse is not true. For example, if $L = H(m)$, where $m \geq 2$, then L is unicentral, but it is not unidegree as $d(L) \neq d^\wedge(L)$ by Proposition 5 and Example 1.

The following example shows that the previous theorem makes some computations easier. □

Example 2. Let L be the Lie algebra $\langle x, y \mid [x, y] = x\rangle$. We show that $\mathcal{M}(L)$ is trivial and $d^\wedge(L) = q^2 + q - 1/q^3$ for all $q \geq 2$. First, we compute the Schur multiplier of L by the used method of Hardy and Stitzinger in [3]. Using the notation of [3], we have $[x, y] = x + s_1$, where $\mathcal{M}(L) = \langle s_1\rangle$. By changing the variable as $x' = x + s_1$, we have $s_1 = 0$, and consequently, $\mathcal{M}(L)$ is trivial. So, $d^\wedge(L) = d(L) = q^2 + q - 1/q^3$ by Example 1 and Theorem 6.

We are going to give the precise structure of Lie algebras when $d^\wedge(L) = q^2 + q - 1/q^3$.

First, we need to state the following lemma.

Lemma 7. *Let L_1 and L_2 be two Lie algebras. Then, $d^\wedge(L_1 \oplus L_2) \leq d^\wedge(L_1)d^\wedge(L_2)$.*

Proof. We show that $C^\wedge_{L_1\oplus L_2}((x, y)) \subseteq C^\wedge_{L_1}(x) \oplus C^\wedge_{L_2}(y)$. Assume that $(x_1, y_1) \in C^\wedge_{L_1\oplus L_2}((x, y))$. Then, $(x_1, y_1) \wedge (x, y) = (0, 0)$. The isomorphism

$$(L_1 \oplus L_2) \wedge (L_1 \oplus L_2) \cong (L_1 \wedge L_1) \oplus (L_1 \otimes L_2) \oplus (L_2 \wedge L_2), \tag{18}$$

in ([2], Page 107) implies $x_1 \wedge x = 0$ and $y_1 \wedge y = 0$. Hence, $x_1 \in C^\wedge_{L_1}(x)$ and $y_1 \in C^\wedge_{L_2}(y)$. So, $(x_1, y_1) \in C^\wedge_{L_1}(x) \oplus C^\wedge_{L_2}(y)$. Hence,

$$\begin{aligned} d^\wedge(L_1 \oplus L_2) &= \sum_{(x,y)\in L_1\oplus L_2} \frac{\left|C^\wedge_{L_1\oplus L_2}((x, y))\right|}{\left|L_1 \oplus L_2\right|^2} \leq \sum_{x\in L_1}\sum_{y\in L_2} \frac{\left|C^\wedge_{L_1}(x)\right|\left|C^\wedge_{L_2}(y)\right|}{\left|L_1\right|^2\left|L_2\right|^2} \\ &= \sum_{x\in L_1} \frac{\left|C^\wedge_{L_1}(x)\right|}{\left|L_1\right|^2} \sum_{y\in L_2} \frac{\left|C^\wedge_{L_2}(y)\right|}{\left|L_2\right|^2} = d^\wedge(L_1)d^\wedge(L_2). \end{aligned} \tag{19}$$

The next lemma and proposition are necessary to prove Theorem 10. □

Lemma 8. *Let L be isomorphic to either $\langle x, y \mid [x, y] = x\rangle \oplus A(n)$ or $H(1) \oplus A(n)$, where $n \geq 0$. Then,*

(i) $d^\wedge(\langle x, y \mid [x, y] = x\rangle \oplus A(n)) \leq d^\wedge(\langle x, y \mid [x, y] = x\rangle \oplus A(1))$ such that $n \geq 2$,

(ii) $d^\wedge(H(1) \oplus A(n)) \leq d^\wedge(H(1))$ such that $n \geq 1$.

Proof. (i) We may proceed by induction on n. First, let $n = 2$. Since $d^\wedge(A(1)) = 1$ by part (i) of Proposition 3, we have

$$\begin{aligned} d^\wedge(\langle x, y \mid [x, y] = x\rangle \oplus A(2)) &= d^\wedge((\langle x, y \mid [x, y] = x\rangle \oplus A(1)) \oplus A(1)) \\ &\leq d^\wedge(\langle x, y \mid [x, y] = x\rangle \oplus A(1))d^\wedge(A(1)) = d^\wedge(\langle x, y \mid [x, y] = x\rangle \oplus A(1)), \end{aligned} \tag{20}$$

by Lemma 7. The result follows by induction hypnotises and part (i) of Proposition 3. (ii) The proof of this part is similar to part (i). □

Proposition 9. *Let L be isomorphic to $\langle x, y \mid [x, y] = x\rangle \oplus \langle z\rangle$. Then,*

(i) $\mathcal{M}(L) \cong A(1)$ and $L \wedge L = \langle x \wedge y, y \wedge z\rangle \cong A(2)$,

(ii) L is capable,

(iii) $d^\wedge(L) = 2q^2 - 1/q^4$.

Proof. (*i*) We know that $\mathcal{M}(\langle x, y \mid [x, y] = x\rangle)$ and $\mathcal{M}(\langle z\rangle)$ are trivial by Proposition 3.2 and ([7], Lemma 23), respectively. On the other hand,

$$\dim \mathcal{M}(L) = \dim \mathcal{M}(\langle x, y \mid [x, y] = x\rangle) + \dim \mathcal{M}(\langle z\rangle) + \dim(A(1) \otimes A(1)), \tag{21}$$

by ([3], Theorem 1) and consequently $\mathcal{M}(L) \cong A(1)$. By ([2], Theorem 35 (iii)), we have $\dim L \wedge L = \dim L^2 + \dim \mathcal{M}(L) = 2$. Also,

$$\begin{aligned} x \wedge z &= [x, y] \wedge z = x \wedge [y, z] + y \wedge [z, x]) = 0, \\ [x \wedge y, y \wedge z] &= [x, y] \wedge [y, z] = 0, \end{aligned} \quad (22)$$

imply that $L \wedge L = \langle x \wedge y, y \wedge z \rangle \cong A(2)$, as required. (ii) Since $L/Z(L) \cong \langle x, y \mid [x, y] = x \rangle$, we have $\mathcal{M}(L/Z(L)) = 0$ by Proposition 3.2. The ρ: $\mathcal{M}(L) \longrightarrow \mathcal{M}(L/Z(L))$ is not a monomorphism and so L is capable by ([5], Theorem 4.4). (iii) Assume that $a \in L/Z^{\wedge}(L)$. Thus, $a = \alpha_1 x + \alpha_2 y + \alpha_3 z$ for all $\alpha_i \in \mathbb{F}_q$ such that $1 \leq i \leq 3$. Let $\alpha_1 \neq 0$. Then, $\mathrm{Im}\varphi_a = \langle \alpha_2 (y \wedge x), \alpha_1 (x \wedge y) + \alpha_3 (z \wedge y), \alpha_2 (y \wedge z) \rangle$. If $\alpha_2 = 0$, then $\dim\mathrm{Im}\varphi_a = 1$ and the number of such elements is $(q-1)q$. If $\alpha_2 \neq 0$ and α_2 is nonzero, then $\dim\mathrm{Im}\varphi_a = 2$ and the number of these elements is equal to $(q-1)^2 q$. If $a = \alpha_2 y + \alpha_3 z$ with $\alpha_2 \neq 0$, then $\mathrm{Im}\varphi_a = \langle \alpha_2 (y \wedge x), \alpha_3 (z \wedge y), \alpha_2 (y \wedge z) \rangle$. Again, $\dim\mathrm{Im}\varphi_a = 2$ and the number of elements in this case is equal to $(q-1)q$. By a similar method, if $a = \alpha_3 z$ with $\alpha_3 \neq 0$, then $\dim\mathrm{Im}\varphi_a = 1$ and the number of such elements is equal to $q - 1$. Also, by part (ii), L is capable, which implies that $Z^{\wedge}(L) = 0$. Therefore, by Lemma 2,

$$\begin{aligned} d^{\wedge}(L) &= \frac{1}{|L|} \sum_{a \in L} \frac{1}{|\mathrm{Im}\varphi_a|} = \frac{1}{|L|} + \frac{1}{|L|} \sum_{a \neq 0} \frac{1}{|\mathrm{Im}\varphi_a|} \\ &= \frac{1}{q^3} + \frac{1}{q^3} \left(\frac{q^2 - 1}{q} + \frac{q^3 - q^2}{q^2} \right) = \frac{2q^2 - 1}{q^4}. \end{aligned} \quad (23)$$

Now, we are ready to give the structure of Lie algebras when $d^{\wedge}(L) = q^2 + q - 1/q^3$. □

Theorem 10. *Let L be a Lie algebra. Then, $d^{\wedge}(L) = q^2 + q - 1/q^3$ if and only if L is isomorphic to $\langle x, y \mid [x, y] = x \rangle$.*

Proof. Assume that $d^{\wedge}(L) = q^2 + q - 1/q^3$ thus $d^{\wedge}(L) \leq d(L) \leq q^2 + q - 1/q^3$, by Theorem 6. It implies that $d(L) = q^2 + q - 1/q^3$. On the other hand, $d(L) = q^2 + q - 1/q^3$ if and only if L is isomorphic to $\langle x, y, z \mid [x, y] = z \rangle = H(1) \oplus A(n)$ or $\langle x, y \mid [x, y] = x \rangle \oplus A(n)$, where $n \geq 0$, by using Example 1 and [[8], Page 20]. Since $d^{\wedge}(L) = q^2 + q - 1/q^3$, then L is isomorphic to $\langle x, y \mid [x, y] = x \rangle$ by Lemma 8 and Propositions 5 and 9, as required. The converse follows from Example 2 directly. □

For future research studies, we may pose the following problem.

Problem. Which rational number can be the exterior degree of some Lie algebras? For such rational numbers, we describe the structure of Lie algebras with the exterior degree.

Data Availability

Data sharing is not applicable to this article as no data were collected or analysed in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] P. Niroomand and R. Rezaei, "On the exterior degree of finite groups," *Communications in Algebra*, vol. 39, no. 1, pp. 335–343, 2011.

[2] G. J. Ellis, "A non-abelian tensor product of Lie algebras," *Glasgow Mathematical Journal*, vol. 33, no. 1, pp. 101–120, 1991.

[3] P. Batten, K. Moneyhun, and E. Stitzinger, "On characterizing nilpotent Lie algebras by their multipliers," *Communications in Algebra*, vol. 24, no. 14, pp. 4319–4330, 1996.

[4] P. Batten and E. Stitzinger, "On covers of Lie algebras," *Communications in Algebra*, vol. 24, no. 14, pp. 4301–4317, 1996.

[5] A. R. Salemkar, V. Alamian, and H. Mohammadzadeh, "Some properties of the Schur multiplier and covers of Lie algebras," *Communications in Algebra*, vol. 36, no. 2, pp. 697–707, 2008.

[6] P. Niroomand, M. Parvizi, and F. G. Russo, "Some criteria for detecting capable Lie algebras," *Journal of Algebra*, vol. 384, pp. 36–44, 2013.

[7] K. Moneyhun, "Isoclinisms in Lie algebras," *Algebras Groups and Geometries*, vol. 11, no. 1, pp. 9–22, 1994.

[8] K. Erdmann and M. J. Wildon, *Introduction to Lie Algebras*, Springer, London, UK, 2006.