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Research Article

Some New Families of Exact Solitary Wave Solutions for Pseudo-Parabolic Type Nonlinear Models

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The objective of the current study is to provide a variety of families of soliton solutions to pseudo-parabolic equations that arise in nonsteady flows, hydrostatics, and seepage of fluid through fissured material. We investigate a class of such equations, including the one-dimensional Oskolkov (1D OSK), the Benjamin-Bona-Mahony (BBM), and the Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation. The Exp (-$\phi(\xi)$)-expansion method is used for new hyperbolic, trigonometric, rational, exponential, and polynomial function-based solutions. These solutions of the pseudo-parabolic class of partial differential equations (PDEs) studied here are new and novel and have not been reported in the literature. These solutions depict the hydrodynamics of various soliton shapes that can be utilized to study the nature of traveling wave solutions of other nonlinear PDE's.

1. Introduction

Nonlinear problems have always been of interest to researchers [1–3] due to various applications to practical problems. While several approximate and numerical methods are available, analytical solutions always provide a benchmark for such methods. Several ansatz-based methods like mapping method [4, 5], the Jacobi elliptic function method [6], the new extended direct algebraic method [7], the tanh-coth method [8], the simple equation method [9], the symmetry method [10, 11], and many others are used for handling exact solutions to nonlinear PDEs. Pseudo-parabolic equations, which appear in many branches of mathematics and physics, have a one-time derivative in the highest-order term. They arise, for example, in the study of the flow of fluid in fractured rocks, the consolidation of clay, the shear of second-order fluids, thermodynamics, and the propagation of long, low-amplitude waves.

To solve a nonlinear pseudo-parabolic equation, a numerical approach has been established [12]. The stability of numerical approximations to backward-time parabolic and pseudo-parabolic problems and a relationship between parabolic and pseudo-parabolic difference schemes were also explored, along with a relationship between parabolic and pseudo-parabolic difference schemes. The approach suggested by Sobolev [13] may be used to identify the source of new issues in mathematical physics, and [14] explained why these problems are referred to as pseudo-parabolic problems. Pseudo-parabolic equations apply to the study of a variety of significant physical phenomena, including the seepage of homogeneous fluids through a fissured rock [15], the accumulation of populations, and the conduction of heat between bodies kept at two different temperatures. They are characterized by the occurrence of a time derivative appearing in the highest-order term [16]. Such equations, which comprise the nonlinear pseudo-parabolic differential, are used in many branches of physics and mathematics to explain many physical implications [17]. In this direction, Camassa [18] has obtained some soliton solutions for the pseudo-parabolic equations. Wazwaz [19, 20] has considered analytical solutions to the same class of equations. Johnson [21] discussed some models arising from water waves. The authors [22, 23] discussed the trigonometric and hyperbolic-type soliton solutions of the same class. To provide readers with additional information, reference [24] is cited.

The generalized pseudo-parabolic equations are one specific instance of this class. We consider the following generalized form of Benjamin Bona Mahony equation:

$$q_t - q_{xxt} - \alpha q_{xx} + \gamma q_x + g(q) = 0, \tag{1}$$

where $g(q)$ is a C^2-smooth nonlinear function, α is a positive constant, γ is a real constant, and $q(x,t)$ represents the velocity of fluid in the horizontal direction. Peregrine [25] and Benjamin et al. [26] suggested the regular long-wave equation for the widely used KdV equation as the specific case of $g(q) = qq_x$ with $\alpha = 0, \gamma = 1$ in equation (1), i.e.,

$$q_t + q_{xxt} + q_x + qq_x = 0. \tag{2}$$

The equation for BBMPB is obtained by substituting $g(q) = \theta qq_x + \beta q_{\mathrm{xxx}}$ in the following equation (1) to get

$$q_t - q_{xxt} - \alpha q_{xx} + \gamma q_x + \theta qq_x + \beta q_{xxx} = 0. \tag{3}$$

The following equation is obtained for $\alpha = \beta = 0$ in the above equation, which is the following BBM equation:

$$q_t - q_{xxt} + \gamma q_x + \theta qq_x = 0, \tag{4}$$

in which $\gamma, \theta \in \mathbb{R}$ and $\theta \neq 0$ is a parameter. We also consider the one-dimensional OSK equation which models incompressible viscoelastic Kelvin-Voigt fluid [27]

$$q_t - \lambda q_{xxt} - \alpha q_{xx} + qq_x = 0. \tag{5}$$

This article aims to find new families of exact soliton solutions for the nonlinear pseudo-parabolic type models arising in mathematical physics using the Exp (-$\phi(\xi)$)-expansion method [28–30]. This method is an extremely powerful tool for dealing with soliton solutions of nonlinear PDEs. It provides the hyperbolic, trigonometric, rational, exponential, and polynomial functions-based soliton solutions to the nonlinear PDEs. This is the actual limitation of the used methodology here. For the more general soliton solutions, we need to enhance our methodology also. Akcagil et al. [31] reported the solutions to the class of trigonometric, hyperbolic, and rational soliton solutions. We want to build on earlier research to advance our quest for a wealth of fresh traveling wave solutions. Our findings include the dark, bright, rational, exponential, polynomial, and solitary wave solutions. The solutions provided in this research are exclusively novel and valid and have not been previously presented for this class of equations.

The structure of this paper is as follows: In Section 2, we present a layout for the Exp (-$\phi(\xi)$)-expansion method. In Sections 3–5, the BBMPB, the 1D OSK and BBM equations respectively have been studied to obtain various solutions using this method. The graphical representations of these solutions are presented in Section 6. In Section 7 the conclusion and some possible directions of future study are mentioned.

2. Floor Plan for the Exp ($-\phi(\xi)$)-Expansion Method

This section deals with the brief floor plan for the Exp ($-\phi(\xi)$)-expansion method [28–30] to find the explicit soliton solutions. We give here the main steps of the method. We consider an explicit form of the nonlinear PDE as follows:

$$P(q, q_x, q_t, q_{xt} \ldots) = 0, \tag{6}$$

where q is the dependent variable.

Step 1. To reduce the number of independent variables of the equation (6), we introduce the following traveling wave transformation:

$$\begin{aligned} q(x,t) &= V(\xi), \\ \xi &= x - wt, \end{aligned} \tag{7}$$

where w is a nonzero real parameter indicating wave speed. Then, by adopting the traveling wave transformation, the nonlinear PDE (6) becomes

$$Q(V, V', V'', \cdots) = 0. \tag{8}$$

Note that for cases, the invariance of the transformation (7) serves as the existence criterion for the traveling wave solution. Also, the term traveling wave is due to the time behavior of the dependent variable.

Step 2. The general solution to (8) is appropriated as a polynomial in exp ($-\phi(\xi)$)

$$V(\xi) = \sum_{i=0}^{\mathscr{M}} a_i \left(e^{-\phi(\xi)}\right)^i, \tag{9}$$

where $a_i\,(0 < i \leq \mathscr{M})$ are the coefficients to be determined later, and $\phi(\xi)$ is the solution of the following equation:

$$\phi'(\xi) = e^{-\phi(\xi)} + \lambda_1 e^{\phi(\xi)} + \lambda_2, \tag{10}$$

where a_i, λ_1, and λ_2 are real constant parameters and $\mathscr{M}$ can be determined by the homogeneous balance principle. There are five cases for the $\phi(\xi)$;

Case I: When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\phi(\xi) = \ln\left[\frac{1}{2\lambda_1}\left(-\sqrt{\lambda_1^2 - 4\lambda_2}\tanh\left(\frac{\sqrt{\lambda_1^2 - 4\lambda_2}}{2}(\xi + c_1)\right) - \lambda_2\right)\right]. \quad (11)$$

Case II: For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$\phi(\xi) = \ln\left[\frac{1}{2\lambda_1}\left(\sqrt{4\lambda_2 - \lambda_1^2}\tan\left(\frac{\sqrt{4\lambda_2 - \lambda_1^2}}{2}(\xi + c_1)\right) - \lambda_2\right)\right]. \quad (12)$$

Case III: For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$\phi(\xi) = -\ln\left[\frac{\lambda_2}{e^{\lambda_2(\xi + c_1)} - 1}\right]. \quad (13)$$

Case IV: For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$\phi(\xi) = \ln\left[\frac{2\lambda_2(\xi + c_1) + 4}{\lambda_2^2(\xi + c_1)}\right]. \quad (14)$$

Case V: For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$\phi(\xi) = \ln(\xi + c_1). \quad (15)$$

Step 3. Substituting the equation (9) into (8), the left hand side of the ODE (8) becomes a polynomial in $e^{-\phi(\xi)}$. By comparing the coefficients of both sides, we get a system of algebraic equations that is solvable by some symbolic software like Mathematica 11.3 or Maple.

Step 4. Putting the values of $\phi(\xi)$ from equations (11)–(15) one by one in (9), we get solutions for ODE (8). Replacing ξ by $x - wt$, we get solutions for our PDE (6).

Now we are going to apply this method for some important nonlinear PDEs to have explicit soliton solutions.

3. The Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) Equation

Here, we first look at the equation BBMPB given by

$$q_t - q_{xxt} - \alpha q_{xx} + \gamma q_x + \theta q q_x + \beta q_{xxx} = 0. \quad (16)$$

By using traveling wave transformation $q(x,t) = V(\xi), \xi = x - wt$, we obtain the following nonlinear ODE:

$$(\gamma - w)V - \alpha V' + \frac{\theta}{2}V^2 + (\beta + w)V'' = 0. \quad (17)$$

By comparing the highest order of linear term V'' with the highest degree of nonlinear term V^2 in (17), we decide the order of V as $O(V) = 2$. Based on this order, we deduce that the solution of (17) is of the following type:

$$V(\xi) = a_0 + a_1\left(e^{-\phi(\xi)}\right) + a_1\left(e^{-\phi(\xi)}\right)^2. \quad (18)$$

Such that $\phi(\xi)$ satisfies (10). By substituting (18) into (17), we transform the right-hand side of the equation into a polynomial in $e^{-\phi(\xi)}$, and then by comparing coefficients the following system arises:

$$\begin{aligned}
a_0(\gamma - w) + \alpha a_1\lambda_1 + a_0^2\frac{\theta}{2} + \lambda_1(w + \beta)(a_1\lambda_2 + 2a_2\lambda_1) &= 0,\\
a_1(\gamma - w) + \alpha(a_1\lambda_2 + 2a_2\lambda_1) + \theta a_0 a_1 + (w + \beta)\left(2a_1\lambda_1 + 6a_2\lambda_1\lambda_2 + a_1\lambda_2^2\right) &= 0,\\
a_2(\gamma - w) + 2\alpha(a_1 + 2a_2\lambda_2) + \theta\left(\frac{a_1^2}{2} + a_0a_2\right) + (w + \beta)(3a_1\lambda_2 + 7a_2\lambda_1\lambda_2 + 4a_2\lambda_2) &= 0,\\
2\alpha a_2 + \theta a_1 a_2 + 2(w + \beta)(a_1 + 5a_2\lambda_2) &= 0,\\
\frac{\theta}{2}a_2^2 + 6a_2(w + \beta) &= 0.
\end{aligned} \quad (19)$$

We solve this system with Mathematica 11.3 to get the following set of parameters.

First set

$$\alpha=\frac{5a_2(\gamma-w)}{6(a_1-a_2\lambda_2)},$$
$$\lambda_1=\frac{a_1(2a_2\lambda_2-a_1)}{4a_2^2},$$
$$\theta=\frac{2a_2(w-\gamma)}{(a_1-a_2\lambda_2)^2}, \tag{20}$$
$$\beta=\frac{-6wa_2^2\lambda_2^2+12wa_1a_2\lambda_2+\gamma a_2^2-wa_1^2-wa_2^2}{6(a_1-a_2\lambda_2)^2},$$
$$a_0=\frac{a_1^2}{4a_2}.$$

Inserting these parameters into (18), we obtain

$$V(\xi)=\frac{a_1^2}{4a_2}+a_1\left(e^{-\phi(\xi)}\right)+a_2\left(e^{-\phi(\xi)}\right)^2. \tag{21}$$

Thus, the solutions for (10) becomes

Case I: When $\lambda_1^2-4\lambda_2>0$ and $\lambda_1\neq 0$,

$$\phi=\ln\left[\frac{1}{2\lambda_1}\left(-\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{4a_2^2}}(\xi+c_1)\right)-\lambda_2\right)\right]. \tag{22}$$

Case II: For $\lambda_1^2-4\lambda_2<0$ and $\lambda_1\neq 0$,

$$\phi=\ln\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{a_2^2}}\tan\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{4a_2^2}}(\xi+c_1)\right)-\lambda_2\right)\right]. \tag{23}$$

Case III: For $\lambda_1=0$ and $\lambda_2\neq 0$,

$$\phi=-\ln\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)}-1}\right]. \tag{24}$$

Case IV: For $\lambda_1\neq 0$, $\lambda_2\neq 0$, and $\lambda_1^2-4\lambda_2=0$,

$$\phi=\ln\left[\frac{\lambda_2(\xi+c_1)+4}{\left(\lambda_2^2(\xi+c_1)\right.}\right]. \tag{25}$$

Case V: For $\lambda_1=0$, $\lambda_2=0$, (or $\lambda_1^2-4\lambda_2=0$),

$$\phi=\ln(\xi+c_1)\cdot \tag{26}$$

Now by putting (22)–(26) one by one into (21) the solutions for our ODE are given by

Family 5. When $\lambda_1^2-4\lambda_2>0$ and $\lambda_1\neq 0$,

$$\begin{aligned}V_{11}(x,t)=\frac{a_1^2}{4a_2}&-a_1\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{4a_2^2}}(\xi+c_1)\right)+\lambda_2\right)\right]^{-1}\\&-a_2\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{4a_2^2}}(\xi+c_1)\right)+\lambda_2\right)\right]^{-2}.\end{aligned} \tag{27}$$

Family 6. For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$V_{12}(x,t) = \frac{a_1^2}{4a_2} + a_1\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{a_1^2 + 2a_1a_2\lambda_2 - \lambda_2^2a_2^2}{a_2^2}}\tan\left(\sqrt{\frac{a_1^2 + 2a_1a_2\lambda_2 - \lambda_2^2a_2^2}{4a_2^2}}(\xi + c_1)\right) - \lambda_2\right)\right]^{-1} + a_2\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{a_1^2 + 2a_1a_2\lambda_2 - \lambda_2^2a_2^2}{a_2^2}}\tan\left(\sqrt{\frac{a_1^2 + 2a_1a_2\lambda_2 - \lambda_2^2a_2^2}{4a_2^2}}(\xi + c_1)\right) - \lambda_2\right)\right]^{-2}. \quad (28)$$

Family 7. For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$V_{13}(x,t) = \frac{a_1^2}{4a_2} + a_1\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)} - 1}\right] + a_2\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)} - 1}\right]^2. \quad (29)$$

Family 8. For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_{14}(x,t) = \frac{a_1^2}{4a_2} + a_1\left[\frac{\lambda_2(\xi + c_1) + 4}{\left(\lambda_2^2(\xi + c_1)\right)}\right]^{-1} + a_2\left[\frac{\lambda_2(\xi + c_1) + 4}{\left(\lambda_2^2(\xi + c_1)\right)}\right]^{-2}. \quad (30)$$

Family 9. For $\lambda_1 = 0$, $\lambda_2 = 0$, (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$V_{15}(x,t) = \frac{a_1^2}{4a_2} + a_1(\xi + c_1)^{-1} + a_2(\xi + c_1)^{-2}. \quad (31)$$

where $\xi = x - wt$ in all above cases.

Second set is as follows:

$$\alpha = \frac{5a_2(w - \gamma)}{6(a_1 - a_2\lambda_2)},$$
$$\lambda_1 = \frac{a_1(2a_2\lambda_2 - a_1)}{4a_2^2},$$
$$\theta = \frac{2a_2(\gamma - w)}{(a_1 - a_2\lambda_2)^2}, \quad (32)$$
$$\beta = \frac{-6wa_2^2\lambda_2^2 + 12wa_1a_2\lambda_2 - \gamma a_2^2 - 6wa_1^2 - wa_2^2}{6(a_1 - a_2\lambda_2)^2},$$
$$a_0 = \frac{8a_1a_2\lambda_2 - 4a_1^2\lambda_2^2 - 3a_1^2}{4a_2}.$$

Inserting these parameters into (18), we obtain.

$$V(\xi) = \frac{8a_1a_2\lambda_2 - 4a_1^2\lambda_2^2 - 3a_1^2}{4a_2} + a_1\left(e^{-\phi(\xi)}\right) + a_2\left(e^{-\phi(\xi)}\right)^2. \quad (33)$$

Thus, the solutions for (10) becomes

Case I: When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[\frac{1}{2\lambda_2}\left(-\sqrt{\frac{\lambda_2^2a_2^2 - 2a_1a_2\lambda_2 - a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_2^2 - 2a_1a_2\lambda_2 - a_1^2}{4a_2^2}}(\xi + c_1)\right) - \lambda_2\right)\right]. \quad (34)$$

Case II: For $\lambda_1^2-4\lambda_2<0$ and $\lambda_1\neq 0$,

$$\phi=\ln\left[\frac{1}{2\lambda_2}\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{a_2^2}}\tan\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{4a_2^2}}(\xi+c_1)\right)-\lambda_2\right)\right]. \tag{35}$$

Case III: For $\lambda_1=0$ and $\lambda_2\neq 0$,

$$\phi=-\ln\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)}-1}\right]. \tag{36}$$

Case IV: For $\lambda_1\neq 0$, $\lambda_2\neq 0$ and $\lambda_1^2-4\lambda_2=0$,

$$\phi=\ln\left[\frac{\lambda_2(\xi+c_1)+4}{\left(\lambda_2^2(\xi+c_1)\right.}\right]. \tag{37}$$

Case V: For $\lambda_1=0$, $\lambda_2=0$ (or $\lambda_1^2-4\lambda_2=0$),

$$\phi=\ln(\xi+c_1). \tag{38}$$

Now by putting (34)–(38) one by one into (33) the solutions for our ODE are given as follows.

Family 10. When $\lambda_1^2-4\lambda_2>0$ and $\lambda_1\neq 0$,

$$\begin{aligned}V_{16}(x,t)={}&\frac{8a_1a_2\lambda_2-4a_1^2\lambda_2^2-3a_1^2}{4a_2}-a_1\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{4a_2^2}}(\xi+c_1)\right)+\lambda_2\right)\right]^{-1}\\&-a_2\left[\frac{1}{2\lambda_2}\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{a_2^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_2^2-2a_1a_2\lambda_2-a_1^2}{4a_2^2}}(\xi+c_1)\right)+\lambda_2\right)\right]^{-2}.\end{aligned} \tag{39}$$

Family 11. For $\lambda_1^2-4\lambda_2<0$ and $\lambda_1\neq 0$,

$$\begin{aligned}V_{17}(x,t)={}&\frac{8a_1a_2\lambda_2-4a_1^2\lambda_2^2-3a_1^2}{4a_2}+a_1\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{a_2^2}}\tan\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{4a_2^2}}(\xi+c_1)\right)-\lambda_2\right)\right]^{-1}\\&+a_2\left[\frac{1}{2\lambda_2}\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{a_2^2}}\tan\left(\sqrt{\frac{a_1^2+2a_1a_2\lambda_2-\lambda_2^2a_2^2}{4a_2^2}}(\xi+c_1)\right)-\lambda_2\right)\right]^{-2}\end{aligned} \tag{40}$$

Family 12. For $\lambda_1=0$ and $\lambda_2\neq 0$,

$$V_{18}(x,t)=\frac{8a_1a_2\lambda_2-4a_1^2\lambda_2^2-3a_1^2}{4a_2}+a_1\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)}-1}\right]+a_2\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)}-1}\right]^2. \tag{41}$$

Family 13. For $\lambda_1\neq 0$, $\lambda_2\neq 0$ and $\lambda_1^2-4\lambda_2=0$,

$$V_{19}(x,t)=\frac{8a_1a_2\lambda_2-4a_1^2\lambda_2^2-3a_1^2}{4a_2}+a_1\left[\frac{\lambda_2(\xi+c_1)+4}{\left(\lambda_2^2(\xi+c_1)\right.}\right]^{-1}+a_2\left[\frac{\lambda_2(\xi+c_1)+4}{\left(\lambda_2^2(\xi+c_1)\right.}\right]^{-2}. \tag{42}$$

Family 14. For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$V_{110}(x,t) = -\frac{3a_1^2}{4a_2} + a_1(\xi + c_1)^{-1} + a_2(\xi + c_1)^{-2}. \quad (43)$$

where $\xi = x - wt$ in all above cases.

Third set is as follows:

$$\begin{aligned}
\alpha &= -\frac{\theta a_1}{2},\\
\lambda_1 &= \frac{a_0(a_1\lambda_2 - a_0)}{a_1^2},\\
w &= -\frac{\theta a_1\lambda_2}{2} + \theta a_0 + \gamma,\\
\beta &= \frac{\theta a_1\lambda_2}{2} - \theta a_0 - \gamma,\\
a_0 &= a_0,\\
a_1 &= a_1,\\
a_2 &= 0,\\
\lambda_2 &= \lambda_2,\\
\theta &= \theta.
\end{aligned} \quad (44)$$

Inserting these parameters into (18), we obtain

$$V(\xi) = a_0 + a_1\left(e^{-\phi(\xi)}\right). \quad (45)$$

Thus, the solutions for (10) becomes

Case I: When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[\frac{1}{2\lambda_1}\left(-\sqrt{\frac{\lambda_2^2a_1^2 - 4a_1a_0\lambda_2 + 4a_0^2}{a_1^2}}\tanh\left(\sqrt{\frac{\lambda_2^2a_1^2 - 4a_1a_0\lambda_2 + 4a_0^2}{a_1^2}}(\xi + c_1)\right) - \lambda_2\right)\right]. \quad (46)$$

Case II: For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{4a_1a_0\lambda_2 - \lambda_2^2a_1^2 - 4a_0^2}{a_1^2}}\tan\left(\sqrt{\frac{4a_1a_0\lambda_2 - \lambda_2^2a_1^2 - 4a_0^2}{a_1^2}}(\xi + c_1)\right) - \lambda_2\right)\right]. \quad (47)$$

Case III: For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$\phi = -\ln\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)} - 1}\right]. \quad (48)$$

Case IV: For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$\phi = \ln\left[\frac{\lambda_2(\xi + c_1) + 4}{\left(\lambda_2^2(\xi + c_1)\right)}\right]. \quad (49)$$

Case V: For $\lambda_1 = 0$, $\lambda_2 = 0$, (or $\lambda_1^2 - 4\lambda_2 = 0$)

$$\phi = \ln(\xi + c_1). \quad (50)$$

Now by putting (34)–(38) one by one into (33) the solutions for our ODE are given by

Family 15. When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$V_{111}(x,t) = a_0 - a_1\left[\frac{1}{2\lambda_1}\left(-\sqrt{\frac{\lambda_2^2 a_1^2 - 4a_1 a_0 \lambda_2 + 4a_0^2}{a_1^2}}\tanh\left(\sqrt{\frac{\lambda_2^2 a_1^2 - 4a_1 a_0 \lambda_2 + 4a_0^2}{a_1^2}}(\xi + c_1)\right) - \lambda_2\right)\right]^{-1}. \quad (51)$$

Family 16. For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$V_{112}(x,t) = a_0 - a_1\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{4a_1 a_0 \lambda_2 - \lambda_2^2 a_1^2 - 4a_0^2}{a_1^2}}\tan\left(\sqrt{\frac{4a_1 a_0 \lambda_2 - \lambda_2^2 a_1^2 - 4a_0^2}{a_1^2}}(\xi + c_1)\right) - \lambda_2\right)\right]^{-1}. \quad (52)$$

Family 17. For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$V_{113}(x,t) = a_0 + a_1\left[\frac{\lambda_2}{e^{\lambda_2(\xi+c_1)} - 1}\right]. \quad (53)$$

Family 18. For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_{114}(x,t) = a_0 + a_1\left[\frac{\lambda_2(\xi + c_1) + 4}{\left(\lambda_2^2(\xi + c_1)\right)}\right]^{-1}. \quad (54)$$

Family 19. For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$V_{115}(x,t) = a_0 + a_1(\xi + c_1)^{-1}. \quad (55)$$

where $\xi = x - wt$ in all above cases.

4. The One-Dimensional Oskolkov (OSK) Equation

The second model, OSK, is given by

$$q_t - \lambda q_{xxt} - \alpha q_{xx} + q q_x = 0. \quad (56)$$

By using the wave transformation $q(x,t)$, $\xi = x - wt$, we obtain the following nonlinear ODE:

$$-wV - \alpha V' + \frac{1}{2}V^2 + \lambda V'' = 0. \quad (57)$$

By comparing the highest order of linear term V'' with the highest degree of nonlinear term V^2 in (57), we decide the order of V as $O(V) = 2$. Based on this order, we deduce that the solution of (57) is of the type.

$$V(\xi) = a_0 + a_1\left(e^{-\phi(\xi)}\right) + a_2\left(e^{-\phi(\xi)}\right)^2. \quad (58)$$

Such that $\phi(\xi)$ satisfies (10). By substituting (58) into (57), we transform the right-hand side of the equation into a polynomial in $e^{-\phi(\xi)}$, and then by comparing coefficients the following system arises:

$$\begin{aligned}
-wa_0 + \alpha a_1\lambda_1 + \frac{a_0^2}{2} + \lambda\lambda_1(a_1\lambda_2 + 2a_2\lambda_1) &= 0,\\
-wa_1 + \alpha(a_1\lambda_2 + 2a_2\lambda_1) + a_0 a_1 + \lambda\left(2a_1\lambda_1 + 6a_2\lambda_1\lambda_2 + a_1\lambda_2^2\right) &= 0,\\
-wa_2 + \alpha(a_1 + 2a_2\lambda_2) + \frac{a_1^2}{2} + a_0 a_2 + \lambda(3a_1\lambda_2 + 7a_2\lambda_1\lambda_2 + 4a_2\lambda_2) &= 0,\\
2\alpha a_2 + a_1 a_2 + 2\lambda(a_1 + 5a_2\lambda_2) &= 0,\\
\frac{a_2^2}{2} + 6a_2\lambda &= 0.
\end{aligned} \quad (59)$$

We solve this system for desired constants with Mathematica 11.3 that leads to the following set of parameters:

$$\begin{aligned}
\alpha &= -5\lambda w\lambda_2 + \frac{5}{6}\left(6\lambda\lambda_2 - \sqrt{6\lambda}\right)w,\\
\lambda_1 &= \frac{6\lambda\lambda_2^2 - 1}{\lambda},\\
a_2 &= -12\lambda w,\\
a_0 &= 3\lambda w\lambda_2^2 - \left(6\lambda\lambda_2 \pm \sqrt{6\lambda}\right)w\lambda_2 + \frac{3}{2}w,\\
a_1 &= -2\left(6\lambda\lambda_2 + \sqrt{6\lambda}\right)w.
\end{aligned} \tag{60}$$

Inserting these parameters into (58), we obtain:

$$V(\xi) = 3\lambda w\lambda_2^2 - \left(6\lambda\lambda_2 \pm \sqrt{6\lambda}\right)w\lambda_2 + \frac{3}{2}w - 2w\left(6\lambda\lambda_2 + \sqrt{6\lambda}\right)\left(e^{-\phi(\xi)}\right) - 12\lambda w\left(e^{-\phi(\xi)}\right)^2. \tag{61}$$

Thus, the solutions for (10) becomes

Case I: When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\phi(\xi) = \ln\left[\frac{1}{2\lambda_1}\left(-\sqrt{\frac{1-23\lambda\lambda_2^2}{\lambda}}\tanh\left(\sqrt{\frac{1-23\lambda\lambda_2^2}{4\lambda}}\left(\xi + c_1\right)\right) - \lambda_2\right)\right]. \tag{62}$$

Case II: For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$\phi(\xi) = \ln\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{23\lambda\lambda_2^2 - 1}{\lambda}}\tan\left(\sqrt{\frac{23\lambda\lambda_2^2 - 1}{4\lambda}}\left(\xi + c_1\right)\right) - \lambda_2\right)\right]. \tag{63}$$

Case III: For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$\phi(\xi) = -\ln\left[\frac{\lambda_2}{e^{\lambda_2(\xi + c_1)} - 1}\right]. \tag{64}$$

Case IV: For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$\phi(\xi) = \ln\left[\frac{2\lambda_2(\xi + c_1) + 4}{\lambda_2^2(\xi + c_1)}\right]. \tag{65}$$

Case V: For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$\phi(\xi) = \ln\left(\xi + c_1\right). \tag{66}$$

Now by putting (62)–(66) one by one into (61) the solutions for our ODE are given by

Family 20. When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\begin{aligned}
V_{21}(x,t) = \epsilon_0 - \epsilon_1&\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{1-23\lambda\lambda_2^2}{\lambda}}\tanh\left(\sqrt{\frac{1-23\lambda\lambda_2^2}{4\lambda}}\left(\xi + c_1\right)\right) + \lambda_2\right)\right]^{-1}\\
&- 12\lambda w\left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{1-23\lambda\lambda_2^2}{\lambda}}\tanh\left(\sqrt{\frac{1-23\lambda\lambda_2^2}{4\lambda}}\left(\xi + c_1\right)\right) + \lambda_2\right)\right]^{-2}.
\end{aligned} \tag{67}$$

Family 21. For. $\lambda_1^2 - 4\lambda_2 < 0$. and $\lambda_1 \neq 0$,

$$V_{22}(x,t) = \epsilon_0 + \epsilon_1 \left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{23\lambda\lambda_2^2 - 1}{\lambda}} \tan\left(\sqrt{\frac{23\lambda\lambda_2^2 - 1}{4\lambda}}(\xi + c_1)\right) - \lambda_2\right)\right]^{-1} - 12\lambda w \left[\frac{1}{2\lambda_1}\left(\sqrt{\frac{23\lambda\lambda_2^2 - 1}{\lambda}} \tan\left(\sqrt{\frac{23\lambda\lambda_2^2 - 1}{4\lambda}}(\xi + c_1)\right) - \lambda_2\right)\right]^{-2}. \tag{68}$$

Family 22. For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$V_{23}(x,t) = \epsilon_0 + \epsilon_1 \left[\frac{\lambda_2}{e^{\lambda_2(\xi + c_1)} - 1}\right] - 12\lambda w \left[\frac{\lambda_2}{e^{\lambda_2(\xi + c_1)} - 1}\right]^2. \tag{69}$$

Family 23. For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_{24}(x,t) = \epsilon_0 + \epsilon_1 \left[\frac{2\lambda_2(\xi + c_1) + 4}{\lambda_2^2(\xi + c_1)}\right]^{-1} - 12\lambda w \left[\frac{2\lambda_2(\xi + c_1) + 4}{\lambda_2^2(\xi + c_1)}\right]^{-2}. \tag{70}$$

Family 24. For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$V_{25}(x,t) = \epsilon_0 + \epsilon_1 (\xi + c_1)^{-1} - 12\lambda w (\xi + c_1)^{-2}, \tag{71}$$

where $\xi = x - wt$ in all above cases and $\epsilon_0 = 3\lambda w\lambda_2^2 - (6\lambda\lambda_2 \pm \sqrt{6\lambda})w\lambda_2 + 3/2w, \epsilon_1 = -2(6\lambda\lambda_2 + \sqrt{6\lambda})w$ for all families.

5. The Benjamin-Bona-Mahony (BBM) Equation

The third equation, BBM, is given by

$$q_t - q_{xxt} + \gamma q_x + \theta q q_x = 0. \tag{72}$$

By using the traveling wave transformation $q(x,t) = V(\xi), \xi = x - wt$, we obtain the following nonlinear ODE:

$$(w - \gamma)V + \frac{\theta}{2}V^2 + wV'' = 0. \tag{73}$$

By comparing the highest order of linear term V'' with the highest degree of nonlinear term V^2 in (73), we decide the order of V as $O(V) = 2$. Based on this order, we deduce that the solution of (73) is of the following type:

$$V(\xi) = a_0 + a_1\left(e^{-\phi(\xi)}\right) + a_2\left(e^{-\phi(\xi)}\right)^2, \tag{74}$$

Such that $\phi(\xi)$ satisfies (10). By substituting (74) into (73), we transform the right-hand side of the equation into a polynomial in $e^{-\phi(\xi)}$, and then by comparing coefficients the following system arises:

$$\begin{aligned}
a_0(\gamma - w) - a_0^2\frac{\theta}{2} + \lambda_1 w(a_1\lambda_2 + 2a_2\lambda_1) &= 0,\\
a_1(\gamma - w) - \theta a_0 a_1 + w\left(2a_1\lambda_1 + 6a_2\lambda_1\lambda_2 + a_1\lambda_2^2\right) &= 0,\\
a_2(\gamma - w) - \frac{\theta}{2}a_1^2 - \theta a_0 a_2 + w\left(3a_1\lambda_2 + 8a_2\lambda_1\lambda_2 + 4a_2\lambda_2^2\right) &= 0,\\
\theta a_1 a_2 - 2w(a_1 + 5a_2\lambda_2) &= 0,\\
\frac{\theta}{2}a_2^2 - 6a_2 w &= 0.
\end{aligned} \tag{75}$$

We solve this system with Mathematica 11.3 for desired constants that lead to the following two sets of parameters.

First set is as follows:

$$\begin{aligned} a_0 &= \frac{2(\gamma - 6w\lambda_1 - w)}{\theta}, \\ a_1 &= -\frac{12}{\theta}\sqrt{\frac{\gamma - 4w\lambda_1 - w}{w}}w, \\ a_2 &= -\frac{12w}{\theta}, \\ w &= w, \\ \theta &= \theta, \\ \lambda_1 &= \lambda_1, \\ \lambda_2 &= \sqrt{\frac{\gamma + 4w\lambda_1 - w}{w}}. \end{aligned} \tag{76}$$

Inserting these parameters into (74), we obtain

$$V(\xi) = \frac{2(\gamma - 6w\lambda_1 - w)}{\theta} - \frac{12}{\theta}\sqrt{\frac{\gamma - 4w\lambda_1 - w}{w}}w\left(e^{-\phi(\xi)}\right) - \frac{12w}{\theta}\left(e^{-\phi(\xi)}\right)^2. \tag{77}$$

Thus, the solution for (10) becomes

Case I: When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[-\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4\lambda_1^2 w}}\tanh\left(\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4w}}(\xi + c_1)\right) - \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]. \tag{78}$$

Case II: For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4\lambda_1^2 w}}\tan\left(\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4w}}(\xi + c_1)\right) - \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]. \tag{79}$$

Case III: For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$\phi = -\ln\left[\frac{\sqrt{\gamma + 4w\lambda_1 - w/w}}{e^{\sqrt{\gamma + 4w\lambda_1 - w/w}\,(\xi + c_1)} - 1}\right]. \tag{80}$$

Case IV: For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$\phi = \ln\left[\frac{2\sqrt{\gamma + 4w\lambda_1 - w/w}\,(\xi + c_1) + 4}{\gamma + 4w\lambda_1 - w/w(\xi + c_1)}\right]. \tag{81}$$

Case V: For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$\phi = \ln(\xi + c_1). \tag{82}$$

Now by putting (78)–(82) one by one into (77) the solutions for our ODE are given by

Family 25. When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$V_{31}(x,t)=\frac{2(\gamma-6w\lambda_1-w)}{\theta}+\delta_1\left[\sqrt{\frac{\gamma-4\lambda_1(w-1)-w}{4\lambda_1^2w}}\tanh\left(\sqrt{\frac{\gamma-4\lambda_1(w-1)-w}{4w}}(\xi+c_1)\right)+\sqrt{\frac{\gamma+4w\lambda_1-w}{4\lambda_1^2w}}\right)\right]^{-1}$$
$$+\delta_2\left[\sqrt{\frac{\gamma-4\lambda_1(w-1)-w}{4\lambda_1^2w}}\tanh\left(\sqrt{\frac{\gamma-4\lambda_1(w-1)-w}{4w}}(\xi+c_1)\right)+\sqrt{\frac{\gamma+4w\lambda_1-w}{4\lambda_1^2w}}\right)\right]^{-2}. \tag{83}$$

Family 26. For $\lambda_1^2-4\lambda_2<0$ and $\lambda_1\neq 0$,

$$V_{32}(x,t)=\frac{2(\gamma-6w\lambda_1-w)}{\theta}-\delta_1\left[\sqrt{\frac{w+4\lambda_1(w-1)-\gamma}{4\lambda_1^2w}}\tan\left(\sqrt{\frac{w+4\lambda_1(w-1)-\gamma}{4w}}(\xi+c_1)\right)-\sqrt{\frac{\gamma+4w\lambda_1-w}{4\lambda_1^2w}}\right)\right]^{-1}$$
$$-\delta_2\left[\sqrt{\frac{w+4\lambda_1(w-1)-\gamma}{4\lambda_1^2w}}\tan\left(\sqrt{\frac{w+4\lambda_1(w-1)-\gamma}{4w}}(\xi+c_1)\right)-\sqrt{\frac{\gamma+4w\lambda_1-w}{4\lambda_1^2w}}\right)\right]^{-2}. \tag{84}$$

Family 27. For $\lambda_1=0$ and $\lambda_2\neq 0$,

$$V_{33}(x,t)=\frac{2(\gamma-w)}{\theta}-\frac{12}{\theta}\sqrt{\frac{\gamma-w}{w}}w\left[\frac{e^{\sqrt{\gamma-w/w}\,(\xi+c_1)}-1}{\sqrt{\gamma-w/w}}\right]^{1}-\frac{12w}{\theta}\left[\frac{e^{\sqrt{\gamma-w/w}\,(\xi+c_1)}-1}{\sqrt{\gamma-w/w}}\right]^{2}. \tag{85}$$

Family 28. For $\lambda_1\neq 0$, $\lambda_2\neq 0$ and $\lambda_1^2-4\lambda_2=0$,

$$V_{34}(x,t)=\frac{2(\gamma-w)}{\theta}-\frac{12}{\theta}\sqrt{\frac{\gamma-4w\lambda_1-w}{w}}w\left[\frac{\gamma+4w\lambda_1-w/w(\xi+c_1)}{2\sqrt{\gamma+4w\lambda_1-w/4}\,(\xi+c_1)+4}\right]^{-1}-\frac{12w}{\theta}\left[\frac{\gamma+4w\lambda_1-w/w(\xi+c_1)}{2\sqrt{\gamma+4w\lambda_1-w/w}\,(\xi+c_1)+4}\right]^{2}. \tag{86}$$

Family 29. For $\lambda_1=0$, $\lambda_2=0$ (or $\lambda_1^2-4\lambda_2=0$),

$$V_{35}(x,t)=\frac{2(\gamma-w)}{\theta}-\frac{12}{\theta}\sqrt{\frac{\gamma-w}{w}}w(\xi+c_1)^{-1}-\frac{12w}{\theta}(\xi+c_1)^{-2}. \tag{87}$$

where $\xi = x - wt, \delta_1 = 12/\theta\sqrt{\gamma - 4w\lambda_1 - w/w}w$ and $\delta_2 = 12w/\theta$ in all above cases.

Second set is as follows:

$$\begin{aligned} a_0 &= -\frac{12w\lambda_1}{\theta}, \\ a_1 &= -\frac{12}{\theta}\sqrt{\frac{\gamma - 4w\lambda_1 - w}{w}}w, \\ a_2 &= -\frac{12w}{\theta}, \\ w &= w, \\ \theta &= \theta, \\ \lambda_1 &= \lambda_1, \\ \lambda_2 &= \sqrt{\frac{\gamma + 4w\lambda_1 - w}{w}}. \end{aligned} \tag{88}$$

Inserting these parameters into (88), we obtain

$$V(\xi) = -\frac{12w\lambda_1}{\theta} - \frac{12}{\theta}\sqrt{\frac{\gamma - 4w\lambda_1 - w}{w}}w\left(e^{-\phi(\xi)}\right) - \frac{12w}{\theta}\left(e^{-\phi(\xi)}\right)^2. \tag{89}$$

Thus, the solution for (10) becomes as follows:

Case I: When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[-\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4\lambda_1^2 w}}\tanh\left(\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4w}}(\xi + c_1)\right) - \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]. \tag{90}$$

Case II: For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$\phi = \ln\left[\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4\lambda_1^2 w}}\tan\left(\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4w}}(\xi + c_1)\right) - \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]. \tag{91}$$

Case III: For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$\phi = -\ln\left[\frac{\sqrt{\gamma + 4w\lambda_1 - w/w}}{e^{\sqrt{\gamma + 4w\lambda_1 - w/w}\,(\xi + c_1)} - 1}\right]. \tag{92}$$

Case IV: For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and $\lambda_1^2 - 4\lambda_2 = 0$,

$$\phi = \ln\left[\frac{2\sqrt{\gamma + 4w\lambda_1 - w/w}\,(\xi + c_1) + 4}{\gamma + 4w\lambda_1 - w/w(\xi + c_1)}\right]. \tag{93}$$

Case V: For $\lambda_1 = 0$, $\lambda_2 = 0$, (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$\phi = \ln(\xi + c_1). \tag{94}$$

Now by putting (90)–(94) one by one into (89) the solutions for our ODE are given by

Family 30. When $\lambda_1^2 - 4\lambda_2 > 0$ and $\lambda_1 \neq 0$,

$$V_{36}(x,t) = -\frac{12w\lambda_1}{\theta} + \delta_1\left[\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4\lambda_1^2 w}}\tanh\left(\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4w}}(\xi + c_1)\right) + \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right)\right]^{-1}$$
$$+ \delta_2\left[\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4\lambda_1^2 w}}\tanh\left(\sqrt{\frac{\gamma - 4\lambda_1(w-1) - w}{4w}}(\xi + c_1)\right) + \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]^{-2}. \tag{95}$$

Family 31. For $\lambda_1^2 - 4\lambda_2 < 0$ and $\lambda_1 \neq 0$,

$$V_{37}(x,t) = -\frac{12w\lambda_1}{\theta} - \delta_1\left[\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4\lambda_1^2 w}}\tan\left(\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4w}}(\xi + c_1)\right) - \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]^{-1}$$
$$- \delta_2\left[\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4\lambda_1^2 w}}\tan\left(\sqrt{\frac{w + 4\lambda_1(w-1) - \gamma}{4w}}(\xi + c_1)\right) - \sqrt{\frac{\gamma + 4w\lambda_1 - w}{4\lambda_1^2 w}}\right]^{-2}. \tag{96}$$

Family 32. For $\lambda_1 = 0$ and $\lambda_2 \neq 0$,

$$V_{38}(x,t) = -\frac{12}{\theta}\sqrt{\frac{\gamma - w}{w}}w\left[\frac{e^{\sqrt{\gamma - w/w}\,(\xi + c_1)} - 1}{\sqrt{\gamma - w/w}}\right]^{1} - \frac{12w}{\theta}\left[\frac{e^{\sqrt{\gamma - w/w}\,(\xi + c_1)} - 1}{\sqrt{\gamma - w/w}}\right]^{2}. \tag{97}$$

Family 33. For $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_{39}(x,t) = -\frac{12w\lambda_1}{\theta} - \frac{12}{\theta}\sqrt{\frac{\gamma - 4w\lambda_1 - w}{w}}w\left[\frac{\gamma + 4w\lambda_1 - w/w(\xi + c_1)}{2\sqrt{\gamma + 4w\lambda_1 - w/w}\,(\xi + c_1) + 4}\right]^{-1} - \frac{12w}{\theta}\left[\frac{\gamma + 4w\lambda_1 - w/w(\xi + c_1)}{2\sqrt{\gamma + 4w\lambda_1 - w/w}\,(\xi + c_1) + 4}\right]^{2}. \tag{98}$$

Family 34. For $\lambda_1 = 0$, $\lambda_2 = 0$ (or $\lambda_1^2 - 4\lambda_2 = 0$),

$$V_{310}(x,t) = -\frac{12}{\theta}\sqrt{\frac{\gamma - w}{w}}w(\xi + c_1)^{-1} - \frac{12w}{\theta}(\xi + c_1)^{-2}, \tag{99}$$

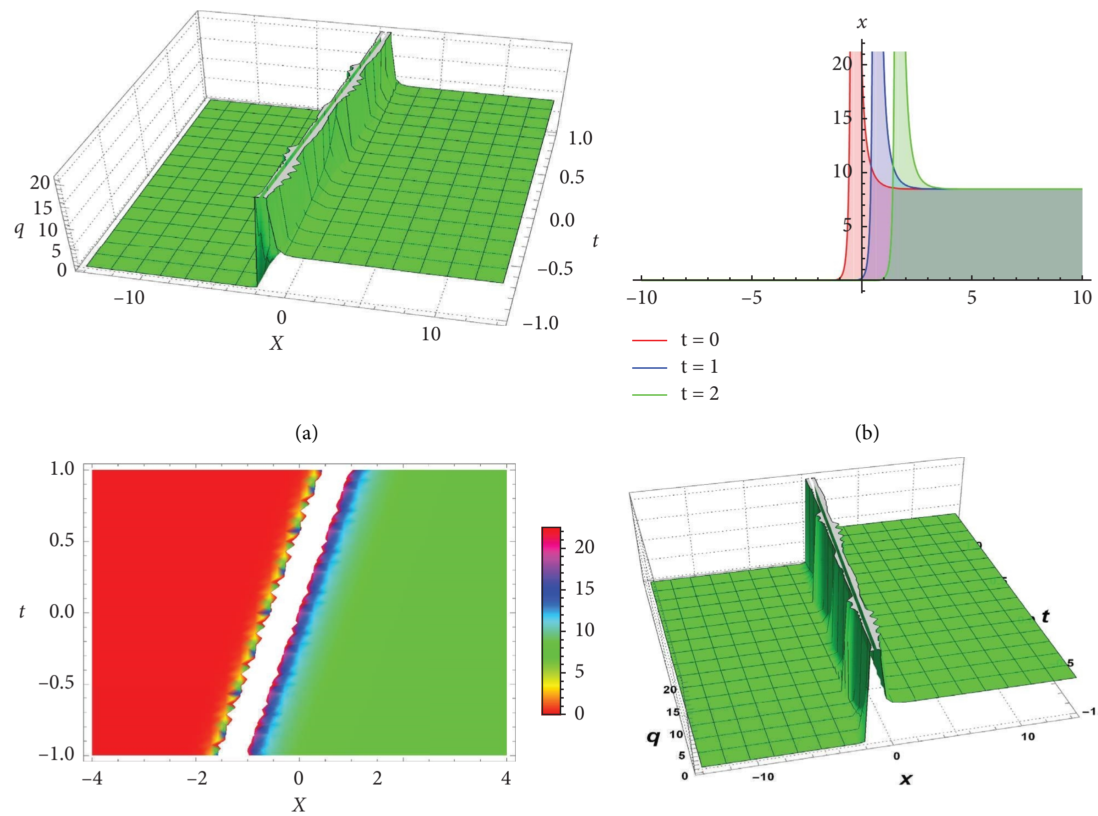
where $\xi = x - wt, \delta_1 = 12/\theta\sqrt{\gamma - 4w\lambda_1 - w/w}w$ and $\delta_2 = 12w/\theta$ in all above cases.

6. Graphical Structures of Some Solitons

This section offers a brief graphical summary of the solutions to the pseudo-parabolic equations discussed here. The wave profile of the BBMPB equation, the BBM equation, and the OSK equation are the main topics of our discussion. Our newly developed families of soliton solutions for nonlinear pseudo-parabolic models represent hyperbolic, trigonometric, rational, exponential, and polynomial functions. The waveform characteristics conform to the properties of some known solitons including the solitary waves, dark, bright, rational, exponential, and polynomial solutions provided in this research. For a particular set of parameters that are listed alongside, Wolfram Mathematica 11.3 simulations were used to create all of these visualizations. The plots include 3D, 2D, contour plots, and density plots. The plots of the solutions are shown in Figures 1–5.

7. Discussion of the Obtained Results

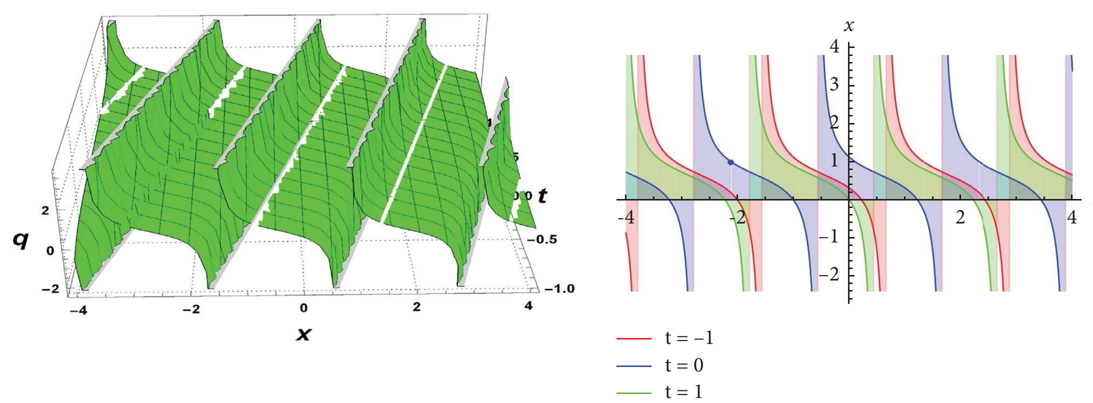
Using the Exp $(-\phi(\xi))$-expansion method, new solitonic families for nonlinear pseudo-parabolic type models have been effectively found. Our focus was on the BBMPB equation, the BBM equation, and the OSK equation. In this study, we have successfully derived the new families of soliton solutions for the nonlinear pseudo-parabolic models. We have obtained the solutions hyperbolic, trigonometric, rational, exponential, and polynomial functions. The dynamics of the solutions show that the obtained solitons are solitary waves, dark, bright, rational, exponential, and polynomial functions-based. The results obtained from the authors [22, 23] contain only dark and bright solitons. So,



(a) (b)

(c) (d)

FIGURE 1: Physical characteristics of the BBMPB equation (16) by equation (27), where $\lambda_2 = -1$ and all remaining are 1. (a) 3D plot. (b) 2D plot. (c) Density plot (d) 3D plot.



(a) (b)

FIGURE 2: Continued.

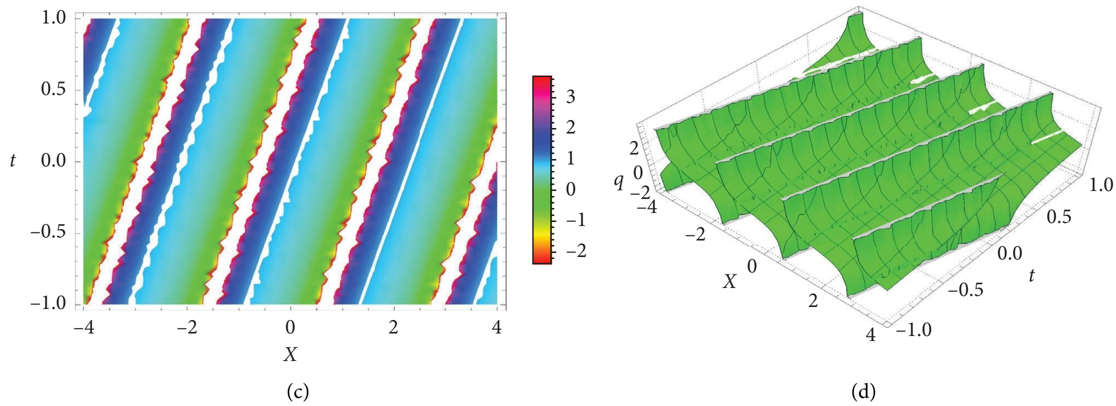


Figure 2: Physical characteristics of the BBMPB equation (16) by equation (27), all the parameters taken to be 1. (a) 3D plot. (b) 2D plot. (c) Density plot. (d) 3D plot.

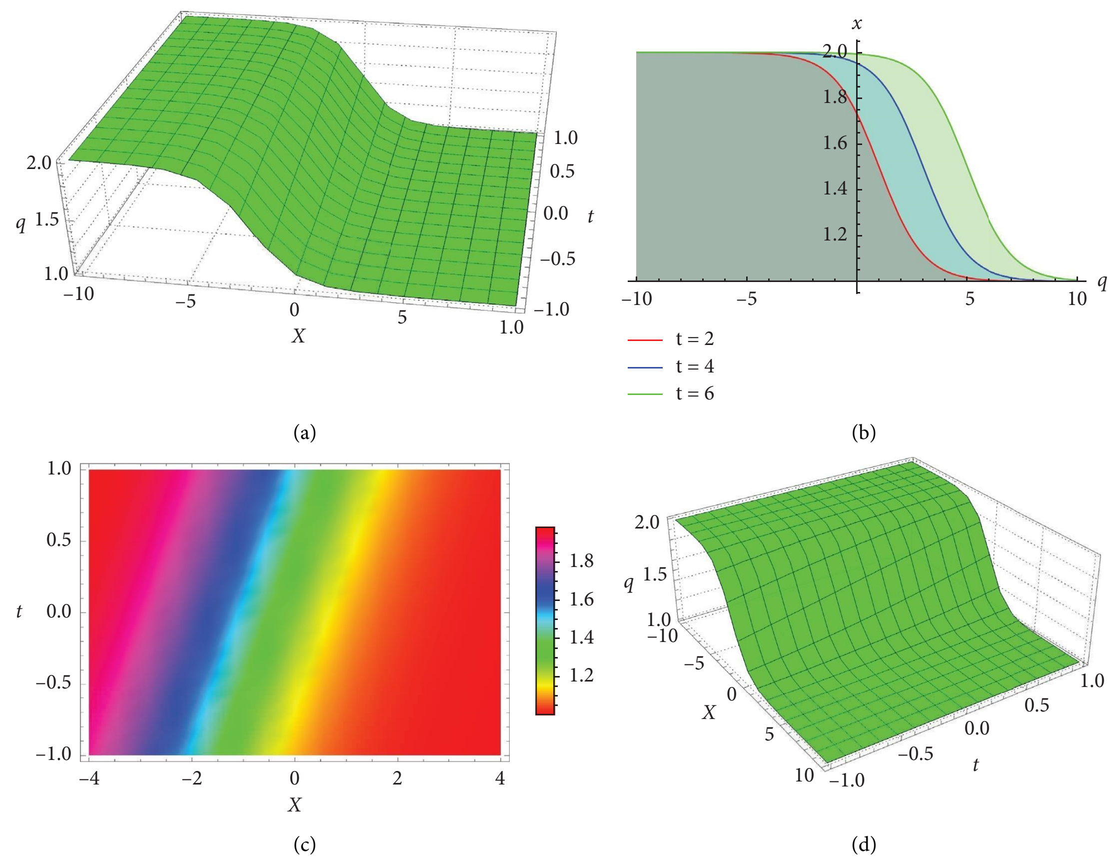


Figure 3: Physical characteristics of the BBMPB equation (16) by equation (48), all the parameters taken to be 1. (a) 3D plot. (b) 2D plot. (c) Density plot. (d) 3D plot.

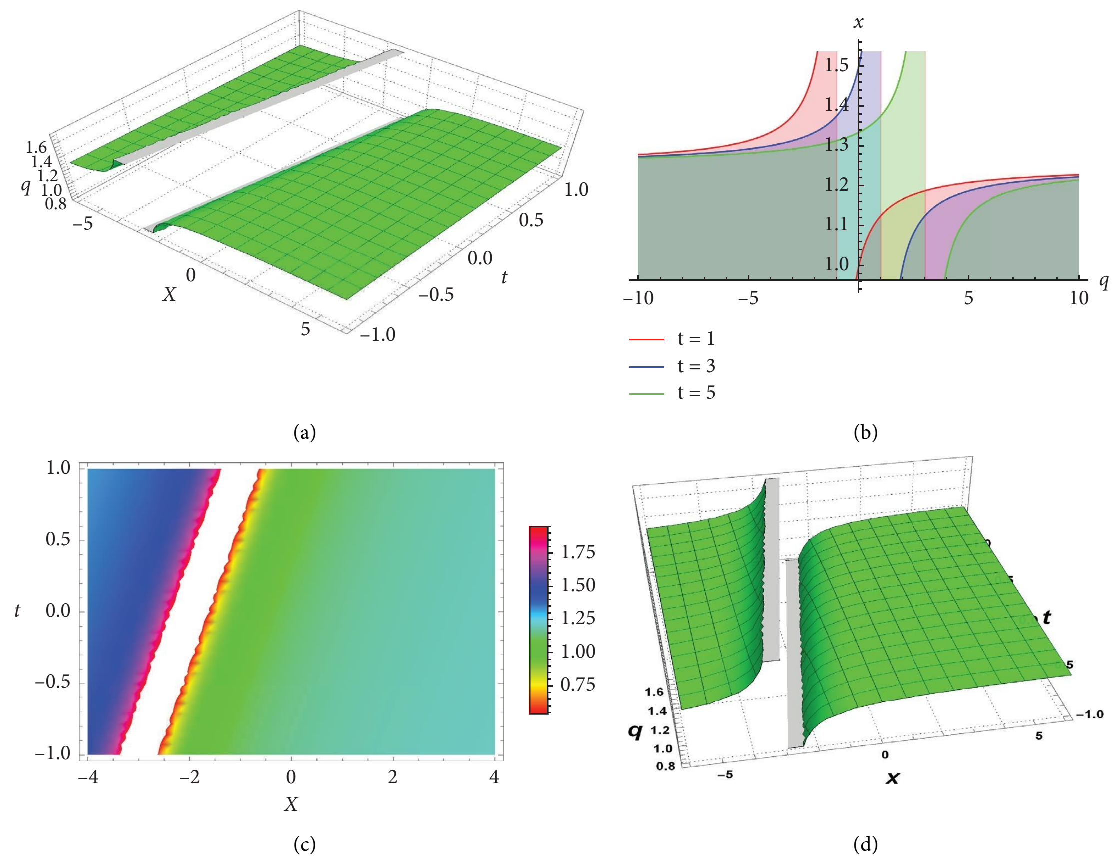


Figure 4: Physical characteristics of the OSK equation by (50), where $\lambda_1 = 2$ and all remaining are 1. (a) 3D plot. (b) 2D plot. (c) Density plot. (d) 3D plot.

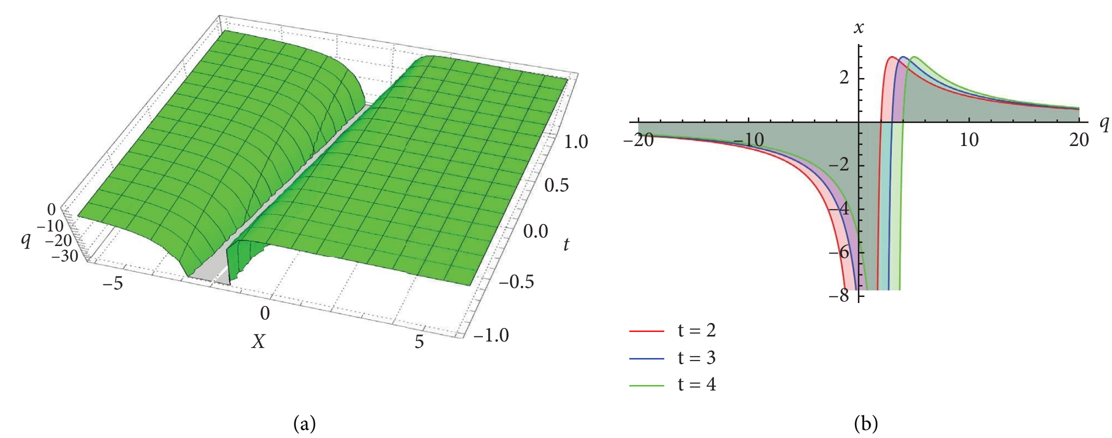
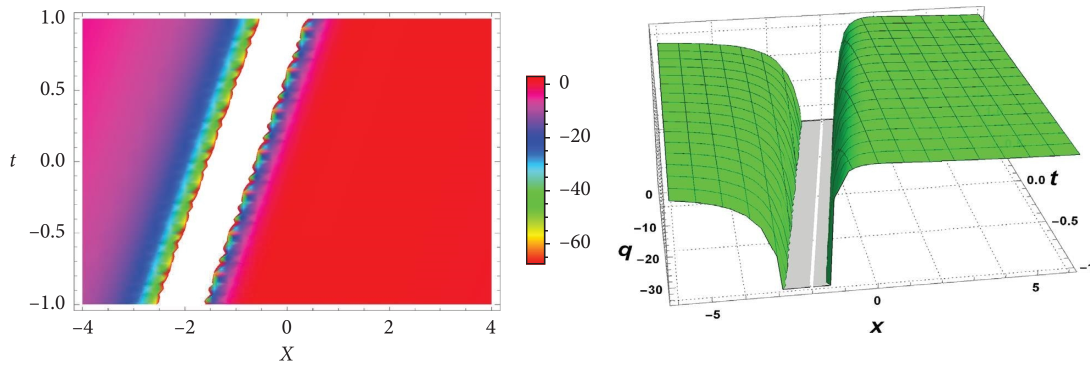


Figure 5: Continued.



(c) (d)

FIGURE 5: Physical characteristics of the BBM equation by (99), where $\gamma_1 = 2$ and all remaining are 1. (a) 3D plot. (b) 2D plot. (c) Density plot. (d) 3D plot.

our results are more general and new and have many applications in the field of fluid dynamics, nonsteady flows, seepage of fluid through fissured material, ocean engineering, and physical sciences.

8. Conclusions

To integrate the pseudo-parabolic type equations in this study, new applications of the Exp (-$\phi(\xi)$)-expansion method were used. It proved to be effective to apply the Exp (-$\phi(\xi)$)-expansion method to find new analytical solutions to the pseudo-parabolic equations. This technique establishes the solutions of the pseudo-parabolic equations in terms of hyperbolic, trigonometric, rational, exponential, and polynomial functions. Previously, only hyperbolic function-based solutions were given by the authors [22, 23], while the solutions found in [31] were trigonometric, hyperbolic, and rational functions. These solutions exhibited the characteristics of dark and bright soliton solutions. However, our results showed that the solitary waves, dark, bright, rational, exponential, and polynomial solutions provided in this research are exclusively novel and valid and have not been previously presented for this class of equations. These precise solutions capture the dynamics of various soliton wave shapes, and they may be used to evaluate, compare, and numerical studies in the area. Additionally, the approach utilized in this work can be applied to other mathematical and physics-related problems. It appears that more study is necessary for the advancement of fresh, effective analytical techniques for solving partial differential equations. By exploiting the improved capacities of such techniques, more problems in science and engineering that occur in the real world can be solved effectively. This can serve as one of the finest incentives for scientists to concentrate more on this remarkable area of study.

Data Availability

All data generated or analyzed during this study are included in this published article [and its supplementary information files].

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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