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Research Article

Performance Analysis of Two Different Types of Waiting Queues with Working Vacations

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This work examines a new class of working vacation queueing models that contain regular (original) and retrial waiting queues. Upon arrival, a customer either starts their service instantly if the server is available, or they join the regular queue if the server is occupied. When it is empty, the server departs the system to take a working vacation (WV). The server provides services more slowly during the WV period. New customers join the retry queue (orbit), if the server is on vacation. The supplementary variable technique (SVT) examines the steady-state probability generating functions (PGFs) of queue size for different server states. Several system performances are numerically displayed, including system state probabilities, mean busy cycles, mean queue lengths, sensitivity analysis, and cost optimization values. The motivation for this model in a pandemic situation is to analyze new healthcare service systems and reflect the characteristics of patient services.

1. Introduction

The importance of managing queues is apparent in the real world. All network communications, healthcare systems, and industries use these queues. Several researchers have investigated the idea of repeated tries (retrial queues), which means that a new customer must leave the service area and repeat his request after a specified period, called retrial time if the server is busy when he arrives. In between trials, a blocked customer in a retrial group has been considered as being in orbit. Artalejo [1] and Falin [2] provide a complete survey.

Many of the authors have investigated concepts in recent years for an emergency service mechanism that would charge different rates during a period of vacation or breakdown. During the absence (vacation) time, the primary server provides service at various rates, known as working vacations or working breakdowns. These services are primarily helpful in communication networks and healthcare systems. Chandrasekaran et al. [3] surveyed working vacation queueing models briefly.

In real life, there are two types of queues: normal and retry. Such queues can be found in computer networks, networks of communication, medical care systems, and many other systems. Most works are covered in queueing theory, which states that there is only one waiting queue, i.e., when a server is busy, customers will either wait in orbit (the retrial queue) or in front of the server. This type of service may often affect the hospital system. Some emergency cases arise in person; when doctors are unavailable, it is a risk. Motivated by this factor, this research developed a model of both queues (original and orbit) into a single server model that incorporates the idea of working vacations ($M^{[1],[2]}/G/1/WV$This model is necessary for the hospital system because patients can receive treatments in person or online.

1.1. Literature Review. Retrial queues in queueing theory have shown to be an interesting area of study, as indicated by the surveys [1, 2]. See Artalejo and Corral [4] and Atencia et al. [5] for additional details regarding the retrial queues.

To get more information on vacation queueing approaches, researchers may study the works of Doshi [6] and Tian and Zhang [7]. The concept of two categories of customers in the vacation model and balking has been investigated by Baruah et al. [8]. The M/G/1 queue with a single working vacation and general service times has been researched by Zhang and Hou [9]. Afterward, vacation interruption was included in a Bernoulli schedule that Gao and Liu [10] considered.

An approach to a retrial queueing system with a single working vacation was suggested by Arivudainambi et al. [11]. In connection to working vacations and vacation interruptions, a retrial queue with general retrial times was created by Gao et al. [12]. Kalidass and Ramanath [13] are identified with introducing the idea of M/M/1 queueing models with working breakdowns. An M/G/1 queueing system with catastrophes and working breakdown services has been discussed by Kim and Lee [14]. Furthermore, a preemptive priority queueing model has been developed by Ammar and Rajadurai [15].

Recently, reservice or feedback service or different types of queues in service is highly required for service quality. In 2009, Krishnakumar et al. [16] considered a multiserver feedback retrial queue with a finite buffer queue. Here the model was framed by the demand for reservice for served customers. Charan et al. [17] have studied an optional reservice retrial queueing model in Bernoulli vacation queues. A model in a multiserver retrial queueing system with the presence of discouraged customers, failures, and vacations was recently investigated by Saravanan et al. [18]. Rismawati et al. [19] developed a priority queueing model with multiple vacations and vacation interruptions.

In 2020, Gao and Zhang [20] studied a queueing system with two types of waiting queues with vacation and general retrial times. In this model, both the normal waiting queue and the retried orbit queue are considered. Recently, Vaezi et al. [21] investigated a real-world case study of prioritizing and queueing emergency department patients utilizing a novel data-driven decision-making technique.

1.2. Methodology and Advantages

1.2.1. Model. Motivated by the work of [20], this investigation proposes a new form of queueing system with two waiting queues (original queue and orbit queue) due to server working vacations ($M^{[1],[2]}$/G/1/WV). The main aim of this model is to investigate new healthcare service systems in the case of a pandemic and to represent different aspects of patient services.

1.2.2. Methodology and Results. We develop a Markovian process for the system using all of the elapsed durations for retrial, service, and working vacation as supplementary variables. We further adopt the generating function approach to obtain system performance measures.

1.2.3. Numerical Illustrations. In system performance measures and cost-effectiveness, the numerical influence of parameters is demonstrated.

1.2.4. Advantages. The number of patients visiting the emergency department is increasing during the COVID-19 pandemic. Suppose the capacity of the queueing system increases, and the number of patients in the system increases for various reasons, such as inappropriate allocation of resources or servers. In such instances, it reduces social isolation and the transmission of infectious illnesses to patients and other staff members in healthcare facilities. To decrease waiting times, the number of patients in emergency departments, and the risk of infection, an additional server should be added or provided with extra services at various rates. We concluded that it was important to keep patients waiting for as a short time as possible based on the literature research.

2. Notations and Probabilities

Notations:

(1) $N_1(t)$ = customers present in the original queue at time t
(2) $N_2(t)$ = customers present in the orbit queue at time t
(3) $I(t)$ = server's state at time t
(4) λ = arrival rate
(5) θ = vacation rate
(6) $\zeta_1^0(t)$ = elapsed retrial time
(7) $\zeta_2^0(t)$ = elapsed service time
(8) $\zeta_3^0(t)$ = elapsed working vacation time
(9) $a(x) \equiv$ the hazard rate (HR or completion rate) related to the retrial, i.e., $a(x)dx = (dA(x)/1 - A(x))$
(10) $\mu_b(x) \equiv$ the HR related to the service, i.e., $\mu_b(x)dx = (dS_b(x)/1 - S_b(x))$
(11) $\mu_v(x) \equiv$ the HR related to the slower rate service, i.e., $\mu_v(x)dx = (dS_v(x)/1 - S_v(x))$
(12) $A^*(\vartheta) \equiv$ Laplace–Stieltjes transform (LST) for retrial
(13) $S_b^*(\vartheta) \equiv$ LST for the regular busy state
(14) $S_v^*(\vartheta) \equiv$ LST for WV
(15) $\beta^{(i)} \equiv$ moments for regular busy period ($i = 1, 2$)
(16) $E(B)$ = expected busy period
(17) $E(W)$ = expected working vacation period
(18) $E(I)$ = expected length of idle time

Probabilities:

(1) $D_0(t) \equiv$ the chance that the server is working vacation (lower speed service) and the system is empty at time t (also known as an idle state).

(2) $D_1(x, n_2, t) \equiv$ it is possible that, at time t, there are precisely n $(n_2 = n)$ customers in the orbit, with the test customer's retry time elapsed and lying between x and $x + dx$ (referred to as the retrial state).

(3) $D_2(x, n_1, n_2, t) \equiv$ the possibility that, at time t, the test customer receiving service has exactly n $(n_1 = n_2 = n)$ customers in the original queue and orbit, with the test customer's elapsed normal service time falling between x and $x + dx$ (referred to as the busy state).

(4) $D_3(x, n_2, t) \equiv$ the chance that, with the test customer's elapsed slower service time falling between x and $x + dx$ (a working vacation state), there are precisely n $(n_2 = n)$ customers in the orbit at time t.

2.1. Detailed Description of the Mathematical Model. We consider a new type of queueing system in two waiting queues (original queue and orbit queue) with working vacations ($M^{[1],[2]}$/G/1/WV). Figure 1 represents the system considered in this paper. The system specifications are given in the following:

(i) The arrival process: according to a Poisson process, customers arrive individually.

(ii) The regular service process: the service time follows a general distribution in normal busy times.

(iii) The working vacations process: Every time the orbit becomes empty, the server starts vacation, and vacation time has an exponential distribution with the parameter θ. If any customer joins in while the server is on vacation (also known as a "working vacation"), the server continues to run at a slower speed service rate $(\mu_v < \mu_b)$. Processes proceed more slowly during the working vacation period. Assume that during a low-speed service completion instant, any customers are in orbit. Vacation interruption occurs when the server stops its vacation and resumes during regular operating mode. If not, it continues the vacation. After a vacation, if there are still customers in orbit, the server continues as usual. The server begins a fresh vacation if it does not.

(iv) The system and retrial process: There are two queues in the system—the orbit queue and the original queue. When the server is busy, arriving customers will join the original queue; otherwise, they will join the orbit queue if the server is on vacation. Both queues follow FCFS rules. The suggested approach has been used to execute the general retrial policy [20].

(v) It is assumed that the random variables in every state are independent.

2.2. A Real-Life Example of the Suggested Model. One of the most essential difficulties in recent situations, such as the COVID-19 pandemic, is the considerable increase in the number of people requiring emergency department services in hospitals. One of the issues that hospital management confronts in emergency departments is establishing the allocation of resources at each time such that the cost of providing services and the expense of patients waiting in queue are equal. In emergency departments, doctors, nurses, paramedics, medical staff, medical equipment, and other resources serve as additional servers in queueing systems to which patients refer themselves. Afterward, patients wait to receive services from these servers and leave the system.

Patients request treatment from the chief doctors (server); if the doctors are already attending to another patient, they enter the queue (original queue) and wait for their turn. There are no patient waits, and the chief doctor goes on a secondary job (vacation). During secondary jobs, if patients request treatment, the chief doctor provides additional treatments like essential first aid treatments or collecting information about the illness (working vacation). Allocating excessive resources significantly costs the system without improving waiting time. When infected patients arrive, if the doctor is in an additional service stage, they will join another queue (retrial queue) and continue to try. After additional service completion, if any patients are waiting for treatment, the doctors return to regular treatment (vacation interruption). It appears reasonable to optimize healthcare queueing systems to lower infection rates among patients based on their health status so patients at higher risk of infection leave the queue sooner. Service options should be increased as much as feasible to decrease the length and density of the emergency department's queues, the risk of disease transmission, and the length of time patients must wait. This idea makes us consider the queue of retry customers resulting from server vacations, which includes the concept of working vacations. Administrators of hospitals will significantly benefit from the results of this model.

3. Steady-State Analysis of the System

This section develops the steady-state difference-differential equations for the retrial queueing system by using SVT. Following that, we calculate the PGF of the orbit size for the system and various server states.

3.1. System Analysis. For further development of this model $M^{[1],[2]}$/G/1/WV, let us define the random variables ζ_i, $(i = 1,2,3)$ to obtain the Markov process $\{(I(t), N_1(t), N_2(t), \zeta_i(t)), t \geq 0\}$. The server states of $\{I(t) = 0, 1, 2, 3\}$ are idle, free, busy, and working vacation. Figure 2 presents a transition state diagram of the model.

We analyze the ergodicity of the embedded Markov chain at departure or vacation epochs. Assume that $\{t_n; n = 1, 2, \ldots\}$ represents the n^{th} sequence of epochs during which a regular service or a lower service period completion epoch occurs. $N_{1,n} = N_1(t_n)$ and $N_{2,n} = N_2(t_n)$ represent the number of customers in the original queue and orbit, respectively, at time t_n. Then, the process $\mathbb{Z}_n = \{(N_{1,n}, N_{2,n}), n \geq 0\}$ is a Markov chain with state space $\mathbb{N} \times \mathbb{N}$. Using the method of embedded Markov chain [20] (see Appendix A), the system is ergodic (stable) if and only if $\rho < 1$, where $\rho = \lambda\beta^{(1)}$.

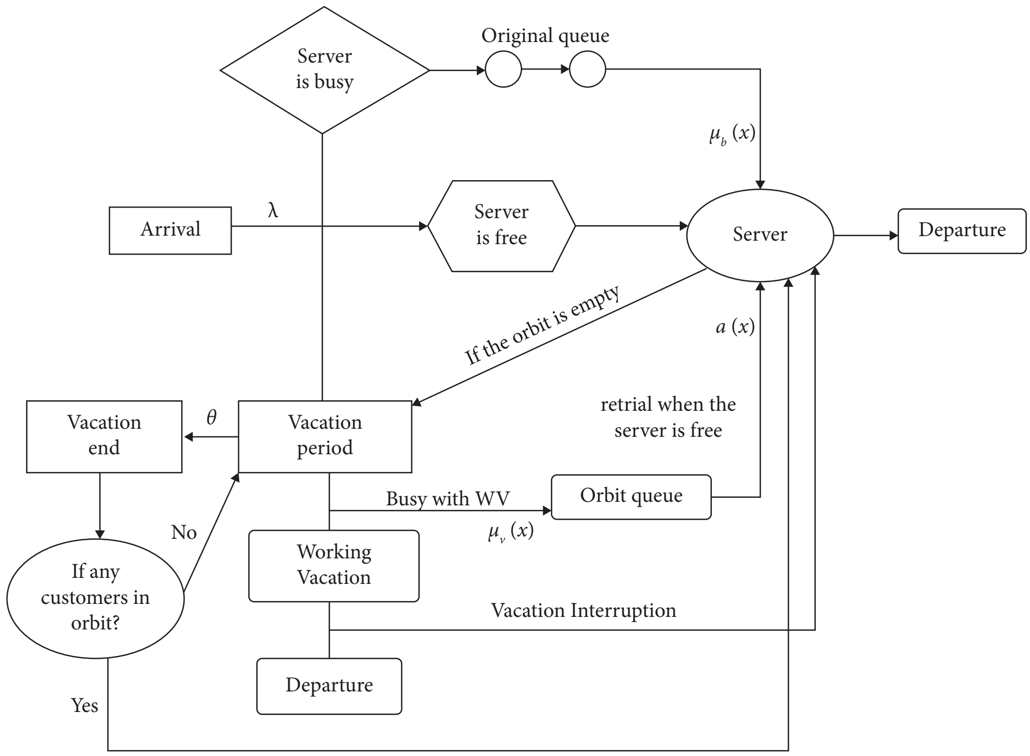

Figure 1: Two waiting queues with working vacations.

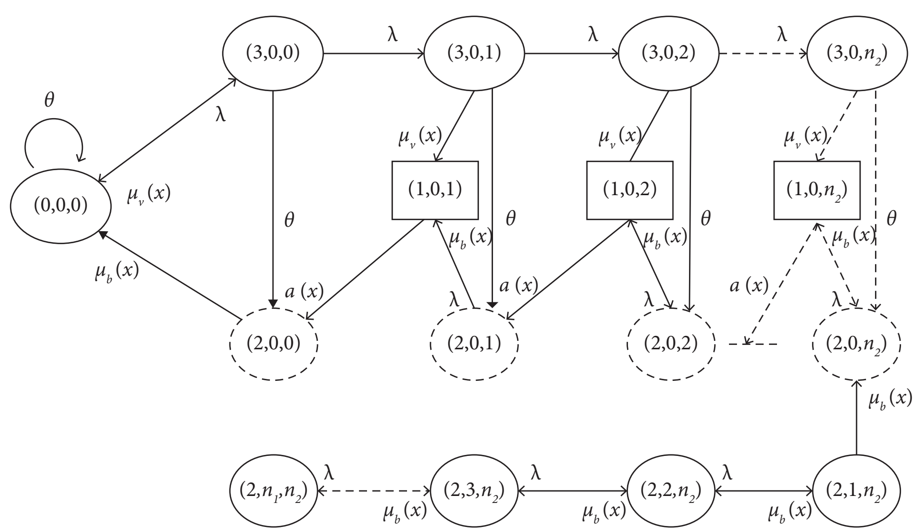

Figure 2: Transition state diagram of the model.

The following densities of probability are defined as

$$
\begin{aligned}
D_0 &= P\{I(t) = 0, N_1(t) = 0, N_2(t) = 0\},\\
D_1(x, n_2)dx &= P\{I(t) = 1, N_1(t) = 0, N_2(t) = n_2, x \le \zeta_1(t) \le x + dx\}, \quad n_2 \ge 1,\\
D_2(x, n_1, n_2)dx &= P\{I(t) = 2, N_1(t) = n_1, N_2(t) = n_2, x \le \zeta_2(t) \le x + dx\}, \quad n_1 \ge 0, n_2 \ge 0,\\
D_3(x, n_2)dx &= P\{I(t) = 3, N_1(t) = 0, N_2(t) = n_2, x \le \zeta_3(t) \le x + dx\}, \quad n_2 \ge 0.
\end{aligned} \tag{1}
$$

The following state's equations can be obtained by applying the SVT method (see Appendix):

$$(\lambda+\theta)D_0=\int_0^{\infty}D_2(x,0,0)\mu_b(x)dx+\int_0^{\infty}D_3(x,0)\mu_v(x)dx+\theta D_0, \tag{2}$$

$$\frac{dD_1(x,n_2)}{dx}+(\lambda+a(x))D_1(x,n_2)=0,\quad n_2\geq 1, \tag{3}$$

$$\frac{dD_2(x,n_1,n_2)}{dx}+(\lambda+\mu_b(x))D_2(x,n_1,n_2)=\lambda D_2(x,n_1-1,n_2),\quad n_1\geq 1, n_2\geq 1, \tag{4}$$

$$\frac{dD_3(x,n_2)}{dx}+(\lambda+\theta+\mu_v(x))D_3(x,n_2)=\lambda D_3(x,n_2-1),\quad n_2\geq 1. \tag{5}$$

The boundary conditions are at $x=0$:

$$D_1(0,n_2)=\int_0^{\infty}D_2(x,0,n_2)\mu_b(x)dx+\int_0^{\infty}D_3(x,n_2)\mu_v(x)dx,\quad n_2\geq 1, \tag{6}$$

$$\begin{aligned}D_2(0,n_1,n_2)=&\int_0^{\infty}D_2(x,n_1+1,n_2)\mu_b(x)dx+\int_0^{\infty}D_1(x,n_2+1)a(x)dx+\lambda\int_0^{\infty}D_1(x,n_2)dx\\&+\theta\int_0^{\infty}D_3(x,n_2)dx,\quad n_1\geq 1, n_2\geq 1,\end{aligned} \tag{7}$$

$$D_3(0,n_2)=\begin{cases}\lambda D_0, & n_2=0,\\ 0, & n_2\geq 1.\end{cases} \tag{8}$$

The normalized condition is

$$D_0+\sum_{n_2=1}^{\infty}\int_0^{\infty}D_1(x,n_2)dx+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\int_0^{\infty}D_2(x,n_1,n_2)dx+\sum_{n_2=1}^{\infty}\int_0^{\infty}D_3(x,n_2)dx=1. \tag{9}$$

3.2. System's Steady-State Solutions. The generating functions (GFs) are defined to solve equations (3)–(8), for $|z_i|\leq 1$, as follows:

$$D_1(x,z_2)=\sum_{n_2=1}^{\infty}D_1(x,n_2)z^{n_2};D_2(x,z_1,z_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}D_2(x,n_1,n_2)z^{n_1}z^{n_2};D_3(x,z_2)=\sum_{n_2=0}^{\infty}D_3(x,n_2)z^{n_2}. \tag{10}$$

Multiplying equations of (3)–(8) by z^n and summing over n to obtain the generating functions,

$$\frac{\partial D_1(x,z_2)}{\partial x}+(\lambda+a(x))D_1(x,z_2)=0, \tag{11}$$

$$\frac{\partial D_2(x,z_1,z_2)}{\partial x}+(\lambda(1-z_1)+\mu_b(x))D_2(x,z_1,z_2)=0, \tag{12}$$

$$\frac{\partial D_3(x,z_2)}{\partial x}+(\lambda(1-z_2)+\theta+\mu_v(x))D_3(x,z_2)=0, \tag{13}$$

$$\begin{aligned} D_1(0,z_2) &= \int_0^\infty D_2(x,0,z_2)\mu_b(x)dx \\ &\quad+\int_0^\infty D_3(x,z_2)\mu_v(x)dx-\lambda D_0, \end{aligned} \tag{14}$$

$$\begin{aligned} D_2(0,z_1,z_2) &= \frac{1}{z_1}\int_0^\infty D_2(x,z_1,z_2)\mu_b(x)dx \\ &\quad+\frac{1}{z_2}\int_0^\infty D_1(x,z_2)a(x)dx+\lambda\int_0^\infty D_1(x,z_2)dx \\ &\quad+\theta\int_0^\infty D_3(x,z_2)dx-\frac{1}{z_1}\int_0^\infty D_2(0,0,z_2)\mu_b(x)dx, \end{aligned} \tag{15}$$

$$D_3(0,z_2)=\lambda D_0. \tag{16}$$

Solving the partial differential equations (11)–(13), it follows that

$$D_1(x,z_2)=D_1(0,z_2)[1-A(x)]e^{-\lambda x}, \tag{17}$$

$$D_2(x,z_1,z_2)=D_3(0,z_1,z_2)[1-S_b(x)]e^{-\lambda(1-z_1)x}, \tag{18}$$

$$D_3(x,z_2)=D_3(0,z_2)[1-S_v(x)]e^{-\lambda(1-z_2)x}. \tag{19}$$

Using equations (17)–(19) in (15) and making some calculations, we get

$$D_2(0,z_1,z_2)=\frac{D_1(0,z_2)}{z_2}(A^*(\lambda)+z(1-A^*(\lambda))+\frac{1}{z_1}D_2(0,z_1,z_2)S_b^*(\lambda-\lambda z_1)-\frac{1}{z_1}D_2(0,\,0,z_2)+\lambda D_0V(z), \tag{20}$$

where $A(z_2)=A^*(\lambda)+z_2(1-A^*(\lambda)$ and $V(z)=(\theta[1-S_v^*(\theta+\lambda-\lambda z_2)]/\theta+\lambda(1-z_2))$.

Using equations (17)–(20) and (14)–(16) after making some manipulations, we get

$$D_1(0,z_2)=D_2(0,\ 0,z_2)+D_3(0,z_2)S_v^*(\theta+\lambda-\lambda z_2)-\lambda D_0, \tag{21}$$

$$D_2(0,\ 0,z_2)=\frac{D_1(0,z_2)A(z_2)}{z_2}+\lambda D_0 V(z), \tag{22}$$

$$D_3(0,z_2)=\lambda D_0. \tag{23}$$

From equations (20)–(23) in (17)–(19), we get

$$D_1(x,z_2)=\frac{z_2\lambda D_0(S_v^*(\theta+\lambda-\lambda z_2)-1+V(z))[1-A(x)]e^{-\lambda x}}{R^*(\lambda)(z_2-1)}, \tag{24}$$

$$D_2(x,z_1,z_2)=\frac{(z_1-1)\lambda D_0}{R^*(\lambda)(z_2-1)}\left\{\frac{((S_v^*(\theta+\lambda-\lambda z_2)-1)A(z_2)+z_2V(z))(1-S_b(x))e^{-\lambda(1-z_1)x}}{(z_1-S_b^*(\lambda-\lambda z_1))}\right\}, \tag{25}$$

$$D_3(x,z_2)=\lambda D_0(1-S_v(x))e^{-\lambda(1-z_2)x}. \tag{26}$$

Theorem 1. *The marginal GFs of the number of customers in the orbit and queue are as follows under the stability condition $\rho<1$:*

(i) During the retrial, if the server is idle,

$$D_1(z_2)=\frac{z_2D_0(1-A^*(\lambda))(S_v^*(\theta+\lambda-\lambda z_2)-1+V(z))}{R^*(\lambda)(z_2-1)}. \tag{27}$$

(ii) When the server is busy,

$$D_2(z_1,z_2)=\frac{(S_b^*(\lambda-\lambda z_1)-1)D_0}{R^*(\lambda)(z_2-1)}\left\{\frac{((S_v^*(\theta+\lambda-\lambda z_2)-1)A(z_2)+z_2V(z))}{(z_1-S_b^*(\lambda-\lambda z_1))}\right\}. \tag{28}$$

(iii) While the server is on a working vacation,

$$D_3(z_2)=\frac{\lambda D_0V(z)}{\theta}. \tag{29}$$

(iv) The idle probability of the server:

$$D_0=\frac{A^*(\lambda)\left(1-\lambda\beta^{(1)}\right)}{\left\{(\lambda/\theta)(1-S_v^*(\theta))+A^*(\lambda)\left(1-\lambda\beta^{(1)}S_v^*(\theta)\right)\right\}}. \tag{30}$$

Proof. From equations (24)–(26), we define the partial PGFs as

$$D_1(z_2)=\int_0^\infty D_1(x,z_2)dx, D_2(z_1,z_2)=\int_0^\infty D_2(x,z_1,z_2)dx, D_3(z_2)=\int_0^\infty D_3(x,z_2)dx. \tag{31}$$

By setting $(z_1, z_2) = (1, 1)$ in (24)–(26) and using l'Hôspital's rule whenever necessary, we can calculate the probability that the server is idle (D_0) when there are no customers in the system using the normalizing condition. This gives us $D_0 + D_1(1) + D_2(1,1) + D_3(1) = 1$. □

Corollary 2. *If the system in $\rho < 1$, then*

(i) The PGFs of the orbit size is

$$K_s(z) = D_0 + D_1(z_2) + D_2(z_1, z_2) + D_3(z_2),$$
$$K_s(z) = D_0\left(\frac{Nr1(z) + Nr2(z)}{Dr1(z)}\right), \tag{32}$$

where

$$\begin{aligned} Nr1(z) &= \left(z_1 - S_b^*(\lambda - \lambda z_1)\right)\left[R^*(\lambda)(z_2 - 1)\left(\left(\frac{\lambda}{\theta}\right)V(z) + 1\right) + z_2(1 - A^*(\lambda))\left(S_v^*(\theta + \lambda - \lambda z_2) - 1 + V(z)\right)\right],\\ Nr2(z) &= \left(S_b^*(\lambda - \lambda z_1) - 1\right)\left((S_v^*(\theta + \lambda - \lambda z_2) - 1)A(z_2) + z_2 V(z)\right),\\ Dr1(z) &= R^*(\lambda)(z_2 - 1)\left(z_1 - S_b^*(\lambda - \lambda z_1)\right). \end{aligned} \tag{33}$$

(ii) The PGF of the system size is

$$K_O(z) = D_0 + D_1(z_2) + z_2\left(D_2(z_1, z_2) + D_3(z_2)\right),$$
$$K_O(z) = D_0\left(\frac{Nr3(z) + Nr4(z)}{Dr1(z)}\right), \tag{34}$$

where

$$\begin{aligned} Nr3(z) &= \left(z_1 - S_b^*(\lambda - \lambda z_1)\right)\left[R^*(\lambda)(z_2 - 1)\left(z_2\left(\frac{\lambda}{\theta}\right)V(z) + 1\right) + z_2(1 - A^*(\lambda))\left(S_v^*(\theta + \lambda - \lambda z_2) - 1 + V(z)\right)\right],\\ Nr4(z) &= z_2\left(S_b^*(\lambda - \lambda z_1) - 1\right)\left((S_v^*(\theta + \lambda - \lambda z_2) - 1)A(z_2) + z_2 V(z)\right), \end{aligned} \tag{35}$$

where D_0 is given in equation (30).

4. Measures of System Performance

In this section, some important system measures of the two waiting queues due to server working vacations ($M^{[1],[2]}/G/1/WVs$) are derived.

4.1. Probabilities of System States. By fixing the limit functions $(z_1, z_2) \longrightarrow (1, 1)$ and using the l'Hôspital's rule as necessary, we obtain the following probabilities for system states from equations (27)–(29):

(i) During the retrial time, the idle probability (D_1) is given by

$$D_1 = D_1(1) = \frac{\lambda D_0}{\theta}\left\{\frac{(1 - A^*(\lambda))(1 - S_v^*(\theta))}{A^*(\lambda)}\right\}. \tag{36}$$

(ii) The idle probability of the system (D) is given by

$$D = D_0 + D_1(1) = \frac{D_0\{A^*(\lambda) + (\lambda/\theta)(1 - A^*(\lambda))(1 - S_v^*(\theta))\}}{A^*(\lambda)}. \tag{37}$$

(iii) The server's busy probability (D_2) is given by

$$D_2 = D_2(1,\,1) = \frac{\lambda D_0\beta^{(1)}((\lambda/\theta) + A^*(\lambda))(1 - S_v^*(\theta))}{A^*(\lambda)\left(1 - \lambda\beta^{(1)}\right)}. \tag{38}$$

(iv) The server's slower service (working vacation) probability (D_3) is given by

$$D_3 = D_3(1) = \frac{\lambda D_0}{\theta}(1 - S_v^*(\theta)). \tag{39}$$

4.2. Computation of Mean Queue Size and Orbit Size

(i) The expected regular queue size (L_1) is given by

$$L_1 = \lim_{z_1\longrightarrow 1}\frac{d}{dz}D_2(z_1, 1),$$
$$L_1 = \frac{D_0\lambda^2\beta^{(2)}(1 - S_v^*(\theta))((\lambda/\theta) + A^*(\lambda))}{2\left(1 - \lambda\beta^{(1)}\right)^2}. \tag{40}$$

(ii) The expected orbit queue size (L_2) is given by

$$L_2 = \lim_{z_2\longrightarrow 1}\frac{d}{dz}D_1(z_2) + \lim_{z_2\longrightarrow 1}\frac{d}{dz}D_2(1,z_2) + \lim_{z_2\longrightarrow 1}\frac{d}{dz}D_3(z_2),$$
$$L_2 = \frac{D_0}{A^*(\lambda)}\left\{\frac{\lambda}{\theta}\left\{(1 - A^*(\lambda))(1 - S_v^*(\theta)) + V'\left(A^*(\lambda) + \frac{\lambda^2\beta^{(2)}}{(1-\rho)^2}\right)\right\} + \frac{\lambda^2\beta^{(2)}}{(1-\rho)^2}\left(V' - \lambda(1 - A^*(\lambda))S_v^{*'}(\theta)\right)\right\}, \tag{41}$$

where $V' = (\lambda/\theta)(1 - S_v^*(\theta) + \theta S_v^{*'}(\theta))$.

(iii) The average number of customers in the system is represented as L_s and it consists of the customers in the orbit queue, the original queue, and the customer being serviced.

$$L_s = D_2 + L_1 + L_2,$$
$$\begin{aligned}L_s = \frac{D_0}{2A^*(\lambda)\left(1 - \lambda\beta^{(1)}\right)^2}&\left\{\left(\frac{\lambda}{\theta} + A^*(\lambda)\right)(1 - S_v^*(\theta))\left(2\lambda\beta^{(1)} + A^*(\lambda)\lambda^2\beta^{(2)}\left(1 - \lambda\beta^{(1)}\right)\right)\right.\\ &\left.+\frac{2\lambda}{\theta}\left(1 - \lambda\beta^{(1)}\right)^2\left((1 - A^*(\lambda))(1 - S_v^*(\theta)) + V'\right) + 2\lambda^2\beta^{(2)}\left(\frac{\lambda}{\theta}V' + \frac{\lambda}{\theta}(1 - S_v^*(\theta)) + \lambda A^*(\lambda)S_v^{*'}(\theta)\right)\right\}.\end{aligned} \tag{42}$$

(iv) Using Little's formula, the mean waiting times for the system (W_s) and queue (W_q) are determined.

$$W_s = \frac{L_s}{\lambda},$$
$$W_q = \frac{L_2}{\lambda}. \tag{43}$$

4.3. Mean Busy Time and Regeneration Cycle. Consider a regeneration cycle (H) to be the period between two consecutive working vacations. When the server is idle, busy, or on vacation, the periods are I, B, and W, respectively, and $H = I + B + W$. Using the concept of the alternating renewal process [20], we get

$$\frac{E(W)}{E(H)} = \frac{\lambda(1 - S_v^*(\theta))}{\theta E(H)} = D_3 = \frac{\lambda D_0}{\theta}(1 - S_v^*(\theta)), \tag{44}$$

which leads to $E(H) = (1/D_0)$.

Based on the results above, we then obtain

$$E(I) = E(H)\times D = \frac{\{A^*(\lambda) + (\lambda/\theta)(1 - A^*(\lambda))(1 - S_v^*(\theta))\}}{A^*(\lambda)},$$
$$E(B) = E(H)\times D_2 = \frac{\left\{\lambda\beta^{(1)}((\lambda/\theta) + A^*(\lambda))(1 - S_v^*(\theta))\right\}}{A^*(\lambda)\left(1 - \lambda\beta^{(1)}\right)}, \tag{45}$$
$$E(W) = E(H)\times D_3 = \frac{\lambda(1 - S_v^*(\theta))}{\theta}.$$

5. Significant System Particular Cases

In this section, we highlighted the following significant system-specific situations of the two types of waiting queues that are caused by server working vacations.

5.1. Case (i): No Vacation and No Original Queue. Let (θ, z_1) be (0, 0); our model can be reduced to a retrial queue with a single working vacation, and the solutions matched those of Arivudainambi et al. [11].

In this case, the idle probability of the server and the PGFs of the system size are

$$D_0 = \frac{S_3^*(\lambda)\left[R^*(\lambda) - \lambda\beta_1^{(1)}\right]}{\lambda\beta_3^{(1)} + R^*(\lambda)S_3^*(\lambda)},$$
$$\chi_s(z) = \left[\frac{\zeta_0\{[1 - S_3^*(\lambda(1-z))][z + (1-z)R^*(\lambda)] + (1-z)R^*(\lambda)S_3^*(\lambda)\}}{S_3^*(\lambda)\{S_1^*(\lambda(1-z))[z + (1-z)R^*(\lambda)] - z\}}\right]. \tag{46}$$

5.2. Case (ii): No Original Queue and Multiple Working Vacations. Let $z_1 = \theta = \mu_v = 0$; our model was changed to a single server retrial queue with working vacations, and the solutions agreed with the results of Gao et al. [12].

In this case, the idle probability and the PGF of the system size are rewritten as

$$\zeta_0 = \frac{S_3^*(\lambda)\left[R^*(\lambda) - \lambda\beta_1^{(1)}\right]}{\lambda\beta_3^{(1)} + R^*(\lambda)S_3^*(\lambda)},$$
$$\chi_s(z) = \left[\frac{\zeta_0\{[1 - S_3^*(\lambda(1-z))][z + (1-z)R^*(\lambda)] + (1-z)R^*(\lambda)S_3^*(\lambda)\}}{S_3^*(\lambda)\{S_1^*(\lambda(1-z))[z + (1-z)R^*(\lambda)] - z\}}\right]. \tag{47}$$

5.3. Case (iii): No Working Vacation. Let $\mu_v \longrightarrow 0$ (no lower speed service); our model was reduced to an M/G/1 queue with retrial customers due to server vacation, and the results coincided with Gao and Zhang [20].

In this case, the server's idle probability and the PGF of the queue size when the server is busy are as follows:

$$\zeta_0 = \frac{\lambda R^*(\lambda)(1-\rho)}{\lambda\beta_3^{(1)}(p + qR^*(\lambda)(1-\rho)) + R^*(\lambda)S_3^*(\lambda p)},$$
$$\zeta_4(z_1, z_2) = \frac{(1 - S_1^*(A_b(z_1)))\zeta_0}{\lambda(z_1 - S_1^*(A_b(z_1)))}\left\{\frac{R(z)(S_3^*(\theta + \lambda p(1-z_2)) - 1)}{R^*(\lambda + \delta)(z_2 - 1)}\right\}. \tag{48}$$

6. Sensitivity Analysis

In this section, we used MATLAB software to discuss the system characteristics of various parameters. We present some numerical examples graphically based on the results. We assume Exponential (Exp), Erlang 2-stage (Erl-2S), and Hyper-Exponential (H-Exp) distributions for retrial, service, and working vacation times. The parameters' arbitrary values are selected to meet the stability requirement. $f(x) = ve^{-vx}, x > 0$ is the Exponential distribution, $f(x) = v^2 x e^{-vx}, x > 0$ is the Erlang-2 stage distribution, and $f(x) = cve^{-vx} + (1-c)v^2 e^{-v^2 x}, x > 0.$ is the Hyper-Exponential distribution.

Tables 1–5 show the effect of system performance D_0, L_1, L_2, and L_s on the arrival rate, retrial rate, service rate, lower speed service rate, and vacation rate parameters. Table 1 shows that as the arrival rate increases, the probability of being idle (D_0) decreases, and the mean original queue length (L_1) and mean orbit queue length (L_2) increase for three distributions. Table 2 shows that the

Table 1: Arrival rate (λ) influences D_0, L_1, and L_2.

Retrial		Exp.			Erl-2S			H-Exp.	
λ	D_0	L_1	L_2	D_0	L_1	L_2	D_0	L_1	L_2
1.00	0.8000	0.0037	0.0008	0.5802	0.0129	0.0090	0.7681	0.0026	0.0007
1.50	0.6886	0.0086	0.0071	0.3855	0.0267	0.0810	0.6423	0.0057	0.0060
2.00	0.5818	0.0159	0.0345	0.2430	0.0453	0.3916	0.5280	0.0099	0.0276
2.50	0.4848	0.0258	0.1190	0.1481	0.0698	1.3757	0.4301	0.0152	0.0903
3.00	0.4000	0.0386	0.3305	0.0874	0.1043	4.0658	0.3493	0.0214	0.2374

Table 2: Retrial rate (a) influences D_0, L_1, and L_2.

Retrial		Exp.			Erl-2S			H-Exp.	
a	D_0	L_1	L_2	D_0	L_1	L_2	D_0	L_1	L_2
5.00	0.6531	0.0211	0.0247	0.3659	0.0875	0.2601	0.6217	0.0141	0.0189
6.00	0.6621	0.0220	0.0235	0.3831	0.0957	0.2410	0.6340	0.0148	0.0177
7.00	0.6687	0.0226	0.0225	0.3960	0.1023	0.2268	0.6431	0.0154	0.0168
8.00	0.6737	0.0232	0.0218	0.4058	0.1077	0.2157	0.6500	0.0158	0.0161
9.00	0.6776	0.0236	0.0212	0.4136	0.1123	0.2069	0.6555	0.0162	0.0156

Table 3: Influence of service rate (μ) on D_0, L_1, and L_2.

Service		Exp.			Erl-2S			H-Exp.	
μ_b	D_0	L_1	L_2	D_0	L_1	L_2	D_0	L_1	L_2
5.00	0.4800	0.0933	0.2023	0.0865	0.5797	5.0164	0.4549	0.0520	0.1453
6.00	0.5161	0.0565	0.1224	0.1410	0.2362	2.0441	0.4808	0.0327	0.0913
7.00	0.5405	0.0378	0.0820	0.1784	0.1329	1.1500	0.4983	0.0225	0.0628
8.00	0.5581	0.0271	0.0588	0.2058	0.0862	0.7460	0.5110	0.0164	0.0459
9.00	0.5714	0.0204	0.0442	0.2266	0.0608	0.5258	0.5205	0.0126	0.0351

Table 4: Lower service rate's (μ_v) effects on D_0, L_1, and L_s.

Vacation		Exp.			Erl-2S			H-Exp.	
μ_v	D_0	L_1	L_s	D_0	L_1	L_s	D_0	L_1	L_s
5.00	0.5817	0.0159	0.1777	0.2430	0.0453	0.7989	0.5280	0.0099	0.1408
6.00	0.6102	0.0148	0.1631	0.2605	0.0442	0.7619	0.5546	0.0093	0.1304
7.00	0.6349	0.0139	0.1508	0.2773	0.0432	0.7268	0.5784	0.0088	0.1215
8.00	0.6567	0.0131	0.1401	0.2935	0.0422	0.6940	0.5999	0.0084	0.1138
9.00	0.6761	0.0123	0.1308	0.3090	0.0413	0.6636	0.6192	0.0080	0.1070

Table 5: Influence of vacation rate (θ) on D_0, L_1, and L_s.

Vacation		Exp.			Erl-2S			H-Exp.	
θ	D_0	L_1	L_s	D_0	L_1	L_s	D_0	L_1	L_s
3.00	0.5818	0.0159	0.0345	0.2430	0.0453	0.3916	0.5280	0.0099	0.0276
4.00	0.6000	0.0167	0.0195	0.2656	0.0459	0.2203	0.5546	0.0103	0.0162
5.00	0.6154	0.0173	0.0125	0.2857	0.0464	0.1423	0.5774	0.0106	0.0107
6.00	0.6286	0.0179	0.0088	0.3038	0.0469	0.1000	0.5969	0.0108	0.0076
7.00	0.6400	0.0183	0.0065	0.3200	0.0472	0.0745	0.6140	0.0110	0.0057

retrial rate (a) increases, D_0 and L_1 increase, and L_2 decreases. Tables 3 and 4 demonstrate that increasing both service rates (μ and μ_v) decreases the value of L_1,L_2, and L_s while increasing the value of D_0. Table 5 illustrates that the vacation rate (θ) tends to increase for D_0 and L_1, but it tends to decrease for L_s.

Figures 3–14 graphically depict many important features of the solutions, such as D_0, L_1, L_2, and L_s, to provide a clear understanding of the present scenario. Figures 3 and 4 depict that with an increasing value of the retrial rate (a), the idle probability (D_0) increases, and the mean orbit queue size (L_2) decreases.

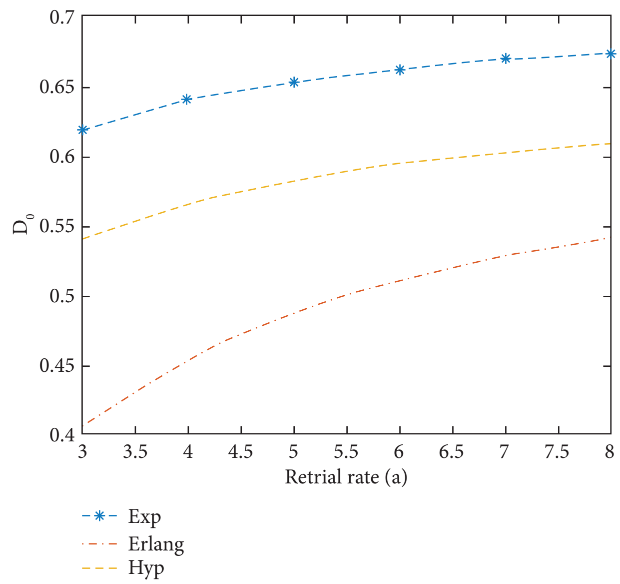

Figure 3: a's effects on D_0.

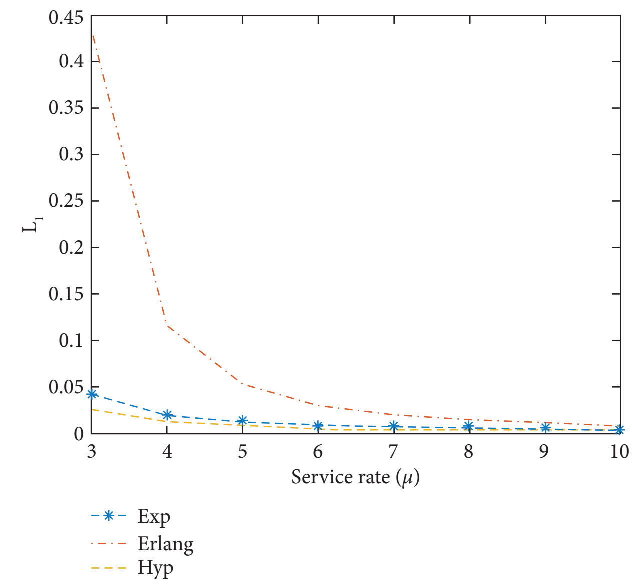

Figure 5: μ_b's effects on L_1.

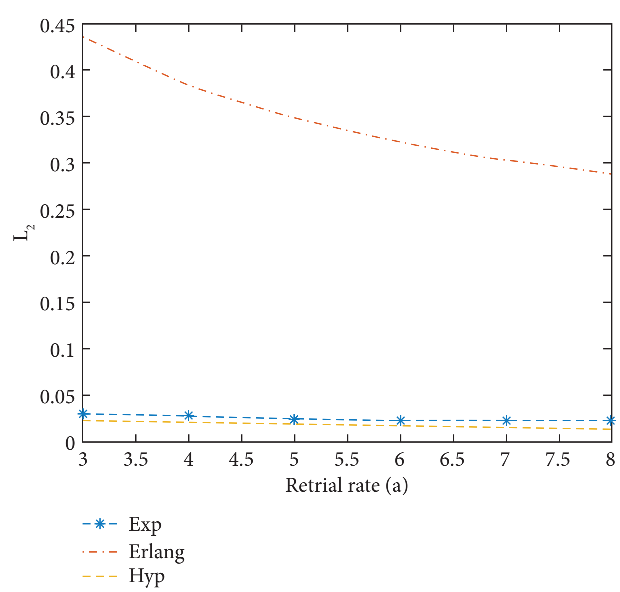

Figure 4: a's effects on L_2.

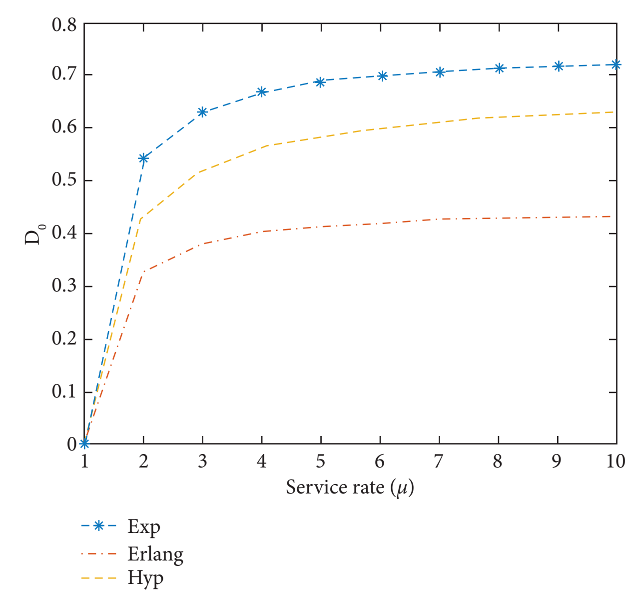

Figure 6: μ_b's effects on D_0.

Figures 5 and 6 illustrate that the increase in service rate (μ_b) increases the original queue size (L_1), which reduces the idle probability (D_0). As shown in Figures 7 and 8, as the vacation rate (θ) increases, D_0 also increases, while L_2 decreases. Figures 9 and 10 demonstrate that the working vacation rate (μ_v) increases as the mean system size (L_s) increases, but the idle (D_0) probability decreases.

The combined impact of several variables on system performance is depicted in Figures 11–14. An increasing trend in λ and a value for L_1 is seen in Figure 11. In Figure 12, the nature of L_s declines to increase the value of the vacation rate (θ) and lower the speed service rate (μ_v). Figure 13 shows that as the service rate (μ_b) and working vacation rate (μ_v) grow, then D_0 rises. The vacation rate (θ) and retry rate (a) increase as the nature of L_s drops in Figure 14.

7. Cost Analysis

In this section, the two waiting queues due to server working vacations ($M^{[1],[2]}/G/1/WVs$) are addressed to optimize the design [22]. After constructing a cost function to make the system as cost-effective as feasible at the lowest possible cost, the findings of Yang and Wu [23] and Gao and Zhang [20]

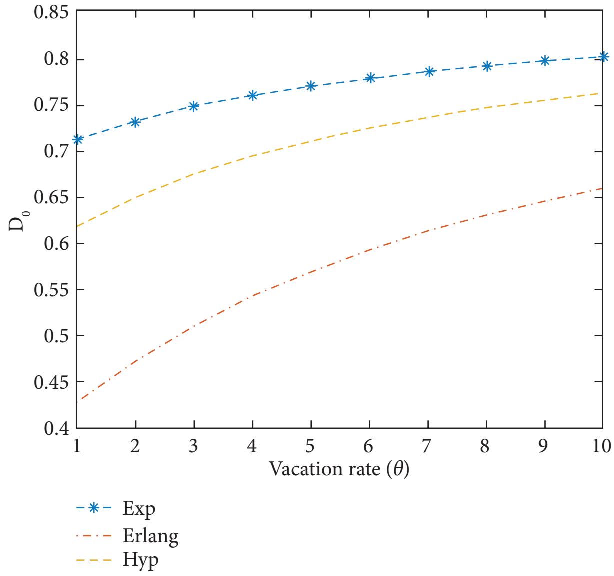

Figure 7: θ's effects on D_0.

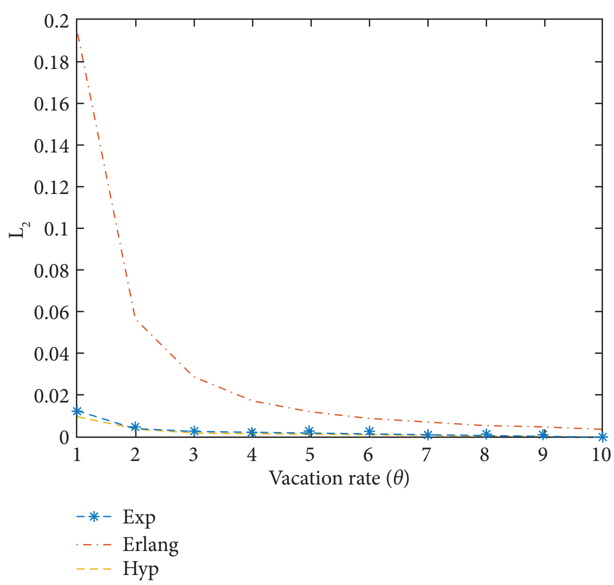

Figure 8: θ's effects on L_2.

are consulted. Parameters are set arbitrarily to satisfy the steady-state requirement.

We performed a cost analysis based on our earlier findings ($E(I)$, $E(B)$, $E(W)$, $E(H)$, L_1, L_2, L_s). Let us define the following costs:

C_q = unit time cost of every customer in the original queue

C_o = unit time cost of every customer in the orbit queue

C_i = unit time cost for keeping the server idle

C_b = unit time cost for keeping the server busy

C_w = unit time cost for keeping the server on working vacation

C_s = setup cost per cycle

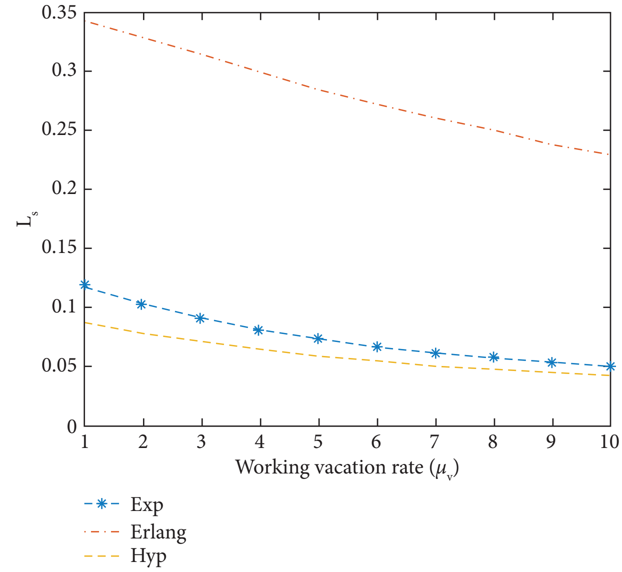

Figure 9: μ_v's effects on L_s.

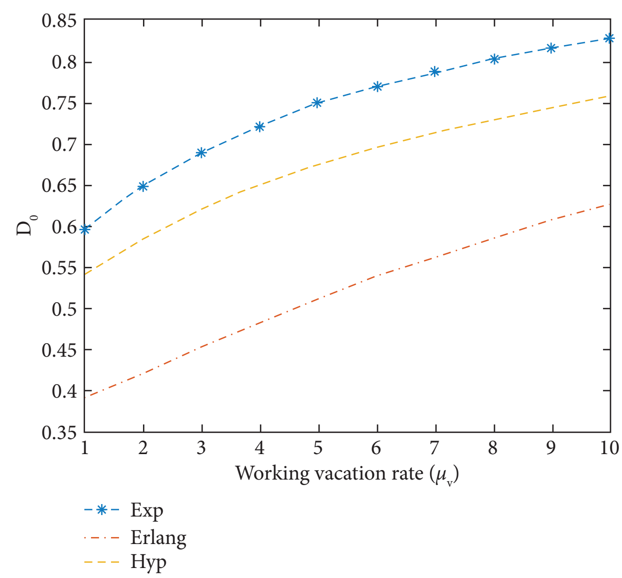

Figure 10: μ_v's effects on D_0.

The expected cost function in the linear cost technique is provided by

$$\mathrm{ETC} = C_q L_1 + C_o L_2 + C_i E(I) + C_b E(B) + C_w E(W) + C_s\left(\frac{1}{E(H)}\right). \tag{49}$$

By fixing the cost and other characteristics, we hope to obtain the total expected cost per unit of time (ETC). For all server state parameters, we use an exponential distribution. The values of the cost elements and other parameters such as $\lambda = 2$; $a = 1$; $\mu = 10$; $\mu_v = 5$; $\theta = 3$; $C_q = 50$; $C_o = 25$; $C_i = 25$; $C_b = 60$; $C_w = 30$; and $C_s = 150$ are chosen to satisfy the

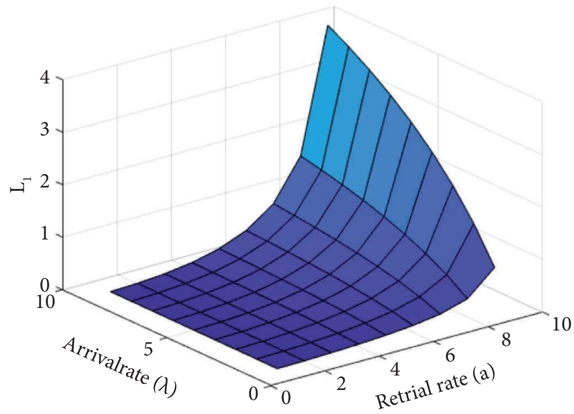

Figure 11: L_1 vs λ and a.

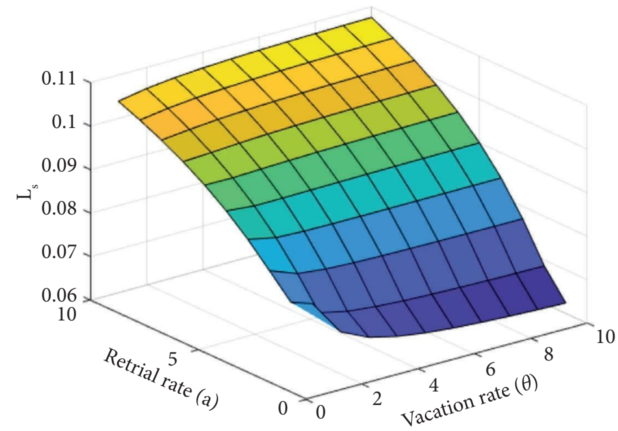

Figure 14: L_s vs a and θ.

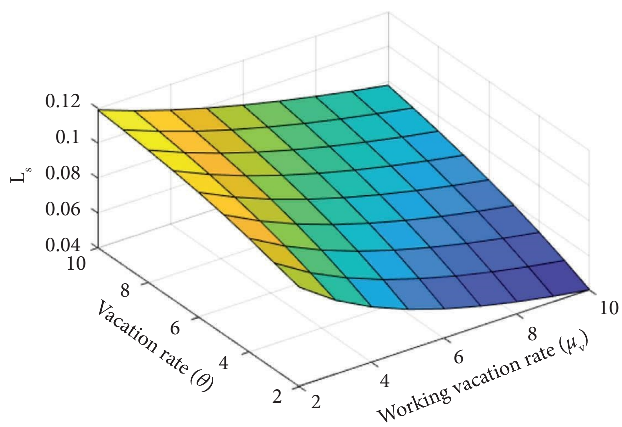

Figure 12: L_s vs θ and μ_v.

Table 6: Impact of (θ, μ_v) on ETC.

$\mu_v = 1$		$\mu_v = 2$		$\mu_v = 3$	
θ	ETC	θ	ETC	θ	ETC
6.00	111.4959	4.00	108.4510	4.00	113.5280
6.50	113.3034	4.50	110.5344	4.50	115.2365
7.00	114.9654	5.00	112.4578	5.00	116.8255
7.50	116.4978	5.50	114.2305	5.50	118.2999
8.00	117.9144	6.00	115.8649	6.00	119.6679

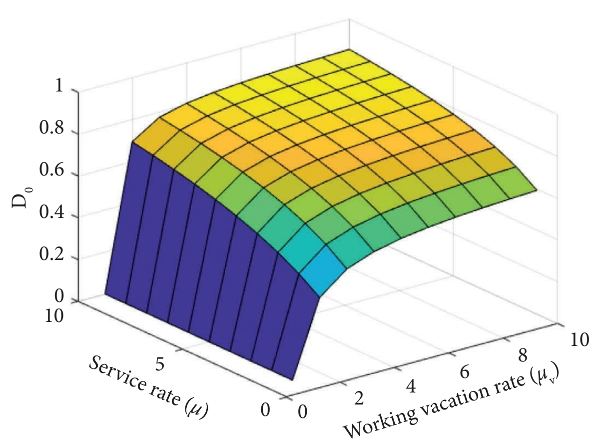

Figure 13: D_0 vs μ_b and μ_v.

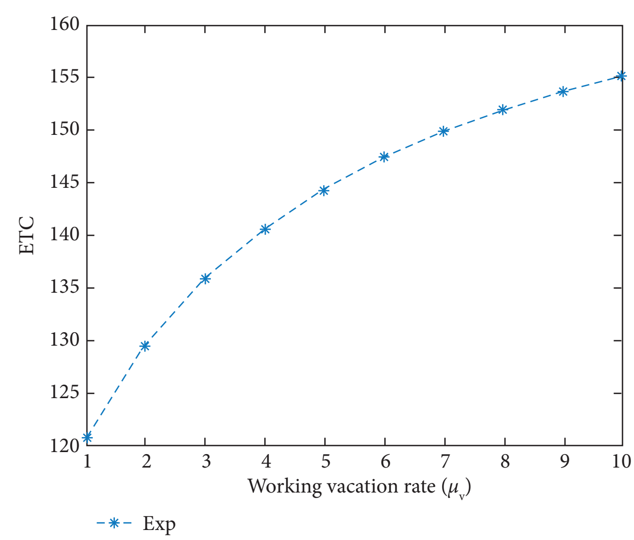

Figure 15: The influence of μ_v on ETC.

stability criteria. We obtain ETC = 115.87 from the parameter values in the estimated total cost per unit of time.

Table 6 and Figures 15–18 depict system parameters' numerical and graphical effects on cost functions. Our research directs us to the conclusion that when the cost parameters and decision parameter (θ) rise, the expected values of TC grow linearly. Similarly, a sensitivity analysis of some of the system's parameters can be conducted. The graphs (from Figures 15–18) show the effect of some system parameters (λ, θ, μ, μ_v) on the total expected cost per unit of time.

8. Results and Discussion

To provide a clear insight into the present problem, some special features of the solutions like doctor's idle time, the average number of patients in the original queue and orbit

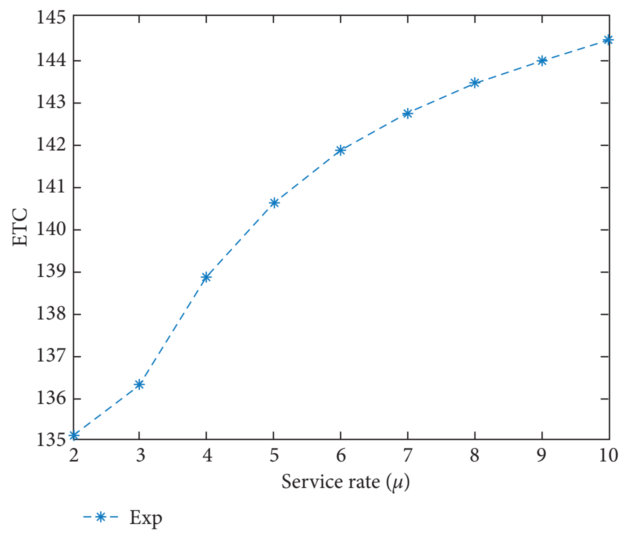

Figure 16: The influence of μ_b on ETC.

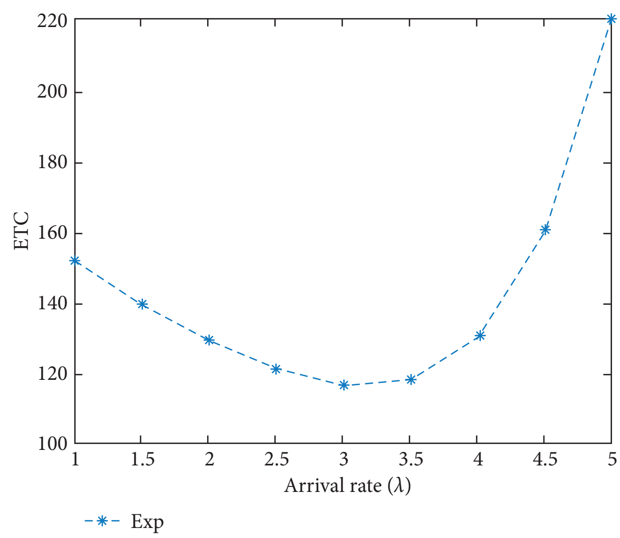

Figure 17: The influence of λ on ETC.

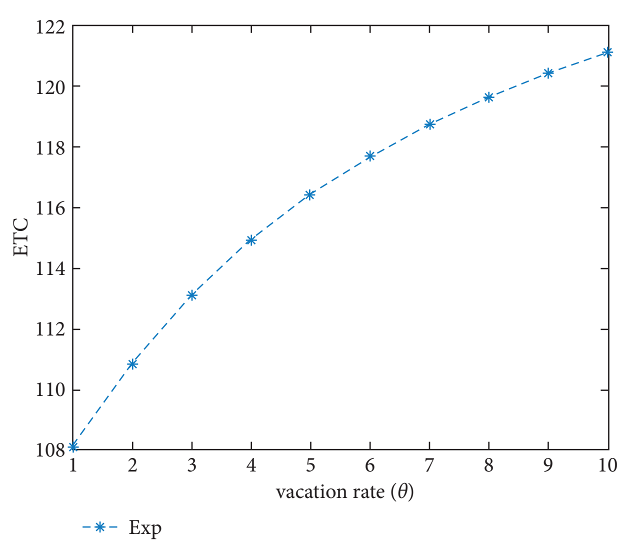

Figure 18: The influence of θ on ETC.

```
Begin
Input: λ, a, μ_b, μ_v and θ.
Compute: the value of ρ
If (ρ < 1)
Compute: Probabilities of system states
Compute: Mean orbit and system size
Compute: Mean busy time
Compute: Expected total cost
Else
Change the input of the parameters by satisfying ρ < 1.
End
Output: D_0, L_1, L_2 and ETC
```

Algorithm 1: Algorithm to compute D_0, L_1, L_2, and ETC.

queue, mean waiting time of the patient in the hospital, and expected total cost of the model are evaluated numerically in Tables 1–6 and presented graphically in form of Figures 3–18 (by Algorithm 1). The main findings of this investigation in the hospital system are summarized as follows:

(i) The chief doctor's resting time (idle time, D_0) decreases by increasing the arrival rate's value of patients for treatment (λ) and additional treatment rate (μ_v). In contrast, it increases for raising the value of rerequesting rate (a), both treatment rates like regular and additional (μ_b and μ_v), and doctor's secondary job time (θ).

(ii) The average number of patients in the regular queue (patients in person, L_1) increases for increasing the value of the arrival rate of patients (λ), the rerequesting rate (a), additional treatment time (μ_v), and secondary job time (θ). At the same time, it decreases by increasing the value of the regular treatment rate (μ_b).

(iii) An increase in the mean number of patients in online requests (patients in online, L_2) increases the patient's arrival rate (λ) and different treatment rates (μ_b and μ_v). It falls for raising the values of retrial rate (a) and secondary job time (θ).

(iv) The expected total cost of this model (ETC) is a linearly increasing trend for the decision parameter (θ), and parameters (arrival and treatment rates, λ, μ_b, μ_v) are increased.

9. Conclusion

In this study, we address a new type of queueing system in two waiting queues (original queue and orbit queue) with working vacations ($M^{[1],[2]}/G/1/WV$). This model's main feature is that it contains two waiting queues with working vacations: the original and orbit queues. Various performance measures, such as the size of the queue length, orbit length, system busy period, and mean waiting times, are derived using the SVT and PGF approaches. Finally, we covered the sensitivity analysis for the system parameters and a cost optimization technique for the suggested model that reduces the average operating cost overall. Additionally, we used MATLAB software to present some numerical examples in graphic form. This model's outcomes are tremendously advantageous to hospital administrators. Researchers may create models in the future to optimize queues and servicing systems for others. They can also explore priority models or a single server queueing system for the multiwaiting station in the presence of working breakdowns, working vacations, etc. Furthermore, it must be focused on a few circumstances of real-life applications for COVID-19 emergencies to govern the patients correctly.

Appendix

A. Proof of Theorem 1

We introduce some notations based on [20] to prove the sufficient ergodicity condition. During a service time, let a_n ($n = 0, 1, 2, \ldots$) be the chance that n customers will be added to the original queue, and let b_n ($n = 0, 1, 2, \ldots$) represent the possibility that n customers will join the obit queue while on vacation, so that we may get

$$\begin{aligned} a_n &= \int_0^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x} dS_b(x), \quad n \geq 0, \\ b_n &= \int_0^\infty \frac{((\lambda + \theta)x)^n}{n!} e^{-(\lambda+\theta)x} dS_v(x), \quad n \geq 0. \end{aligned} \tag{A.1}$$

The GFs of a_n and b_n are as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} z^n a_n &= S_b^*(\lambda(1-z)), \\ \sum_{n=0}^{\infty} z^n b_n &= S_v^*(\theta + \lambda(1-z)). \end{aligned} \tag{A.2}$$

Let $P_{(k,j)(m,i)}$ represent the probability of a one-step transition; then, we have that $P_{(k,j)(m,i)} = P(N_{1,n+1} = m, N_{2,k} = i / N_{1,n} = k, N_{2,n} = j)$; then, define the following conditions.

(i) Case (i): $k=0$ and $j=0$,

$$P_{(0,0)(m,i)} = a_m\left(S_v^*(\theta+\lambda)\delta_{i,0} + A^*(\lambda)b_{(i+1)} + (1-A^*(\lambda))\left(1-\delta_{i,0}\right)b_i\right), \quad m\geq 0, i\geq 0. \tag{A.3}$$

(ii) Case (ii): $k=0$ and $j\geq 1$,

$$P_{(0,j)(m,i)} = \begin{cases} a_m\left(A^*(\lambda)\delta_{i,j-1} + (1-A^*(\lambda))\left(1-\delta_{i,j-1}\right)\right), & m\geq 0, i\geq j-1,\\ 0, & \text{Otherwise.}\end{cases} \tag{A.4}$$

(iii) Case (iii): $k\geq 0$ and $j\geq 0$,

$$P_{(k,j)(m,i)} = \begin{cases} a_{m-k+1,} & m\geq k-1, i=j,\\ 0, & \text{Otherwise.}\end{cases} \tag{A.5}$$

From the above conditions, we can get the following:

$$E\left[\frac{z_1^{N_{1,n+1}} z_2^{N_{2,n+1}}}{N_{1,n}} = k, N_{2,n}=j\right] = \begin{cases} \left(\dfrac{S_b^*(\lambda-\lambda z_1)}{z_2}\right)((S_v^*(\theta+\lambda-\lambda z_2)-1)A(z_2)+z_2V(z)), & k=0, j=0,\\ z_2^{j-1}S_b^*(\lambda-\lambda z_1)A(z_2), & k=0, j\geq 1,\\ z_1^{k-1}z_2^{j}S_b^*(\lambda-\lambda z_1), & k\geq 1, j\geq 0.\end{cases} \tag{A.6}$$

If the function $f(k,j) = k + (j/A^*(\lambda))$ is nonnegative, then $(k,j)\in\mathbb{N}\times\mathbb{N}$, and then the mean drift

$$\begin{aligned}\psi_{k,j} &= E\left[\left(N_{1,n+1}-N_{1,n}\right) + \frac{1}{A^*(\lambda)}\left(N_{2,n+1}-N_{2,n}\right)\frac{z_2^{N_{2,n+1}}}{N_{1,n}} = k, N_{2,n}=j\right]\\ &= \begin{cases} \lambda\beta^{(1)} + \left(\dfrac{\lambda}{\theta}\right)(1-S_v^*(\theta)) + \left(\dfrac{\lambda S_v^*(\theta)}{A^*(\lambda)}\right), & k=0, j=0,\\ \lambda\beta^{(1)}-1, & k=0, j\geq 1,\\ \lambda\beta^{(1)}-1, & k\geq 1, j\geq 0.\end{cases}\end{aligned} \tag{A.7}$$

Using Foster's criteria is highly feasible (see Pakes [24]), and the inequality $\rho < 1$ is a necessary and sufficient condition for ergodicity and the system to be stable. It is ensured by $D_0 > 0$.

B. Proof of Steady-State Equations

The steady-state equations are presented from equations (2)–(5), and they can be obtained by applying the SVT method. For a better reading, here we explained the derivation of equations (3) and (4) by the basic assumption of queueing literature:

$$\begin{aligned}
D_0(t+\Delta t) &= (1-(\lambda+\theta)\Delta t)D_0(t) + \int_0^\infty D_2(x,0,0,t)\mu_b(x)\Delta t dx + \int_0^\infty D_3(x,0,t)\mu_v(x)\Delta t dx - \theta\Delta t D_0(t),\\
D_1(x,n_2,t+\Delta t) &= (1-(\lambda+a(x))\Delta t)D_1(x,n_2,t), \quad n_2 \ge 1,\\
D_2(x,n_1,n_2,t+\Delta t) &= (1-(\lambda+\mu_b(x))\Delta t)D_2(x,n_1,n_2,t) - \lambda\Delta t D_2(x,n_1-1,n_2,t), \quad n_1 \ge 1, n_2 \ge 1,\\
D_3(x,n_2,t+\Delta t) &= (1-(\lambda+\theta+\mu_v(x))\Delta t)D_3(x,n_2,t) - \lambda\Delta t D_3(x,n_2-1,t), \quad n_2 \ge 1.
\end{aligned} \tag{B.1}$$

Applying the limiting behavior $\Delta t \longrightarrow 0$ in equation (B.1), then we get

$$\frac{dD_0(t)}{dt} = -(\lambda+\theta)D_0(t) + \int_0^\infty D_2(x,0,0,t)\mu_b(x)dx + \int_0^\infty D_3(x,0,t)\mu_v(x)dx - \theta D_0(t), \tag{B.2}$$

$$\frac{\partial D_1(x,n_2,t)}{\partial t} + \frac{\partial D_1(x,n_2,t)}{\partial x} = -(\lambda+a(x))D_1(x,n_2,t), \quad n_2 \ge 1, \tag{B.3}$$

$$\frac{\partial D_2(x,n_1,n_2,t)}{\partial t} + \frac{\partial D_2(x,n_1,n_2,t)}{\partial x} = -(\lambda+\mu_b(x))D_2(x,n_1,n_2,t) + \lambda D_2(x,n_1-1,n_2,t), \quad n_1 \ge 1,\ n_2 \ge 1, \tag{B.4}$$

$$\frac{\partial D_3(x,n_2,t)}{\partial t} + \frac{\partial D_3(x,n_2,t)}{\partial x} = -(\lambda+\theta+\mu_v(x))D_3(x,n_2,t) + \lambda D_3(x,n_2-1,t), \quad n_2 \ge 1. \tag{B.5}$$

Applying the steady-state behavior limit $t \longrightarrow \infty$ from equations (B2)–(B5), we get the steady-state differential equations (2)–(5).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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