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Research Article

Study on the Solutions of Impulsive Integrodifferential Equations of Mixed Type Based on Infectious Disease Dynamical Systems

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Since ancient times, infectious diseases have been a major source of harm to human health. Therefore, scientists have established many mathematical models in the history of fighting infectious diseases to study the law of infection and then analyzed the practicability and effectiveness of various prevention and control measures, providing a scientific basis for human prevention and research of infectious diseases. However, due to the great differences in the transmission mechanisms and modes of many diseases, there are many kinds of infectious disease dynamic models, which make the research more and more difficult. With the continuous progress of infectious disease research technology, people have adopted more ways to prevent and interfere with the derivation and spread of infectious disease, which will make the state of infectious disease system change in an instant. The mutation of this state can be described more scientifically and reasonably by the mathematical impulse dynamic system, which makes the research more practical. Based on this, a time-delay differential system model of infectious disease under impulse effect was established by means of impulse differential equation theory. A class of periodic boundary value problems for impulsive integrodifferential equations of mixed type with integral boundary conditions was studied. The existence of periodic solutions of these equations was obtained by using the comparison theorem, upper and lower solution methods, and the monotone iteration technique. Finally, combined with the practical application, the established time-delay differential system model was applied to the prediction of the stability and persistence of the infectious disease dynamic system, and the correctness of the conclusion was further verified. This study provides some reference for the prevention and treatment of infectious diseases.

1. Introduction

Impulsive differential equation is a basic tool to study process state transition and has important applications in life science. Compared to differential equations without pulses, impulsive differential equations can more truly and accurately reflect the motion laws of nature and scientific fields, so they are widely used in the research of population dynamic systems, infectious disease dynamic models, microbial models, medical chemotherapy, and neural network systems [1–7]. In the history of human development, infectious disease is a major pain point affecting human health and life span. For this reason, scientists have established numerous mathematical models to quantitatively study the transmission law of infectious diseases in the history of continuous fight against infectious diseases and have made brilliant achievements, which have extraordinary practical significance for the control of infectious diseases. The use of mathematical models to study the law of the spread of infectious diseases first began with En'ko in 1889, and then in 1927, Kermack and Mckendrick established an infectious disease model and studied its spread law and epidemic trend by using dynamic methods. In recent years, the international research on infectious diseases and other biological systems has developed rapidly, and a large number of mathematical models have been used to analyze various infectious diseases. In the prevention and treatment of infectious diseases such as the novel coronavirus, influenza, SAS, and avian flu based on the mathematical modeling analysis, it is found that prevention and interference can be carried out through vaccination, wearing masks, medical treatment, and eliminating environmental pollution, thus making the state of the infectious disease system change instantly. The sudden change of this state can be described more reasonably and realistically by using the impulsive dynamic system in mathematics, so the differential equation model with an impulsive effect has been widely studied by scholars at home and abroad [7–10].

At the same time, the boundary value problem of differential equations is an important research field, and scholars have been very active in this field [2–4, 7–10]. In the previous theory of differential equations, people always assumed that the state of dynamic systems changes continuously with time; however, in the real world, there are many natural systems that change at certain intervals. For some reasons, the state of the system is often subjected to some temporary interference, which makes the state of the system change greatly in a short time. For example, when fish farmers release fish or catch fish, the fish in the pond will suddenly increase or decrease; when spraying agricultural and forestry pesticides, the number of pests will be greatly reduced. In the prevention and control of the novel coronavirus pneumonia and other infectious diseases, the number of infected people in this region has rapidly decreased through prevention and control and isolation and vaccination. Such problems, scientists found that in mathematical modeling, such disturbances that make the system change dramatically instantaneously can be expressed in the form of pulses, and this discovery quickly made the evolution process of pulse disturbances widely applied to the study of various infectious disease dynamic systems. Using the impulsive differential system to establish a more scientific and reasonable model, studying the transmission law and epidemic trend of the virus, and proposing prevention, immunization, and control strategies have quickly become the focus of research at home and abroad. For example, Yang et al. [10] gave the sufficient conditions for the elimination and persistence of the disease by using the persistence theory of the pulse system and disturbance technology in the SIS epidemic model with pulse inoculation and isolation and verified the reliability of the conclusion through numerical simulation. Haque et al. [9] analyzed four impulsive differential equation models in the study of Clonorchiasis sinensis, obtained an effective method to control the number of snails through fish inoculation, and analyzed the stability of the control time of Clonorchiasis sinensis. Cooke and Driessche [11] studied the existence, uniqueness, and stability of the solution of the integral-differential equation in the SEIRS disease transmission model of the population structure, thus providing the threshold and stability conditions of the model. In [12], Wen Wang studied the dynamic behavior of the SEIRS epidemic model with delayed pulses and obtained sufficient conditions for the local and global stability of the equilibrium state of the endemic system. Dietz [13] studied the change of age structure of host population caused by parasite death in the new model of microparasite and macroparasite transmission and predicted the change trend of the critical community of parasite population by deducing the results of the formula.

With the increasingly wide application of impulsive differential equations in the process of solving practical problems, many achievements have been made [14–26] and many new research methods have been obtained. For example, in literature [4, 14, 15], the authors studied the existence of positive solutions to these problems by using the fixed point index theory, cone stretching, and cone compression fixed point theorems. However, through the review of many literature studies, it is found that the common feature of these literature studies is that the equations studied contain relatively few cases of integral boundary conditions or integral impulsive conditions and most literature only contains one of them [17–20]. In particular, the periodic boundary value problems involving mixed tybe impulsive Integro-differential equation are rarely studied. In addition, the epidemic model of impulse interference is less studied in recent years, especially the model with integral impulse has not been studied. However, in the study of the infectious disease dynamic system, we found many cases about the joint impact of impulse vaccination, isolation, treatment, and population movement on the disease dynamics. The solution of such problems requires more complex and accurate impulse differential equation models, which have not been fully studied so far, so this is a very worthy research field. The integral impulsive conditions and integral boundary conditions have not been reported; however, this type of equation often appears in life science research, such as the prevention and vaccination of infectious diseases. For example, the dynamic system of infectious diseases studied in the literature [9, 10] intervenes with the spread of diseases by means of pulse vaccination and isolation. The persistence theory and perturbation technology of impulsive systems are used to study the conditions of the disappearance and persistence of the disease and further discuss the global stability of the infection-free periodic solution, which has theoretical value and practical significance for the prevention and treatment of infectious diseases. In addition, the epidemic model of impulse interference is less studied in recent years; especially, the model with integral impulse has not been studied. However, in the study of the infectious disease dynamic system, we found many cases about the joint impact of impulse vaccination, isolation, treatment, and population movement on the disease dynamics. The solution of such problems requires more complex and accurate impulse differential equation models, which have not been fully studied so far, so this is a very worthy research field. Therefore, in this paper, a class of differential models with integral impulsive conditions and integral boundary conditions was studied with the background of the infectious disease dynamics model. The model considered in this paper has some generality in the prevention and control of infectious diseases. Especially when controlling the epidemic of infectious diseases by means of pulse vaccination and isolation, in order to predict the number of vaccination, control cycle, continuity, and stability, we need to apply the theory and method of the pulse differential equation to study, so as to give more accurate results and the best scheme of rational drug use. The conclusion of this paper generalizes the conclusions of the existing literature to a large extent, and in practical application, it can better reflect the continuity and stability of the control measures under a variety of intervention conditions, which has certain practical significance and application value for the prevention and treatment of infectious diseases.

In this paper, we are concerned with the following impulsive integrodifferential equations of mixed type:

$$\begin{cases} u'(t) = g(t, u(t), u(\alpha(t)), (\mathrm{Au})(t), (\mathrm{Bu})(t)), & t \neq t_k, t \in J = [0, T], \\ \Delta u(t_k) = I_k\left(\int_{t_k-\tau_k}^{t_k} u(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} u(s)\mathrm{d}s\right), & k = 1, 2, \cdots, m, \\ u(0) = u(T) + k_1\int_0^T u(s)\mathrm{d}s + k_2, \end{cases} \tag{1}$$

where $k_1, k_2 \in R, \alpha \in [J, J].0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T.t_k < \alpha(t) \leq t, t \in (t_k, t_{k+1}]$,

$$\begin{gathered} I_k \in C(R, R), \Delta u(t_k) = u(t_{k_+}) - u(t_{k_-}), 0 < \sigma_{k-1} \leq \frac{t_k - t_{k-1}}{2}, 0 < \tau_k \leq \frac{t_k - t_{k-1}}{2}, \quad k = 1, 2, \cdots, m. \\ (\mathrm{Au})(t) = \int_0^t a(t, s)u(s)\mathrm{d}s, \ (Bu)(t) = \int_0^1 b(t, s)u(s)\mathrm{d}s, \end{gathered} \tag{2}$$

where $a \in C[F, R^+], F = \{(t, s) \in J \times J: t \geq s\}, b \in C[J^2, R^+], R^+ = [0, +\infty)$.

In paper [17], Chen and Qin studied the following impulsive integrodifferential equations:

$$\begin{cases} u'(t) = f(t, u(t), (\mathrm{Tu}(t)), (\mathrm{Su})(t), & t \in J'_+, \\ \Delta u|_{t_k} = I_k(u(t_k)), & k = 1, 2, \cdots, \\ u(\infty) = \gamma u(\eta) + \beta u(0), \end{cases} \tag{3}$$

where $J = [0, +\infty), J_+ = (0, +\infty), 0 < t_1 \cdots < t_k < \cdots, t_k \longrightarrow \infty, J'_+ = J_+ \backslash \{t_1, \cdots, t_k, \cdots\}$, $C \in [J_+ \times P_+ \times P_+ \times P_+, P], I_k \in C[P_+, P_+](k = 1, 2, \cdots), 0 \leq \gamma < 1, \beta + \gamma > 1, t_{m-1} < \eta \leq t_m$. The author investigated the multipositive solutions by applying the fixed point theorem of cone expansion-compression.

Zhang et al. [19] used the fixed point theorem of strict set contraction operators to study the following impulsive integrodifferential equations:

$$\begin{cases} x''(t) + w(t)f(t, x(t), x'(t), (Ax)(t), (Bx)(t)) = \theta, & t \neq t_k, t \in J, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \\ \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k), x'(t_k)), & k = 1, 2, \cdots, m, \\ x(0) = x(1) = \int_0^1 v(s)x(s)\mathrm{d}s, \end{cases} \tag{4}$$

where $w \in C(J, [0, +\infty)), f \in C[J \times E \times E \times E \times E, E], J = [0, 1], I_K \in C[E, E]$, and $\overline{I}_k \in C[E \times E, E], v \in L^1[0, 1]$ are nonnegative numbers.

Recently, Chatthai et al. in [20] concerned a class of first-order impulsive integrodifferential equations with multi-point boundary conditions:

$$\begin{cases} x'(t) = f(t, x(t), (Fu)(t), (Su)(t)), & t \in J = [0, T], t \neq t_k, \\ \Delta x(t_k) = I_k\left(\sum_{l=1}^{c_k} \rho_l^k x(\eta_l^k)\right), & k = 1, 2, \cdots, m, \\ x(0) + \mu\sum_{k=1}^m \sum_{l=1}^{c_k} \tau_l^k x(\eta_l^k)\rho_l^k = x(T), \end{cases} \tag{5}$$

where $f \in C(J \times R^3, R), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.

$\Delta x(t_k) = I_k(\int_{t_k-\tau_k}^{t_k} u(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} u(s)ds), k = 1, 2, \cdots, m.$

By using the method of lower and upper solutions in reversed order coupled with the monotone iterative technique, the authors obtained the extremal solutions of the boundary value problem.

In recent years, there have been few studies on mixed impulsive integral-differential equations. Inspired by literature [19, 20] and combined with pulses caused by various disturbance factors of the infectious disease dynamic system, this paper studied the infectious disease model with pulse delay. Compared with the simple time-delay model and impulsive model, the boundary conditions and impulsive integral considered were more complex and the established equation system was more general. This paper established the time-delay differential system model of infectious disease under the pulse effect, gave the existence condition of the solution under pulse action, used upper and lower solution methods and the monotone iteration technique to study the stability of the periodic solution of the system under pulse action, and extended it to the practical application case of infectious disease prevention and control, hoping to provide a new solution for the optimal control and prevention of infectious disease.

2. Preliminaries

Let $J^- = J \backslash \{t_1, t_2, \cdots, t_m\}, PC(J) = \{u: J \to R \mid u(t)$ be continuous at $t \neq t_k$, $u(t_k^-) = u(t_k)$, $k = 1, 2, \cdots, m\}$. $PC'(J) = \{u \in \mathrm{PC}(J): u'$ is continuous in J^-, $u'(0^+), u'(T^-), u'(t_{t_k}^+)$ and $u'(t_{t_k}^-)$ exist, $k = 1, 2, \cdots, m\}$. Let $\tau = \max\{t_{k+1} - t_k\}, k = 0, 1, 2, \cdots, m$.

$\mathrm{PC}(J)$ and $PC'(J)$ are Banach spaces with the following norm:

$$\|u\|_{\mathrm{PC}} = \sup\{|u(t)|: t \in J\}, \|u\|_{PC'} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}. \tag{6}$$

Lemma 1 (Comparison theorem). *There exist $M > 0, M_1, M_2, M_3 \geq 0, 0 \leq D_k < 1, k = 1, 2, \cdots, m$, satisfying*

$$\begin{cases} u'(t) + Mu(t) + M_1 u(\alpha(t)) + M_2(\mathrm{Au})(t) + M_3(\mathrm{Bu})(t) \leq 0, & t \neq t_k, t \in [0, T], \\ \Delta u(t_k) \leq -D_k \displaystyle\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} u(s)ds, & k = 1, 2, \cdots, m, \\ u(0) \leq u(T), \end{cases} \tag{7}$$

where $0 < \sigma_{k-1} \leq (t_k - t_{k-1})/2, 0 < \tau_k \leq (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$ and $\int_0^T [M + M_1 + M_2 \int_0^t a(t,s)ds + M_3 \int_0^T b(t,s)\mathrm{d}s]\mathrm{d}t - 1/M \sum_{k=1}^m D_k (e^{M\tau_k} - e^{M(\tau - \sigma_{k-1})}) \leq 1$. Then $u(t) \leq 0, \forall t \in J$.

Proof. Let $v(t) = u(t)e^{Mt}$, we have

$$\begin{cases} v'(t) \leq -M_1 e^{M(t-s)} v(\alpha(t)) - M_2 \displaystyle\int_0^t a(t,s) e^{M(t-s)} v(s)\mathrm{d}s - M_3 \int_0^T b(t,s) e^{M(t-s)} v(s)\mathrm{d}s, \\ \Delta v(t_k) \leq -D_k \displaystyle\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)} v(s)\mathrm{d}s, \quad k = 1, 2, \cdots, m, \\ v(0) \leq v(T) e^{-\mathrm{M}t}. \end{cases} \tag{8}$$

In order to prove that the conclusion holds, the proof by contradiction is used here. Suppose $v(t) > 0$, for all $t \in J$. It is discussed in the following two cases:

(i) There exists $t^- \in J$, satisfying $v(t^-) > 0$, and $v(t) \geq 0, \forall t \in J$.

(ii) There exists $t_*, t^* \in J$, satisfying $v(t_*) < 0, v(t^*) > 0$.

Case (i): From (8), we have $v'(t) < 0\,(t \in J)$ and $\Delta v(t_k) \le 0$. So, $v(t)$ is nonincreasing, it shows $v(0) \ge v(t^-) > 0$ and $v(T) \le v(0)$. Again by (8), we have $v(T) \le v(0) \le e^{-MT} v(T)$, hence $v(T) \le 0$, namely, $v(0) \le 0$, which is a contradiction.

Case (ii): Let $t_* \in (t_i, t_{i+1}], i \in \{0, 1, 2, \cdots, m\}$, such that $v(t_*) = \inf\{v(t)\colon t \in J\} < 0$, and $t^* \in (t_i, t_{i+1}], j \in \{0, 1, 2, \cdots, m\}$, such that $v(t^*) > 0$.

If $t_* < t^*$, then $i \le j$, by integration of (8), we can get

$$\begin{aligned} v(t^*) &= v(t_*) + \int_{t_*}^{t^*} v'(s)\mathrm{d}s + \sum_{0<t_k<t^*} \Delta v(t_k) \\ &\le v(t_*) - v(t_*)\left[\int_0^T \left[M + M_1 + M_2 \int_0^t a(t,s)\mathrm{d}s + M_3 \int_0^T b(t,s)\mathrm{d}s\right]\mathrm{d}t + \sum_{k=1}^m D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)} ds\right] \\ &\le v(t_*) - v(t_*)\left[\int_0^T \left[M + M_1 + M_2 \int_0^t a(t,s)\mathrm{d}s + M_3 \int_0^T b(t,s)\mathrm{d}s\right]\mathrm{d}t - \frac{1}{M}\sum_{k=1}^m \left(e^{M\tau_k} - D_k e^{M(\tau-\sigma_{k-1})}\right)\right] \\ &\le v(t_*) - v(t_*) \\ &< 0, \end{aligned} \tag{9}$$

which contradicts $v(t^*) > 0$.

If $t_* > t^*$, then it is similar to the above proof. The lemma is proved.

Next, let us consider the following linear equation:

$$\begin{cases} u'(t) + \mathrm{Mu}(t) + M_1 u(\alpha(t)) + M_2(\mathrm{Au})(t) + M_3(\mathrm{Bu})(t) = \sigma(t), \quad t \in J^-, \\ \Delta u(t_k) = -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} u(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \eta(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)\mathrm{d}s\right) + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)\mathrm{d}s, \\ u(0) = \eta(T) + k_1 \int_0^T \eta(s)\mathrm{d}s + k_2, \end{cases} \tag{10}$$

where $M > 0, M_1, M_2, M_3 \ge 0, 0 \le D_k < 1, \sigma(t), \eta(t) \in PC(J)$, $0 < \sigma_{k-1} \le (t_k - t_{k-1})/2$,

$$0 < \tau_k \le \frac{t_k - t_{k-1}}{2}, \quad k = 1, 2, \cdots, m. \tag{11}$$

□

Lemma 2. *Assume that $u \in PC'(J)$ is a solution of (10) if and only if $u \in PC(J)$ is a solution of the following impulsive integral equations:*

$$\begin{aligned} u(t) &= e^{-Mt}\left[\eta(T) + k_1 \int_0^T \eta(s)ds + k_2\right] \\ &+ \int_0^t [\sigma(s) - M_1 u(\alpha(s)) - M_2(Au)(s) - M_3(Bu)(s)] e^{M(s-t)} ds \\ &+ \sum_{0<t_k<t} e^{M(t_k-t)} \cdot \left[-D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} u(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \eta(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)\mathrm{d}s\right) + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)\mathrm{d}s\right]. \end{aligned} \tag{12}$$

Proof. Assume that $u \in PC'(J)$ is a solution of (10). Let $v(t) = u(t)e^{Mt}, v \in PC'(J)$,

$$v'(t) = [\sigma(t) - M_1 u(\alpha(t)) - M_2 (\mathrm{Au})(t) - M_3 (\mathrm{Bu})(t)]e^{\mathrm{Mt}},$$
$$v(t) = v(0) + \int_0^t v'(s)\mathrm{d}s + \sum_{0<t_k<t} [v(t_k^+) - v(t_k)], \tag{13}$$

so

$$v(t) = u(0) + \int_0^t [\sigma(s) - M_1 u(\alpha(s)) - M_2 (Au)(s) - M_3 (Bu)(s)]e^{Ms} ds + \sum_{0<t_k<t} \Delta u(t_k)e^{Mt_k}. \tag{14}$$

Then,

$$\begin{aligned} u(t) = {} & e^{-Mt}\left[\eta(T) + k_1 \int_0^T \eta(s)\mathrm{d}s + k_2\right] \\ & + \int_0^t [\sigma(s) - M_1 u(\alpha(s)) - M_2 (\mathrm{Au})(s) - M_3 (\mathrm{Bu})(s)]e^{M(s-t)}\mathrm{d}s \\ & + \sum_{0<t_k<t} e^{M(t_k-t)} \cdot \left[-D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} u(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \eta(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)\mathrm{d}s\right) + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)\mathrm{d}s\right]. \end{aligned} \tag{15}$$

It is easy to know that $u(0) = \eta(T) + k_1 \int_0^T \eta(s)\mathrm{d}s + k_2, t \in [0,T]$, so $u(t)$ satisfies (12).

Conversely, if $u \in \mathrm{PC}(J)$ is a solution of (12), then by performing the direct differentiation of (12), we can easily verify that $u \in \mathrm{PC}(J)$ satisfies (10). The lemma is proved. □

Lemma 3. *Let $M > 0, M_1, M_2, M_3 \geq 0, 0 \leq D_k < 1, 0 < \sigma_{k-1} \leq (t_k - t_{k-1})/2$, $0 < \tau_k \leq (t_k - t_{k-1})/2, k = 1, 2, \cdots, m$, and*

$$\sup_{t\in J}\left\{\int_0^T\left[M_1 + M_2 \int_0^s a(s,r)\mathrm{d}r + M_3 \int_0^T b(s,r)\mathrm{d}r\right]e^{M(s-t)}ds + \sum_{k=1}^m D_k(t_k - t_{k-1} - \sigma_{k-1} - \tau_k)\right\} < 1, \tag{16}$$

then problem (10) has a unique solution.

Proof. For convenience, we define the operator T:

$$\begin{aligned} (Tu)(t) = {} & e^{-Mt}\left[\eta(T) + k_1 \int_0^T \eta(s)\mathrm{d}s + k_2\right] \\ & + \int_0^t [\sigma(s) - M_1 u(\alpha(s)) - M_2 (\mathrm{Au})(s) - M_3 (\mathrm{Bu})(s)]e^{M(s-t)}\mathrm{d}s \\ & + \sum_{0<t_k<t} e^{M(t_k-t)} \cdot \left[-D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} u(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \eta(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)\mathrm{d}s\right) + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)\mathrm{d}s\right]. \end{aligned} \tag{17}$$

If $u_1, u_2 \in PC'(J)$ are two solutions of (10), from Lemma 2, we have

$$
\begin{aligned}
& u_1(t)=(Tu_1)(t), u_2(t)=(Tu_2)(t),\\
& \| u_1-u_2\|_{PC}=\|(Tu_1)(t)-(Tu_2)(t)\|_{PC}\\
&\quad =\Big\|\int_0^t\left[M_1u_2(\alpha(s))+M_2\int_0^s a(s,r)u_2(r)\mathrm{d}r+M_3\int_0^T b(s,r)u_2(r)\mathrm{d}r\right]e^{M(s-t)}\mathrm{d}s\\
&\qquad +\sum_{0<t_k<t}e^{M(t_k-t)}\cdot D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}u_2(s)\mathrm{d}s\\
&\qquad -\int_0^t\left[M_1u_1(\alpha(s))+M_2\int_0^s a(s,r)u_1(r)\mathrm{d}r+M_3\int_0^T b(s,r)u_1(r)\mathrm{d}r\right]e^{M(s-t)}\mathrm{d}s\\
&\qquad -\sum_{0<t_k<t}e^{M(t_k-t)}\cdot D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}u_1(s)\mathrm{d}s\Big\|\\
&\quad =\sup_{t\in J}\Big|\int_0^t\Big[M_1(u_2(\alpha(s))-u_1(\alpha(s)))+M_2\int_0^s a(s,r)(u_2(r)-u_1(r))\mathrm{d}r\\
&\qquad +M_3\int_0^T b(s,r)(u_2(r)-u_1(r))\mathrm{d}r\Big]e^{M(s-t)}\mathrm{d}s\\
&\qquad +\sum_{0<t_k<t}e^{M(t_k-t)}\cdot D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}(u_2(s)-u_1(s))\mathrm{d}s\Big|\\
&\quad \le\sup_{t\in J}\left\{\int_0^T\left[M_1+M_2\int_0^s a(s,r)\mathrm{d}r+M_3\int_0^T b(s,r)\mathrm{d}r\right]e^{M(s-t)}\mathrm{d}s+\sum_{k=1}^m D_k(t_k-t_{k-1}-\sigma_{k-1}-\tau_k)\right\}\|u_1-u_2\|_{PC}.
\end{aligned}
\tag{18}
$$

So, by the Banach fixed point theorem [27], impulsive integral (12) has a unique solution $u\in \mathrm{PC}(J)$. From Lemma 2, $u\in \mathrm{PC}(J)$ is a unique solution of (10). The lemma is proved. □

3. Main Result

In this section, the authors establish the existence conditions of the solution of (1) by using the upper and lower solution methods and the monotone iteration technique. The following definition is first given.

Definition 4. $\psi\in PC'(J)$ is called a lower solution of (1) if

$$
\begin{cases}
\psi'(t)\le g(t,\psi(t),\psi(\alpha(t)),(A\psi)(t),(B\psi)(t)), & t\ne t_k, t\in J=[0,T],\\
\Delta\psi(t_k)\le I_k\left(\int_{t_k-\tau_k}^{t_k}\psi(s)\mathrm{d}s-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\psi(s)\mathrm{d}s\right), & k=1,2,\cdots,m,\\
\psi(0)\le\psi(T)+k_1\int_0^T\psi(s)\mathrm{d}s+k_2.
\end{cases}
\tag{19}
$$

Identically, $\varphi\in PC'(J)$ is called an upper solution of (1) if

$$
\begin{cases}
\varphi'(t)\ge g(t,\varphi(t),\varphi(\alpha(t)),(A\varphi)(t),(B\varphi)(t)), & t\ne t_k, t\in J=[0,T],\\
\Delta\varphi(t_k)\ge I_k\left(\int_{t_k-\tau_k}^{t_k}\varphi(s)\mathrm{d}s-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\varphi(s)ds\right), & k=1,2,\cdots,m,\\
\varphi(0)\ge\varphi(T)+k_1\int_0^T\varphi(s)\mathrm{d}s+k_2.
\end{cases}
\tag{20}
$$

In addition, we denote $[\psi(t),\varphi(t)]=\{u\in PC(J)\colon \psi(t)\le u(t)\le\varphi(t), t\in J\}$.

Theorem 5. *Suppose the following conditions hold:*

(H_1) $\psi(t),\varphi(t)$ are the lower and upper solutions of (1), respectively, and satisfy $\psi(t)\le\varphi(t)$.

(H_2) $\exists M>0, M_1, M_2, M_3\ge 0$, such that

$$g(t,x,y,z,w)-g(t,\overline{x},\overline{y},\overline{z},\overline{w})\ge -M(x-\overline{x})-M_1(y-\overline{y})-M_2(z-\overline{z})-M_3(w-\overline{w}). \tag{21}$$

For $\psi(t)\le\overline{x}(t)\le x(t)\le\varphi(t)$, $\psi(t)\le\overline{y}(t)\le y(t)\le\varphi(t)$,

$$(A\psi)(t)\le\overline{z}(t)\le z(t)\le(A\varphi)(t),\ (B\psi)(t)\le\overline{w}(t)\le w(t)\le(B\varphi)(t),\quad \forall t\in J^{-}. \tag{22}$$

(H_3) $\exists 0\le D_k<1\,(k=1,2,\cdots,m)$, such that

$$\begin{aligned}&I_k\left(\int_{t_k-\tau_k}^{t_k}x(s)\mathrm{d}s-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}x(s)\mathrm{d}s\right)-I_k\left(\int_{t_k-\tau_k}^{t_k}y(s)\mathrm{d}s-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}y(s)\mathrm{d}s\right)\\&\quad\ge -D_k\cdot\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}(x(s)-y(s))\mathrm{d}s,\\&u(T)-\overline{u}(T)+k_1\left(\int_0^T(u(s)-\overline{u}(s))\mathrm{d}s\right)\ge 0.\end{aligned} \tag{23}$$

Here, $\psi(t)\le y(t)\le x(t)\le\varphi(t), k=1,2,\cdots,m.\psi(t)\le\overline{u}\le u\le\varphi(t), k_1\in R$.

Then, there exist sequences of monotone iterations $\{\psi_n(t)\},\{\varphi_n(t)\}\subset PC'(J)$ satisfying $\psi=\psi_0\le\psi_1\le\psi_2\le\cdots\le\psi_n\le\cdots\le\varphi_n\le\varphi_{n-1}\le\cdots\le\varphi_1\le\varphi_0=\varphi$, such that $\lim_{n\longrightarrow\infty}\psi_n=u_*(t), \lim_{n\longrightarrow\infty}\varphi_n=u^*(t)$ uniformly on J. Here, $u_*(t), u^*(t)$ are the minimum and maximum solutions of (1) on $[\psi(t),\varphi(t)]$, respectively.

Proof. First, we set up two sequences $\{\psi_n(t)\},\{\varphi_n(t)\}$ which satisfy the following equations:

$$\begin{cases}\psi_i'(t)+M\psi_i(t)+M_1\psi_i(\alpha(t))+M_2(A\psi_i)(t)+M_3(B\psi_i)(t))\\=g(t,\psi_{i-1}(t),\psi_{i-1}(\alpha(t)),(A\psi_{i-1})(t),(B\psi_{i-1})(t))\\+M\psi_{i-1}(t)+M_1\psi_{i-1}(\alpha(t))+M_2(A\psi_{i-1})(t)+M_3(B\psi_{i-1})(t))\\\Delta\psi_i(t_k)=-D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\psi_i(s)ds+I_k\left(\int_{t_k-\tau_k}^{t_k}\psi_{i-1}(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\psi_{i-1}(s)\mathrm{d}s\right)\\+D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\psi_{i-1}(s)\mathrm{d}s,\\\psi_i(0)=\psi_{i-1}(T)+k_1\int_0^T\psi_{i-1}(s)\mathrm{d}s+k_2,\end{cases} \tag{24}$$

$$\begin{cases} \varphi_i'(t) + M\varphi_i(t) + M_1\varphi_i(\alpha(t)) + M_2(A\varphi_i)(t) + M_3(B\varphi_i)(t)) \\ = g(t, \varphi_{i-1}(t), \varphi_{i-1}(\alpha(t)), (A\varphi_{i-1})(t), (B\varphi_{i-1})(t)) \\ + M\varphi_{i-1}(t) + M_1\varphi_{i-1}(\alpha(t)) + M_2(A\varphi_{i-1})(t) + M_3(B\varphi_{i-1})(t)) \\ \Delta\varphi_i(t_k) = -D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\varphi_i(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k}\varphi_{i-1}(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\varphi_{i-1}(s)\mathrm{d}s\right) \\ + D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\varphi_{i-1}(s)\mathrm{d}s, \\ \varphi_i(0) = \varphi_{i-1}(T) + k_1\int_0^T\varphi_{i-1}(s)\mathrm{d}s + k_2. \end{cases} \tag{25}$$

Next, the conclusion is proved to hold in three steps. Step one, we show that $\psi_i \le \psi_{i+1}, \varphi_i \le \varphi_{i-1}, i = 1, 2, \cdots, n$.

Let $q(t) = \psi_0(t) - \psi_1(t)$, then

$$\begin{aligned} & q'(t) + Mq(t) + M_1q(\alpha(t)) + M_2(Aq)(t) + M_3(Bq)(t)) \\ & = \psi_0'(t) + M\psi_0(t) + M_1\psi_0(\alpha(t)) + M_2(A\psi_0)(t) + M_3(B\psi_0)(t)) \\ & \quad - [g(t, \psi_0(t), \psi_0(\alpha(t)), (A\psi_0)(t), (B\psi_0)(t)) + M\psi_0(t) + M_1\psi_0(\alpha(t)) + M_2(A\psi_0)(t) + M_3(B\psi_0)(t)] \\ & = \psi_0'(t) - g(t, \psi_0(t), \psi_0(\alpha(t)), (A\psi_0)(t), (B\psi_0)(t)) \\ & \le 0, \\ \Delta q(t_k) & = \Delta\psi_0(t_k) - \Delta\psi_1(t_k) \\ & \le -D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}(\psi_0(s) - \psi_1(s))ds \\ & = -D_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}q(s)\mathrm{d}s, \quad k = 1, 2, \cdots m, \\ q(0) & = \psi_0(0) - \psi_1(0) \le \psi_0(T) - \psi_1(T) = q(T). \end{aligned} \tag{26}$$

From Lemma 1, we have $q(t) \le 0, \psi_0(t) \le \psi_1(t), t \in J$. It is similar to the above proof, we can verify $\varphi_1(t) \ge \varphi_0(t)$. By mathematical induction, we can get

$$\psi_i \le \psi_{i+1}, \varphi_i \le \varphi_{i-1}, \quad i = 1, 2, \cdots, n. \tag{27}$$

Step two, we prove that $\psi_1 < \varphi_1$ if $\psi_0 < \varphi_0$. Let $q(t) = \psi_1(t) - \varphi_1(t), \forall t \in J^-$, we have

$$\begin{aligned} & q'(t) + Mq(t) + M_1q(\alpha(t)) + M_2(Aq)(t) + M_3(Bq)(t)) \\ & = \psi_1'(t) + M\psi_1(t) + M_1\psi_1(\alpha(t)) + M_2(A\psi_1)(t) + M_3(B\psi_1)(t)) \\ & \quad -\varphi_1'(t) - M\varphi_1(t) - M_1\varphi_1(\alpha(t)) - M_2(A\varphi_1)(t) - M_3(B\varphi_1)(t)) \\ & = [g(t, \psi_0(t), \psi_0(\alpha(t)), (A\psi_0)(t), (B\psi_0)(t)) + M\psi_0(t) + M_1\psi_0(\alpha(t)) + M_2(A\psi_0)(t) + M_3(B\psi_0)(t)] \\ & \quad - [g(t, \varphi_0(t), \varphi_0(\alpha(t)), (A\varphi_0)(t), (B\varphi_0)(t)) + M\varphi_0(t) + M_1\varphi_0(\alpha(t)) + M_2(A\varphi_0)(t) + M_3(B\varphi_0)(t)] \\ & \le 0, \\ \Delta q(t_k) & = \Delta\psi_1(t_k) - \Delta\varphi_1(t_k) \end{aligned}$$

$$
\begin{aligned}
&= -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \psi_1(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \psi_0(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \psi_0(s)\mathrm{d}s\right) \\
&\quad + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \psi_0(s)\mathrm{d}s - \left[-D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \varphi_1(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \varphi_0(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \varphi_0(s)\mathrm{d}s\right) + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \varphi_0(s)\mathrm{d}s\right] \\
&\leq -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} (\psi_1(s) - \varphi_1(s))\mathrm{d}s \\
&= -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} q(s)\mathrm{d}s, \quad k = 1, 2, \cdots m, \\
q(0) &= \psi_1(0) - \varphi_1(0) \leq \psi_1(T) - \psi_1(T) = q(T).
\end{aligned} \tag{28}
$$

By Lemma 1, we can get $q(t) \leq 0\,(t \in J)$, i.e. $\psi_1 < \varphi_1$. Still by mathematical induction, we have $\psi_i \leq \varphi_i, i = 1, 2, \cdots, n$.

Step three, from the conclusion of the above two steps, we get

$$
\psi_0 \leq \psi_1 \leq \psi_2 \leq \cdots \leq \psi_n \leq \cdots \leq \varphi_n \leq \varphi_{n-1} \leq \cdots \leq \varphi_1 \leq \varphi_0. \tag{29}
$$

It is obvious that each $\psi_i, \varphi_i\,(i = 1, 2, \cdots)$ satisfies (24) and (25). There exist u_*, u^* such that sequences $\{\psi_i(t)\}, \{\varphi_i(t)\}$ satisfy $\lim_{n\longrightarrow\infty} \psi_n = u_*(t), \lim_{n\longrightarrow\infty} \varphi_n = u^*(t)$ uniformly on J. So, $u_*(t), u^*(t)$ satisfy (1).

Finally, we verify that $u_*(t), u^*(t)$ are extreme solutions of (1).

Assume that $u(t)$ is any solution of (1) and satisfies $\psi_0(t) \leq u(t) \leq \varphi_0(t), t \in J$.

Let $q(t) = \psi_{n+1}(t) - u(t), t \in J^-$, we have

$$
\begin{aligned}
q'(t) &= \psi_{n+1}'(t) - u'(t) \\
&= \left[g(t, \psi_n(t), \psi_n(\alpha(t)), (A\psi_n)(t), (B\psi_n)(t)) \right.\\
&\quad + M\psi_n(t) + M_1\psi_n(\alpha(t)) + M_2(A\psi_n)(t) + M_3(B\psi_n)(t) \\
&\quad \left. -M\psi_{n+1}(t) - M_1\psi_{n+1}(\alpha(t)) - M_2(A\psi_{n+1})(t) - M_3(B\psi_{n+1})(t)\right] \\
&\quad - g(t, u(t), u(\alpha(t)), (\mathrm{Au})(t), (\mathrm{Bu})(t)) \\
&\leq -M(\psi_{n+1}(t) - u(t)) - M_1(\psi_{n+1}(\alpha(t)) - u(\alpha(t))) \\
&\quad -M_2(A(\psi_{n+1} - u))(t) - M_3(B(\psi_{n+1} - u))(t)] \\
&= -Mq(t) - M_1 q(\alpha(t)) - M_2(Aq)(t) - M_3(Bq)(t), \\
\Delta q(t_k) &= \Delta\psi_{n+1}(t_k) - \Delta u(t_k) \\
&= -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \psi_{n+1}(s)\mathrm{d}s + I_k\left(\int_{t_k-\tau_k}^{t_k} \psi_n(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \psi_n(s)\mathrm{d}s\right) \\
&\quad + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \psi_n(s)\mathrm{d}s - I_k\left(\int_{t_k-\tau_k}^{t_k} u(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} u(s)\mathrm{d}s\right) \\
&\leq -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \psi_{n+1}(s)\mathrm{d}s - D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} (\psi_n(s) - u(s))\mathrm{d}s + D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \psi_n(s)\mathrm{d}s \\
&= -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} (\psi_{n+1}(s) - u(s))\mathrm{d}s \\
&= -D_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} q(s)\mathrm{d}s, \quad k = 1, 2, \cdots m, \\
q(0) &= \psi_{n+1}(0) - u(0) \leq \psi_{n+1}(T) - u(T) = q(T).
\end{aligned} \tag{30}
$$

From Lemma 1, we can get $q(t) \le 0\,(t \in J)$, i.e., $\psi_{n+1}(t) \le u(t)\,(t \in J)$. Similarly, it is easy to see that $u \le \varphi_{n+1}(t \in J)$.

Therefore, $\psi_{n+1}(t) \le u(t) \le \varphi_{n+1}(t)$ for any $t \in J$, this implies that $u_*(t) \le u(t) \le u^*(t)$, the theorem is proved. □

4. Application

Impulsive differential equation theory is based on the continuous differential equation. Scientists use dynamics to describe the laws of life phenomena and then build mathematical models based on impulsive differential equation theory to further explore scientific problems, which promotes more understanding of life science. The study of these mathematical models has led to the exploration of a large number of differential equations, and it is found that some phenomena in the optimal control of some life phenomena are not a continuous process, which cannot be studied simply by differential equations. For example, in the dynamic system of infectious diseases, in order to prevent and control the spread of infectious diseases, pulse vaccination is one of the commonly used measures. This kind of time-delay infectious disease model with pulse preventive vaccination is a typical impulsive differential equation model, which needs to apply the theory and method of impulsive differential equation to carry out research and give the optimal control strategy. Therefore, impulsive differential equations are widely used in infectious disease dynamic systems, and it is of great practical significance to study the optimal prevention and control strategy of these systems by using the theory of impulsive differential equations.

Based on the above theoretical derivation and the infectious disease dynamic system model, we consider the following infectious disease system model:

$$\begin{cases} u'(t) = -\dfrac{1}{20}t^4 u(t) + \dfrac{1}{15}t^3\left[t - u\left(\dfrac{t}{2}\right)\right] + \dfrac{1}{500}t\left[t^5 - \displaystyle\int_0^{t/2} 8\mathrm{tsu}(s)\mathrm{d}s\right]^5 & \\ +\dfrac{1}{700}t^2\left[t^2 - \displaystyle\int_0^1 2t^2\mathrm{su}(s)\mathrm{d}s\right]^7, & t \ne \dfrac{1}{2},\ t \in J = [0,1], \\ \Delta u(t_1) = -\dfrac{1}{12}\left(\displaystyle\int_{1/3}^{t_1} u(s)\mathrm{d}s - \int_0^{1/6} u(s)\mathrm{d}s\right), & t_1 = \dfrac{1}{2}, \\ u(0) = u(1) - \dfrac{1}{10}\displaystyle\int_0^1 u(s)\mathrm{d}s + \dfrac{1}{9}, & \end{cases} \tag{31}$$

where $T = 1, t_0 = 0, t_1 = 1/2, \tau = 1/2, \sigma_0 = 1/6, \tau_1 = 1/6. k_1 = -1/10, k_2 = 1/9$.

In this system, $u(t)$ is susceptible to infection at time t and $\Delta u(1/2)$ indicates the change of the infected person at time 1/2.

The following we prove that system (31) is stable.

$$\begin{aligned} g(t,u(t),u(\alpha(t)),(Au)(t),(Bu)(t)) &= -1/20t^4u(t) + 1/15t^3[t - u(t/2)] + 1/500t\left[t^5 - \int_0^{t/2} 8tsu(s)\mathrm{d}s\right]^5 \\ &\quad + \frac{1}{700}t^2\left[t^2 - \int_0^1 2t^2 su(s)\mathrm{d}s\right]^7. \end{aligned} \tag{32}$$

Let $\psi_0(t) = 0, \varphi_0(t) = 1$, It is not hard to verify that $\psi_0(t) = 0$ is the lower solution and $\varphi_0(t) = 1$ is the upper solution of system (31).

By computing,

$$\begin{aligned} g(t,x,y,z,w) - g(t,\overline{x},\overline{y},\overline{z},\overline{w}) &= -\frac{1}{20}t^4(x-\overline{x}) + \frac{1}{15}t^3[(t-y)-(t-\overline{y})] \\ &\quad + \frac{1}{500}t\left[\left(t^5-z\right)^5 - \left(t^5-\overline{z}\right)^5\right] + \frac{1}{700}t^2\left[\left(t^2-w\right)^7 - \left(t^2-\overline{w}\right)^7\right] \\ &\geq -\frac{1}{20}t^4(x-\overline{x}) - \frac{1}{15}t^3(y-\overline{y}) - \frac{1}{100}t(z-\overline{z}) - \frac{1}{100}t^2(w-\overline{w}) \\ &\geq -\frac{1}{20}(x-\overline{x}) - \frac{1}{15}(y-\overline{y}) - \frac{1}{100}(z-\overline{z}) - \frac{1}{100}(w-\overline{w}), \end{aligned} \tag{33}$$

where $\psi_0(t) \leq \overline{x}(t) \leq x(t) \leq \varphi_0(t)$, $\psi_0(\alpha(t)) \leq \overline{y} \leq y \leq \varphi_0(\alpha(t))$,

$$A\psi_0(t) \leq \overline{z} \leq z \leq A\varphi_0(t), B\psi_0(t) \leq \overline{w} \leq w \leq B\varphi_0(t). \tag{34}$$

So, we can get

$$\begin{aligned} &M = \frac{1}{20}, M_1 = \frac{1}{15}, M_2 = \frac{1}{100}, M_3 = \frac{1}{100}, \\ &\int_0^T \left[M + M_1 + M_2\int_0^t a(t,s)\mathrm{d}s + M_3\int_0^T b(t,s)\mathrm{d}s\right]\mathrm{d}t - \frac{1}{M}\sum_{k=1}^m D_k\left(e^{M\tau_k} - e^{M(\tau-\sigma_{k-1})}\right) \\ &= \int_0^1\left[\frac{1}{20} + \frac{1}{15} + \frac{1}{100}\int_0^{t/2} 8ts\mathrm{d}s + \frac{1}{100}\int_0^1 2t^2s\mathrm{d}s\right]\mathrm{d}t - 20\cdot\frac{1}{12}\left(e^{1/120} - e^{1/60}\right) \\ &\leq \frac{1}{20} + \frac{1}{15} + \frac{1}{100} + \frac{1}{100} + \frac{1}{75} \\ &< 1, \end{aligned} \tag{35}$$

we can get

$$\begin{aligned} &I_k\left(\int_{t_k-\tau_k}^{t_k} x(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} x(s)\mathrm{d}s\right) - I_k\left(\int_{t_k-\tau_k}^{t_k} y(s)\mathrm{d}s - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} y(s)\mathrm{d}s\right) \\ &= -\frac{1}{12}\left[\int_{1/3}^{1/2}(x(s)-y(s))\mathrm{d}s - \int_0^{1/6}(x(s)-y(s))\mathrm{d}s\right] \\ &\geq -\frac{1}{12}\int_{1/6}^{1/3}(x(s)-y(s))\mathrm{d}s, \end{aligned} \tag{36}$$

where $\psi_0(t_k) \leq y(t_k) \leq x(t_k) \leq \varphi_0(t_k), D_k = 1/12$.

Obviously,

$$u(T) - \overline{u}(T) + k_1\left(\int_0^T (u(s) - \overline{u}(s))\mathrm{d}s\right) \geq 0, \tag{37}$$

where $\psi_0(t) \leq \overline{u} \leq u \leq \varphi_0(t), t \in [0,T]$.

Therefore, the all conditions of Theorem 5 are satisfied. Thus, system (31) has the extremal system of solutions $u_*, u^* \in [\psi_0, \varphi_0]$, which can be obtained by taking limits from some iterative sequences. Combining the initial conditions of the example and the theoretical derivation in this paper, it can be concluded that the system has certain stability at one time of prevention (interference) at half time; that is, the system will tend to a disease-free equilibrium point at half period, indicating that in the prevention of infectious diseases, the susceptible persons can be reduced by pulse vaccination to reduce the possibility of disease transmission until the disease slowly disappears. In addition, we can further verify that increasing the number of inoculations (interference times), for example, when the impulse term of model (4.1) has three pulse moments ($t_1 = 1/4, t_2 = 1/3, t_3 = 1/2$), the stability conclusion of the system is still true by applying the theory of this paper. However, in this case, the theory of literature [7, 9] is not applicable to the application example in this paper. Therefore, this paper studies an infectious disease model with multiple pulse disturbances. The model studied in this paper is an extension and optimization of the existing infectious disease model, which is more universal and realistic in the prevention and control of infectious diseases.

5. Conclusion

In life science research, the occurrence of many biological phenomena and people's optimal control of some life phenomena are not a continuous process, which will be interfered by some external conditions or human factors, resulting in a short time disturbance of the system state. This disturbance with time delay can be studied by pulsed differential equation theory. Especially in the prevention and control of infectious diseases, this kind of pulse phenomenon with time delay is very common, and the theory of impulsive differential equation is commonly used to study such models in mathematics, so as to give the most scientific prevention strategy of infectious diseases. Based on this, a class of periodic boundary value problem of impulsive integrodifferential equations of mixed type with integral boundary conditions and integral impulsive conditions was studied in this paper. On the basis of the theory of impulsive differential equation, the comparison theorem of impulsive integrodifferential equation was established and the model studied was transformed into the corresponding linear integrodifferential equation. By defining the integral equation of the model, the condition of the unique solution was obtained. On the basis of this theory, the existence of the extreme solutions was further studied by means of the upper and lower solution methods combined with the monotone iteration technique and the stability of the periodic solutions of system model (1) was obtained. Finally, the validity of the mathematical model was verified by an application example of the infectious disease dynamic system model under pulse inoculation interference. This model provides a new way for the optimal control and prevention of the infectious disease model, which has certain theoretical value and practical significance.

Data Availability

All the datasets are provided within the main body of the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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