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Research Article

The Hautus-Type Inequality for Abstract Fractional Cauchy Problems and Its Observability

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In this paper, we investigate the observability of the fractional resolvent family, and we prove two main results: the first result shows a generalization of the Hautus-type test for observable exponentially stable semigroups to the fractional resolvent family and the second result shows the equivalence of the observability and the below boundedness of the linear operator on the wave packet when the generator conforms to a specific form.

1. Introduction

In this paper, we mainly investigate the observability of the following fractional differential equation in Hilbert space:

$$\begin{cases} D_t^{\alpha} u(t) = \mathrm{Au}(t), & t > 0 \\ u(0) = x \in X, & (\text{in addition } u'(0) = 0 \text{ if } \alpha > 1), \end{cases} \tag{1}$$

A is a closed densely defined linear operator on a Banach space X. Our main results can be listed as follows: First, we prove that the exact observability in finite time τ can deduce a Hautus type test with a constant depending on τ, and the assumptions are weaker than the classical one, for example [1], Theorem 6.5.3. Next, we show that if the resolvent is stable and observation operator B is commuted with coefficient operator A, then the Hautus type inequality is enough to prove the approximate observability. Second, we assume that $A = i^{\alpha} C - \epsilon$, where C is a negative self-adjoint operator and ϵ is a constant small enough. We prove that for such an operator A, the exact observability of resolvent generated by A can be deduced from the Hautus type test, and a perturbation result is given as a corollary. Finally, we show that if C has a compact resolvent, then we can use the conditions satisfied by the spectrum of the operator A to fully characterize its observability.

The properties of the fractional resolvent family have been studied very intensively over the past few years; for example, many stability, regularity, and continuity results concerning the fractional resolvent family can be found in [2] and references therein. Also, many pieces of literature on this topic provide applications for Caputo fractional calculus. In [3], authors investigate the stability of fractional evolution systems with memory. In [4–8], authors also did some quantitative analysis related to Caputo fractional calculus, and for many other results about fractional differential equations, we refer to [9–14].

The observability of C_0 semigroup is a classical topic, and there is a lot of literature on this subject; for example [1, 15], these two pieces of literature give extensive results on the observability of C_0 semigroups: the first one focuses on the elementary introduction and classical results and gives many applications and relations to wave equation or Schröndinger's equation, see also [16], while the second one provides many ideals and open problems about this subject. Also, there are some works about observability not only on general Hilbert space but also on some measurable sets, for example [7, 17]. In addition to studying the subject itself, this concept is also a powerful tool for proving the stability of semigroups; for example, in the literature [18], the authors proved that the semigroup generated by $A - \mathrm{BB}^*$ is strongly polynomially stable by assuming that the operators A, B satisfy a nonuniform Hatus-type test. In the literature [19], the observability and stabilization of magnetic Schrödinger equations have been investigated. If we concentrate on certain equations, observability can also be proved, for example [20].

Fractional order differential equations, and evolution systems containing fractional order differentials, are very common in nature, so the question of controllability and observability of this class of equations is also a matter of interest, and this is the motivation of this paper. Comparatively, the observability of fractional resolvent families has been much less studied, and many kinds of literature concentrate on fractional differential equations with boundary value conditions such as [14, 21]. The main difficulty in dealing with the fractional resolvent family is that the fractional derivative does not satisfy the chain rule and the fractional integral operator is a nonlocal operator; therefore, we use a different approach when dealing with fractional resolvent families, and the results are slightly different from the semigroup case.

This paper is organized as follows: in Section 2, some necessary definitions and results on observability, Mittag–Leffler functions, and fractional resolvent families are given. Section 3 is devoted to proving a Hautus-type necessary condition for exact observability. Also, Section 4 deals with the resolvent family generated by operator A which conforms to a specific form and characterizes the observability by the spectrum of A.

2. Preliminaries

In this paper, $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ means the complex plane, the real line, and the set of all integers, respectively. For every complex number x, $\mathrm{Re}(x), \mathrm{Im}(x)$ being its real, imaginary part and $\arg(x)$ being its angle. M, m are two constants that may change from line to line, and the constants $M(\tau, s)$, $m(\tau, s)$ vary with τ and s. In all cases, we assume that X and Y are Hilbert spaces and that A is a densely defined, closed, linear operator on X with $N(A)$, $D(A)$, and $R(A)$ being its kernel, domain, and range, respectively. $\sigma(A)$ and $\rho(A)$ signify the spectrum and resolvents set of A, respectively, and $B \in \mathscr{L}(X, Y)$ is a linear bounded operator map X to Y. Regularly, $*$ stands for the convolution on $\mathbb{R}_+$:

$$(f * g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s. \quad k \in L^1(\mathbb{R}_+), f \in L^1(\mathbb{R}_+, X), \tag{2}$$

and $\widehat{f}(\lambda)$ denote the Laplace transform of an exponentially bounded function $f \in L^1(\mathbb{R}_+, X)$, defined by

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\mathrm{d}t, \tag{3}$$

if this integral is convergent. Also, $\widetilde{f}(\lambda)$ denote the Fourier transform of a function $f \in L^1(\mathbb{R}, X)$, defined by

$$\widetilde{f}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} f(t)\mathrm{d}t, \tag{4}$$

if f is defined on $\mathbb{R}_+$, then we extend it to $\mathbb{R}$ by zero and still denote by f if no confusion occurs.

The sector $\Sigma(\omega, \theta)$ is defined as

$$\Sigma(\omega, \theta) := \{z \mid z \in \mathbb{C}, |\arg(z-\omega)| < \theta\}, \tag{5}$$

for $\omega \in R$ and $\theta \in (0, \pi]$; if $\omega = 0$, then we write $\Sigma(0, \theta) := \Sigma(\theta)$ for convenience.

We now define the observability of the fractional resolvent family.

Definition 1. If A generates a bounded fractional resolvent family $\{S_\alpha(t)\}$, then the pair (A, B, α) is exactly observable in time τ, and there is a constant $\mu > 0$ such that

$$\int_0^\tau \|\mathrm{BS}_\alpha(t)x\|^2 \mathrm{d}t \geq \mu \|x\|^2 \quad \forall x \in D(A). \tag{6}$$

The pair (A, B, α) is approximately observable in time τ if

$$\bigcap_{t \in [0,\tau]} N(\mathrm{BS}_\alpha(t)) = \{0\}. \tag{7}$$

Definition 2. Let $\phi(x) = \mathrm{BS}_\alpha(t)x \in \mathscr{L}(D(A), L^2((0,\infty), Y))$. The pair (A, B, α) is exactly (or approximately) observable in infinite time if ϕ is bounded from below (or $\ker(\phi) = \{0\}$).

Then, we recall the Mittag–Leffler function, which plays an important role in studying fractional differential equations. The properties and applications of this function can be found in the [13, 22, 23].

The Mittag–Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i}\int_{H_a} \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z}\mathrm{d}\mu, \quad z \in \mathbb{C}, \tag{8}$$

where $\alpha, \beta > 0$, H_a is the Hankel contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counterclockwise. We write $E_\alpha(z) := E_{\alpha,1}(z)$ if there is no confusion. The Mittag–Leffler function $E_\alpha(t)$ satisfies the fractional differential equation.

$$D_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha), \tag{9}$$

where D_t^α is the Caputo derivative of α-order (see [13, 24, 25]). The most useful properties of this function are the following integral:

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(st^\alpha)\mathrm{d}t = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - s}, \quad \mathrm{Re}(\lambda) > |s|^{1/\alpha}, \tag{10}$$

and their asymptotic expansion for $0 < \alpha < 2$ and $\beta > 0$:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{1-\beta/\alpha} \exp\left(z^{1/\alpha}\right) + \varepsilon_{\alpha,\beta}(z), \quad |\arg z| \leq \frac{1}{2}\alpha\pi, \tag{11}$$

$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < \frac{1}{2}\alpha\pi, \tag{12}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O\left(|z|^{-N}\right), \tag{13}$$

as $z \longrightarrow \infty$ and $N \in \mathbb{N}$ be a integer bigger than 1. From the asymptotic expansion, one knows that $E_\alpha(-\omega t^\alpha) = O(t^{-\alpha})$ as $t \longrightarrow \infty$ when $\omega > 0$.

Next, we define fractional resolvent families and list some properties here.

Definition 3 (see [24]) [Definition 3]. Let $0 < \alpha \le 2$, a family $\{S_\alpha(t)\}_{t\ge 0} \subset L(X)$ is called an α-times resolvent family generated by A if the following conditions are satisfied:

(a) $S_\alpha(t)$ is strongly continuous for $t \ge 0$ and $S_\alpha(0) = I$

(b) $S_\alpha(t)A \subset AS_\alpha(t)$ for $t \ge 0$; that is, $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for $x \in D(A)$

(c) for $x \in X$, the resolvent equation

$$S_\alpha(t)x = x + A\int_0^t g_\alpha(t-s)S_\alpha(t)x\mathrm{d}s, \tag{14}$$

holds for all $t \ge 0$, where $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$.

It is well known that the α-times resolvent family is uniquely determined by its generator A. If A generates an α-times resolvent family $S_\alpha(t)$, then the solution to the abstract fractional Cauchy problem (1) is given by $u(t) = S_\alpha(t)x$.

Definition 4. An α-times resolvent family $S_\alpha(t)$ is said to be exponentially bounded if there exists a constant $M \ge 1$ and $\omega \ge 0$ such that $\|S_\alpha(t)\| \le Me^{\omega t}$ for every $t \ge 0$. $S_\alpha(t)$ is called bounded if ω can be taken as 0, i.e., $\|S_\alpha(t)\| \le M$ for all $t \ge 0$.

Let $\theta_0 \in (0, \pi/2]$, an α-times resolvent family $S_\alpha(t)$ is called the analytic of angle θ_0 if $S_\alpha(t)$ admits an analytic extension to the sectorial sector $\Sigma(\theta_0)$. An analytic α-times resolvent family $S_\alpha(t)$ is called bounded if $\|S_\alpha(z)\|$ is uniformly bounded for $z \in \Sigma_\theta$ for any $0 < \theta < \theta_0$.

Lemma 5 (see [24]). *Theorems 2.8 and 2.9. Let $0 < \alpha \le 2$. Then, A generates an α-times resolvent family $S_\alpha(t)$ satisfying $\|S_\alpha(t)\| \le \mathrm{M}e^{\omega t}$ for every $t \ge 0$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and*

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda^{\alpha-1}R(\lambda^\alpha, A)\right)\right\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega, n \in \mathbb{N}_0. \tag{15}$$

In this case, $\{\lambda^\alpha\colon \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\mathrm{d}t, \quad \mathrm{Re}\lambda > \omega, \tag{16}$$

for every $x \in X$. In particular, if $S_\alpha(t)$ is bounded, then $\sup_{\lambda>0}\|\lambda R(\lambda, A)\| < \infty$.

3. A Hautus-Type Necessary Condition for Exact Observability

The Hautus-type necessary condition for exponential stable semigroup can be found in many works of literature, such as [1], but the conditions we assume here are different.

Theorem 6. *If A generates a bounded fractional resolvent family $\{S_\alpha(t)\}$,*

$$\|S_\alpha(t)x\| \le M\|x\|, \quad x \in X, t > 0. \tag{17}$$

If pair (A, B, α) is exactly observable in time τ, then for every $s \in \mathbb{C}$ and every $x \in D(A)$,

$$\|(\mathrm{sI} - A)x\|^2 + m(\tau, s)\|\mathrm{Bx}\|^2 \ge \mu\|x\|^2, \tag{18}$$

where $M(\tau, s) := \tau^{1/2}M\|B\|\int_0^\tau \|t^{\alpha-1}E_{\alpha,\alpha}(\mathrm{s}t^\alpha)\|\mathrm{d}t$ and $m(\tau, s) = \int_0^\tau \|E_\alpha(\mathrm{s}t^\alpha)\|^2\mathrm{d}t$.

Proof. Since pair (A, B, α) is exactly observable in time τ, we only need to prove that

$$M(\tau, s)\|(zI - A)x\|^2 + m(\tau, s)\|\mathrm{Bx}\|^2 \ge \|BS_\alpha(t)x\|^2_{L^2((0,\tau),X)}. \tag{19}$$

We choose $s \in C$ and $x \in D(A)$, we denote

$$q = (A - \mathrm{sI})x, \tag{20}$$

then we have

$$D_t^\alpha S_\alpha(t)x = S_\alpha(t)\mathrm{Ax} = sS_\alpha(t)x + S_\alpha(t)q, \tag{21}$$

and then we conclude that

$$S_\alpha(t)x = E_\alpha(\mathrm{s}t^\alpha)x + \int_0^t h(t-\tau)S_\alpha(\tau)\mathrm{q}\mathrm{d}\tau, \tag{22}$$

where $h(t) = t^{\alpha-1}E_{\alpha,\alpha}(\mathrm{s}t^\alpha)$.

Then, we have

$$\mathrm{BS}_\alpha(t)x = E_\alpha(\mathrm{s}t^\alpha)\mathrm{Bx} + (h * \mathrm{BS}_\alpha)(t)q. \tag{23}$$

By using the property of convolutions, we have

$$\begin{aligned}\|\mathrm{BS}_\alpha(t)x\|^2_{L^2((0,\tau),X)} \le{}& \|E_\alpha(\mathrm{s}t^\alpha)\|^2_{L^2((0,\tau),X)}\|\mathrm{Bx}\|^2 \\ &+ \|h(t)\|_{L^1((0,\tau),X)}\|\mathrm{BS}_\alpha(t)\|_{L^2((0,\tau),X)}\|q\|.\end{aligned} \tag{24}$$

So if we denote $M(\tau, s) := \tau^{1/2}M\|B\|\int_0^\tau \|t^{\alpha-1}E_{\alpha,\alpha}(\mathrm{s}t^\alpha)\|\mathrm{d}t$ and $m(\tau, s) = \int_0^\tau \|E_\alpha(\mathrm{s}t^\alpha)\|^2\mathrm{d}t$, then we have

$$\begin{aligned}\mu\|2\|^2 \le \|\,\mathrm{BS}_\alpha(t)x\,\|^2_{L^2((0,\tau),X)} \le{}& m(\tau, s)\|Bx\|^2 \\ &+ M(\tau, s)\|(A - sI)x\|^2.\end{aligned} \tag{25}$$

□

Example 1. (1) Take $= L^2(\mathbb{R})$, $Af(x) = (-\mathrm{d}f/\mathrm{d}x - \lambda f(x))$, $\lambda > 0$. Then, it is well known that A generates a stable C_0-semigroup $\{T(t)\}$,

$$T(t)f(x) = e^{-\lambda t} f(x+t). \tag{26}$$

Thus, it is easy to see that A generates a stable α-times fractional resolvent family $\{S_\alpha(t)\}$,

$$S_\alpha(t)f(x) = \int_0^\infty \Phi_{\alpha,1-\alpha}(t,s)T(s)f(x)\mathrm{ds}, \tag{27}$$

where $\Phi_{\alpha,1-\alpha}(t,s)$ is the Wright-type function, which can be found in many pieces of literature, such as [11, 22, 24].

Now, let $\mathrm{Bf}(x) = f(0)$, the $\mathrm{BT}(t)f(x) = e^{-\lambda t}f(t)$ and

$$\mathrm{BS}_\alpha(t)f(x) = e^{-\lambda t} f(t). \tag{28}$$

So we have

$$\int_0^\tau \|\mathrm{BS}_\alpha(t)f(x)\|^2 \mathrm{dt} = \int_0^\tau \|e^{-\lambda t} f(t)\|^2 \mathrm{dt} \geq e^{-2\lambda\tau}\|f\|^2. \tag{29}$$

Thus, $\mathrm{BS}_\alpha(t)$ is exactly observable in time τ, $\tau > 0$. Then, by this theorem, we deduce that for every $\tau > 0$, the following equation is valid:

$$M(\tau,s)\left\|(s+\lambda)f(x) + \frac{\mathrm{df}}{\mathrm{dx}}\right\|^2_{L^2(\mathbb{R})} + m(\tau,s)\|f(0)\|^2 \geq \mu\|f(x)\|^2_{L^2(\mathbb{R})}. \tag{30}$$

By using Theorem 6, we can prove the following result.

Proposition 7. *If A generates a stable fractional resolvent family $\{S_\alpha(t)\}$ with $\alpha \in (1,2)$, there exists a constant M such that for every $x \in X$,*

$$\|S_\alpha(t)x\| \leq \mathrm{Mt}^{-\alpha}\|x\|. \tag{31}$$

If $\{S_\alpha(t)\}$ commute with B and equation (18) holds for some $\tau > 0$, then pair (A, B, α) is approximately observable in infinite time.

Proof. Since B and $S_\alpha(t)$ commute, we deduce that

$$\|\mathrm{BS}_\alpha(t)S_\alpha(\tau)x\| \leq M\,\|\mathrm{BS}_\alpha(t)x\|. \tag{32}$$

Then, we denote $\phi(x) = \mathrm{BS}_\alpha(t)x$, of course, $\phi(x)$ is well-defined, and we know that $\ker(\phi)$ is an invariant space of $\{S_\alpha(t)\}$, and we need to prove that $\ker(\phi) = \{0\}$. Let $\{\widetilde{S_\alpha}(t)\}$ be the restriction of $\{S_\alpha(t)\}$ on $\ker(\phi)$ which also be a fractional resolvent family with generator $\widetilde{A}$, the restriction of A on $\ker(\phi)$. Then, it is easy to prove that

$$D(\widetilde{A}) \subseteq D(A) \cap \ker(\phi) \subseteq N(B). \tag{33}$$

Now, suppose we have equation (18) with constants $M(\tau,s)$ and $m(\tau,s)$, then, for every $s \notin \Sigma(1/2\alpha\pi)$ and every $x \in D(\widetilde{A})$,

$$M(\tau,s)\,\|(A - sI)x\|^2 \geq \mu\,\|x\|^2, \tag{34}$$

or equivalently, for every $s \in \rho(\widetilde{A})$, $s \notin \Sigma(1/2\alpha\pi)$,

$$\|(sI - A)^{-1}\| \leq \mu^{(1/2)}M(\tau,s). \tag{35}$$

It can be calculated directly that for every given $\tau > 0$, $M(\tau,s)$ is uniformly bounded for $s \in \rho(\widetilde{A})$, $s \notin \Sigma((1/2)\alpha\pi)$. For $s \in \Sigma((1/2)\alpha\pi)$, the uniform boundedness of $(\mathrm{sI} - A)^{-1}$ can be deduced from the property of fractional resolvent family with estimate (31), and then, we obtain that $\|(\mathrm{sI} - A)^{-1}\|$ is uniformly bounded for $s \in \rho(\widetilde{A})$, then by [1], Lemma 6.5.5, $\ker(\phi) = \{0\}$. □

Proposition 8. *If pair (A, B, α) is exactly observable in infinite time and $\{S_\alpha(t)\}$ is stable with rate $t^{-\alpha}$ and $\alpha \in (1,2)$, then it is exactly observable in time τ for τ big enough.*

Proof. For $x \in D(A)$ and $\tau > 0$, we have

$$\int_0^\tau \|\mathrm{BS}_\alpha(t)x\|^2\mathrm{dt} = \int_0^\infty \|\mathrm{BS}_\alpha(t)x\|^2\mathrm{dt} - \int_\tau^\infty \|\mathrm{BS}_\alpha(t)\|^2\mathrm{dt}. \tag{36}$$

Since there are constants m, M, such that

$$\begin{aligned} \int_0^\infty \|\mathrm{BS}_\alpha(t)x\|^2\mathrm{dt} &\geq m\|x\|^2 \\ \int_\tau^\infty \|\mathrm{BS}_\alpha(t)\|^2\mathrm{dt} &\leq M\tau^{1-\alpha}\|x\|^2. \end{aligned} \tag{37}$$

Then, equation (36) reads

$$\int_0^\tau \|\mathrm{BS}_\alpha(t)x\|^2\mathrm{dt} \geq \left(m - M\tau^{1-\alpha}\right)\|x\|^2. \tag{38}$$

If τ big enough such that $m - M\tau^{1-\alpha} > 0$, for such τ, pair (A, B, α) is exactly observable in time τ. □

4. Hautus-Type Tests for Exact Observability with a Special Generator

In this section, we focus on the fractional resolvent family generator of the following form:

$$A = i^\alpha C, \tag{39}$$

where C is a negative self-adjoint operator and satisfies the following condition:

$$\overline{N(E_\alpha(z))} \cap \sigma(i^\alpha C) = \varnothing, \tag{40}$$

where $N(E_\alpha(z))$ is the set consisting of all zeros of $E_\alpha(z)$. Then, A generates a bounded fractional resolvent family $\{S_\alpha(t)\}$, and for every $x, y \in X$, we have

$$\langle S_\alpha(t)x, y\rangle = \int_{\sigma(A)} E_\alpha(t^\alpha s)\mathrm{dE}_{x,y}(s), \tag{41}$$

where E is the resolution of identity corresponding to C.

$$E_{x,y}(s) := \langle E(s)x, y\rangle. \tag{42}$$

This can be checked directly by the Laplace transform, dominant convergence theorem, and asymptotic behavior of the Mittag–Leffler function. More details about the resolution of identity can be found in [26], here we only list one of them.

Lemma 9 (see [26]). *Theorem 13.24. Let E be a resolution of identity on a set Ω, then every measurable function $f\colon \Omega \longrightarrow \mathbb{C}$ corresponds a densely defined, closed operator $\Phi(f)$ with domain $D(\Phi(f)$ such that:*

$$\begin{aligned}\langle \Phi(f)x, y\rangle &= \int_{\Omega} f \mathrm{dE}_{x.y} \quad x \in D(\Phi(f)), y \in X,\\ \|\Phi(f)x\|^2 &= \int_{\Omega} |f|^2 \mathrm{dE}_{x,x}.\end{aligned} \tag{43}$$

By using this lemma, estimate (11), and equation (40), we deduce that

$$M\|x\| \geq \|S_\alpha(t)x\| \geq m\|x\|. \tag{44}$$

Moreover, consider the operator $A_1 = A - \epsilon$ for some $\epsilon > 0$, then A_1 generates a fractional resolvent family $\{S_{\alpha,1}(t)\}$ and

$$\begin{aligned}\|S_{\alpha,1}(t)x\|^2 &= \int_{\sigma(A)} |E_\alpha((s-\epsilon)t^\alpha)|^2 \mathrm{dE}_{x,x}\\ &\leq Mt^{-2\alpha}\|x\|^2, \quad as \quad t \longrightarrow \infty.\end{aligned} \tag{45}$$

Theorem 10. *If there exist constants M and m such that*

$$\|(i^\alpha \omega I - A)z\|^2 + m^2\|Bz\|^2 \geq \|z\|^2 \quad \omega \in \mathbb{R}, z \in D(A). \tag{46}$$

If $\epsilon^2 M < 1$ and $\alpha \in (1/2, 2)$, then pair (A_1, B, α) is exactly observable in infinite time.

Proof. Let $\{S_{\alpha.1}(t)\}$ be the fractional resolvent family generated by A_1, then, for every $z \in D(A_1) = D(A)$, we have

$$D_t^\alpha S_{\alpha,1}(t)z - \mathrm{AS}_{\alpha,1}(t)z = -\epsilon S_{\alpha,1}(t)z. \tag{47}$$

Since $S_\alpha(t)z \in L^2(\mathbb{R}_+, X)$, then

$$((i\lambda)^\alpha - A)\widetilde{S_{\alpha,1}}(\lambda)z = -\epsilon\widetilde{S_{\alpha,1}}(\lambda)z, \tag{48}$$

then by equation (46), we have

$$\begin{aligned}\|\widetilde{S_{\alpha,1}}(\lambda)z\|^2 &\leq \|M((i\lambda)^\alpha - A)\widetilde{S_{\alpha,1}}(\lambda)z\|^2\\ &\quad + m\|B\widetilde{S_{\alpha,1}}(\lambda)z\|^2 \quad \lambda \in \mathbb{R}.\end{aligned} \tag{49}$$

Then, Plancherel theorem implies that

$$\begin{aligned}\int_0^\infty \|S_{\alpha,1}(t)z\|^2 \mathrm{dt} &\leq M\int_0^\infty \|\epsilon S_{\alpha,1}(t)z\|^2 \mathrm{dt}\\ &\quad + m\int_0^\infty \|\mathrm{BS}_{\alpha,1}(t)z\|^2 \mathrm{dt},\end{aligned} \tag{50}$$

thus,

$$(1-\epsilon^2 M)\int_0^\infty \|S_{\alpha,1}(t)z\|^2 \mathrm{dt} \leq m\int_0^\infty \|\mathrm{BS}_{\alpha,1}(t)z\|^2 \mathrm{dt}. \tag{51}$$

Combining equations (44) and (45) and $1 - \epsilon^2 M > 0$, we have

$$L\|z\|^2 \leq \int_{R_+} \|\mathrm{BS}_{\alpha,1}(t)z\|^2 \mathrm{dt}. \tag{52}$$

for some constant L. This completes the proof. □

Remark 11. These results are also valid if A is a normal operator with $\overline{\Sigma(\pi - 1/2\alpha\pi)} \subseteq \rho(A)$ and $|E_\alpha(z)| \geq \epsilon$ for all $z \in \sigma(A)$ and $\epsilon > 0$.

It should be noted that if a pair (A, B, α) is exactly observable in time $\tau < \infty$, then equation (18) is satisfied with constant $M(\tau, s)$ and $m(\tau, s)$, and if $s \in \{z \mid \arg(z) = 1/2\alpha\pi\}$ and $\tau < \infty$, then both $M(\tau, s)$ and $m(\tau, s)$ are bounded and then equation (46) is satisfied. So we conclude then the following corollary.

Corollary 12. *Suppose ϵ small enough and $A_1 = A - \epsilon$. If pair (A, B, α) is exactly observable in time $\tau < \infty$, then pair (A_1, B, α) is exactly observable in infinite time.*

If operator $A = i^\alpha C$ has compact resolvent, then we can use spectral conditions to inscribe the observability of the fractional resolvent family.

Since C is self-adjoint and has compact resolvent, then we denote by $\{\phi_n\}_{n\in\mathbb{Z}_1}$ an orthonormal basis consisting of eigenvectors of A and by $\{i^\alpha \mu_n\}_{n\in\mathbb{Z}_1}$, and the corresponding eigenvalues of A and index set $\mathbb{Z}_1$ are a subset of $\mathbb{Z}$.

For $\omega \in \mathbb{R}$ and $r > 0$, set

$$J(\omega, r) = \{k \mid k \in \mathbb{Z}_1, \; |\mu_k - \omega| < r\}. \tag{53}$$

We call an element $x \in X$ is a wave packet of A of parameters ω and r if z can be represented as

$$x = \Sigma_{k\in J(\omega,r)} x_k \phi_k, \quad x_k \in \mathbb{C}. \tag{54}$$

We first give the following proposition.

Proposition 13. *The following conditions are equivalent:*

(1) $B\phi_n \neq 0$ for all $n \in \mathbb{Z}_1$

(2) The pair (A, B, α) is approximately observable in infinite time

Proof. The proof of this proposition is the same as [1], Proposition 6.9.1, so we omit it here.

The following lemma is a direct extension of the [1], Lemma 6.9.4. □

Lemma 14. *For each $r > 0$ and $\omega \in \mathbb{R}$, we define the subspace $V(\omega, r) \subseteq X$,*

$$V(\omega, r) \coloneqq \{\phi_k \mid k \in J(\omega, r)\}^\perp. \tag{55}$$

Let $A_{\omega,r}$ be the part of A in $V(\omega, r)$. If K is the non-increasing function

$$K(r) \coloneqq \sup_{\mathrm{Re}(s)\geq r} (\mathrm{Re}(s))^{1/2} \|B(sI - A)^{-1}\|, \tag{56}$$

then

$$\|B(i^{\alpha}\omega I - A)^{-1}\|_{\mathscr{L}(V(\omega,r),Y)} \le \frac{K(r)}{\sqrt{r}} \quad \forall \omega \in \mathbb{R}. \tag{57}$$

Now, we prove another main result of this section.

Theorem 15. *If $\alpha \in (1,2)$ and ϵ small enough, $A_1 = A - \epsilon$, then the following statements are equivalent:*

(1) There exist $r, m > 0$ such that for all $\omega \in \mathbb{R}$ and for every wave packet of A of parameters ω and r, denoted by x, we have

$$\|\mathrm{Bx}\|_{Y} \ge \delta \| x \|_{X} \tag{58}$$

(2) Pair (A_1, B, α) is exactly observable in time $\tau < \infty$

Remark 16. Although Theorem 10 does not assume $\alpha > 1$, we need this assumption to ensure the function $K(r)$ is well-defined.

Proof. First, we show that (2) implies (1). Assume that pair (A, B, α) is exactly observable in time τ, then by equation (18) we have

$$\| ((i^{\alpha}\lambda)I - A_1)x\|^2 + m\|\mathrm{Bx}\|^2 \ge \|x\|^2 \quad \lambda \in \mathbb{R}, \tag{59}$$

then choose r such that $(r+\epsilon)^2 = 1/2M$ and $x = \Sigma_{k\in J(\omega,r)} x_k \phi_k \in J(\omega, r)$, we have

$$\begin{aligned} \frac{1}{2}\|x\|^2 + m\|\mathrm{Bx}\|^2 &\ge M\, |\Sigma_{k\in J(\omega,r)} \left(i^{\alpha}(\omega - \mu_k) + \epsilon\right)x_k|^2 \\ &+ m\|\mathrm{Bx}\|^2 \ge \|x\|^2, \end{aligned} \tag{60}$$

then

$$m\|\mathrm{Bx}\|^2 \ge \frac{1}{2}\|x\|^2. \tag{61}$$

This proves the claim.

Next, we show that (1) implies (2). We first prove that (1) implies equation (46), then use Theorem 10 and Proposition 8 to get the desired result.

Take $x \in D(A)$ and represent it on the basis $\{\phi_k\}$, denote by $x = \Sigma_k x_k \phi_k$. Take r, δ, and ω such that (1) holds and decompose $x = x_1 + x_2$, where

$$\begin{aligned} x_1 &= \Sigma_{k\in J(\omega,r)} x_k \phi_k, \\ x_2 &= \Sigma_{k\notin J(\omega,r)} x_k \phi_k. \end{aligned} \tag{62}$$

Then, we have

$$\|\mathrm{Bx}\|^2 = \|\mathrm{Bx}_1\|^2 + \|\|\mathrm{Bx}\|^2 = \|\mathrm{Bx}_1\|^2 + \|^2 + 2\mathrm{Re}\langle \mathrm{Bx}_1, \mathrm{Bx}_2\rangle. \tag{63}$$

By using elementary inequality, we have, for every $\epsilon > 0$,

$$2\mathrm{Re}\langle \mathrm{Bx}_1, \mathrm{Bx}_2\rangle \ge -\epsilon \|\mathrm{Bx}_1\|^2 - \frac{1}{\epsilon}\|\mathrm{Bx}_2\|^2, \tag{64}$$

then we have

$$\|\mathrm{Bx}\|^2 \ge (1-\epsilon)\|\mathrm{Bx}_1\|^2 - \frac{1-\epsilon}{\epsilon}\|\mathrm{Bx}_2\|^2. \tag{65}$$

Then, by using Lemma 14 and (1), we have

$$\begin{aligned} \|\mathrm{Bx}\|^2 &\ge \delta^2(1-\epsilon)\|x_1\|^2 - \frac{1-\epsilon}{\epsilon}\|B(i^{\alpha}\omega I - A)^{-1}(i^{\alpha}\omega I - A)x_2\|^2 \\ &\ge \delta^2(1-\epsilon)\|x_1\|^2 - \frac{1-\epsilon}{\epsilon}\frac{4K(r)^2}{r}\|(i^{\alpha}\omega I - A)x_2\|^2. \end{aligned} \tag{66}$$

Since $A_{\omega,r}$ is the restriction of A in $V(\omega, r)$, then

$$\begin{aligned} &\|(i^{\alpha}\omega I - A)x\|^2 \ge \|\left(i^{\alpha}\omega I - A_{\omega,r}\right)x_2\|^2, \\ &\|\left(i^{\alpha}\omega I - A_{\omega,r}\right)x_2\|^2 \ge r^2\|x_2\|^2. \end{aligned} \tag{67}$$

Combining these equations, we have

$$\begin{aligned} &M^2\|(i^{\alpha}\omega I - A)x\|^2 + m^2\|\mathrm{Bx}\|^2 \\ &\ge m^2\delta^2(1-\epsilon)\|x_1\|^2 \\ &+\left(M^2r^2 - m^2\frac{1-\epsilon}{\epsilon}\frac{4K(r)^2}{r}\right)\|x_2\|^2, \end{aligned} \tag{68}$$

since $\epsilon < 1$, choose $m = 1/\delta(1-\epsilon)^{1/2}$ and M big enough, we have

$$\|(i^{\alpha}\omega I - A)x\|^2 + m^2\|\mathrm{Bx}\|^2 \ge \|x_1\|^2 + \|x_2\|^2 \ge \|x\|^2. \tag{69}$$

This completes the proof. □

5. Discussion

This paper focuses on the observability of fractional differential equations, giving some sufficient and necessary conditions for determining that an equation is observable. In particular, we apply the Hautus-type inequality for the classical case to fractional differential equations, giving an analogous inequality. When the space is a Hilbert space and the operator is a self-adjoint operator, we give a condition that does not depend on the estimate of the resolvent of the operator, which is used to characterize the observability of the equations.

Since this paper focuses on abstract fractional Cauchy problems in general Banach spaces, the methods that are generally used to consider observability are not applicable, for example, considering the dual operator and using the controllability of dual problems to study observability [1]. Because abstract resolvent families have quite a few common features, we can use this approach in the future to study other problems, such as building similar Hautus-type inequalities for general k-regularized resolvent operator families, or even more general (b, l)-regularized resolvent operator families (see definitions of these operator families at [27] and reference therein). And because the study of the observability of equations now focuses on parabolic equations [7], using the resolvent family [28–30] to analyze the equations may provide a way to study nonparabolic equations.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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