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Research Article

Partition Resolvability of Nanosheet and Nanotube Derived from Octagonal Grid

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Chemical graph theory, a branch of computational and applied mathematics, covers a very wide range of topics. As a result, the world of applied sciences heavily relies on graph theory. The concept of partition dimension has significant importance in the field of chemical graph theory. Although certain graphs have bounded partition dimensions, a graph's partition dimension may be constant. In this study, we look at two alternative chemical structures made of an octagonal grid: nanosheets and nanotubes. We determined the partition dimension of an octagonal grid-generated nanosheet to be 3, and the partition dimension of a nanotube to be limited from 4.

1. Introduction

Different scientific disciplines use a variety of methodologies to study chemical structures. To explore various chemical networks and structures, mathematical chemistry in particular offers a variety of tools and methodologies. There are numerous approaches to studying chemical networks in depth and tackling their applications using suggested tools. In addition to chemistry itself, mathematical chemistry also removes barriers in several other disciplines. One such area is physical chemistry, where the study of chemicals is conducted using methods from mathematical chemistry, particularly in the areas of thermodynamics and compound energy [1–3]. Graph theory offers distinct and practical subjects and variations for the study of chemical structures and their topologies in mathematical chemistry. Several publications on the subject of this study are provided in [4–8].

Chemical graph theory is a reliable tool that provides several methods for characterizing the structural properties of molecules, crystals, polymers, and clusters. A vertex in chemical graph theory can be a group of atoms, an orbital, an electron, a molecule, an intermediate, or an atom, among many other things, depending on the context, model, and subject of the implication. An intermolecular bond or any other force, such as the Keesom forces, may be seen as an edge connecting two atoms [9–12].

Three graph theory professionals suggested the concept of analyzing a graph using a distance vector [13, 14]. Various titles were assigned to this notion depending on the field, including resolving set, metric basis, or finding set. In this notion, a subset of vertices is selected so that the other vertices in the overall structure, network, or graph are all arranged in a unique manner. The selected vertices were arranged into sets that were referred to as the locating set [15, 16] in the context of computer-related issues, the resolving set [17] in the context of chemical-related topics, and the metric basis [18, 19] in the context of mathematical studies of pure theoretical graphs.

Various titles for the concept of metric dimension have been used. The idea of metric dimension as locating sets was first suggested in [13, 20]. Later, we proposed the concept using the term metric dimension rather than locating sets [14]. The concept of metric dimension was defined as resolving sets [14]. See for additional information on the revolving set, metric basis, and metric dimension emerged [21–23]. Partition dimension, as described in [24], is the generalized form of metric dimension. The partition dimension of a linked graph is based on the distances between vertices and sets containing vertices, whereas the metric dimension is based on the distances between vertices. It has been demonstrated that figuring out a graph's metric dimension is an NP-hard issue [24]. Finding a graph's partition dimension is similarly an NP-hard task because it is a generalization of determining the metric dimension. Since the partition dimension is a generalization of the metric dimension, the concepts of metric and partition dimension are closely related. We calculate the distance between a vertex and a set rather than between two vertices [25, 26].

Taking into consideration the peculiarities of the partition dimension, it is reasonable to query about the methodology behind the description of the graphs. The question of whether a family of a network's partition dimension is constant, finite, or unbounded is one that researchers are continually attempting to answer. As a consequence of this, much progress has been made in the investigation of identifying the division dimension of a graph, and several results have been found. Such as this idea is presented concerning the book graph [27], the operation of the comb product's produced graph is described in [28], generalized partition dimensions are investigated in [29], the partition dimension of a generalized class of graphs is examined in [30], the families of circulant graphs, multipartite graphs, and chemical graphs of fullerene are explored together with their partition dimension in [31], and convex polytopes' graphs for their partition dimension can be found in [32]. See the references [33–38] for more information.

Consider the graph denoted by H, characterized by the juxtaposition of its vertex and edge sets, namely, $\alpha(H)$ and $\blacklozenge(H)$, respectively. Let the set P, comprising proper elements selected from $\alpha(H)$, be designated as the s-elements' proper set. Now, let $r(\alpha_{\diamondsuit,\diamondsuit} \mid P)$ represent an s-tuple distance identification, denoted by $\zeta(\alpha_{\diamondsuit,\diamondsuit} \mid P_1), \zeta(\alpha_{\diamondsuit,\diamondsuit} \mid P_2), \ldots, \zeta(\alpha_{\diamondsuit,\diamondsuit} \mid P_s)$, elucidating the partition resolving set of the principal node within the structural framework of H. This principal node set is precisely designated as P under the condition that all principal nodes contained therein possess distinct identifiers. The partition dimension (pd) of H is denoted by the subsets in that set of $\alpha(H)$ that have the lowest possible count. The shortest geodesics or path from one vertex to another vertex is known as distance and written as ζ.

There are many fields where resolving partition parameters are applied, including network verification and its detection [39]. In their discussion of resolving partitions in robot navigation, the well-known Djokovic–Winkler relationship is discussed [39, 40], describing the resolving sets taken into account as an application for the mastermind game strategies. Additionally, [41–43] all make use of resolving sets. Moreover, [14, 21, 44] are references to further examine the uses of this notion in networks. We computed the partition dimension of the chemical structures in this paper due to the numerous applications of partition dimension in chemical graph theory.

2. Partition Dimension of Nanosheet Derived from Octagonal Grid

The thickness of the 2D nanostructures is what makes them so significant [45]. They are employed in nanotechnology, nanomedicine, and gene transfer. Their thicknesses range from $1nm$ to $100nm$. Because of their incredibly thin architectures, nanosheets differ from their bulk counterparts. They are best suited for the administration of various medications along with therapeutic DNA and RNAs due to the high surface-to-volume ratio.

The following details the construction of the nanosheet $NS_{\eta,\zeta}$ obtained using the octagonal grid. Red is used in Figure 1 to indicate edges with degree 2 and 3 end points. When an edge has an endpoint of degree 2, it is coloured blue, and when it has an endpoint of degree 3, it is coloured black. A two-degree vertex is represented by the colour green, and a three-degree vertex is represented by the colour black. The focal point of the resolving set is the double-coloured vertices. Due to degree 2, the points $\beta_{1,1}, \beta_{1,4\eta}$ are coloured green and red, respectively, and are components of the resolving set.

Let η and ζ stand for the horizontal and vertical numbers of C_8 and $\eta, \zeta \geq 1, \eta, \zeta \in Z^+$, respectively. The number of nodes with a degree of 3 is $\eta, \zeta \geq 1, \eta, \zeta \in Z^+$, while the number of vertices with a degree of two is $4(\eta + \zeta)$. The order of $NS_{\eta,\zeta}$ is $|O(NS_{\eta,\zeta})| = 8\eta\zeta$, and the size of $NS_{\eta,\zeta}$ is $|E(NS_{\eta,\zeta})| = 12\eta\zeta - 4(\eta + \zeta)$.

In labelling, two parameters η, ζ and two indexes $\diamondsuit, \diamondsuit$ are employed. $\diamondsuit$ changes 4 times with η, and $\diamondsuit$ varies twice with v. Moreover, the labelling described above in vertex and edge sets is shown in Figure 1, and it is used in our major findings. The nanosheet's vertex and edge sets are as follows:

$$
\begin{aligned}
V(\mathrm{NS}) &= \left\{\alpha_{\diamondsuit,\diamondsuit}, \beta_{\diamondsuit,\diamondsuit}; \diamondsuit = 1, 2, \ldots \zeta, \diamondsuit = 1, 2, \ldots 4\eta\right\}, \\
E(\mathrm{NS}) &= \left\{\alpha_{\diamondsuit,\diamondsuit}\alpha_{\diamondsuit,\diamondsuit+1}, \beta_{\diamondsuit,\diamondsuit}\beta_{\diamondsuit,\diamondsuit+1}; \diamondsuit = 1, 2, \ldots \zeta, \diamondsuit = 1, 2, \ldots 4\eta\right\}, \\
&\quad \cup \left\{\alpha_{\diamondsuit,\diamondsuit}\beta_{\diamondsuit,\diamondsuit}; \diamondsuit = 1, 2, \ldots \zeta, \diamondsuit = 0, 1 (\mathrm{mod} 4)\right\}, \\
&\quad \cup \left\{\alpha_{\diamondsuit,\diamondsuit}\beta_{\diamondsuit,\diamondsuit}; \diamondsuit = 1, 2, \ldots \zeta, \diamondsuit = 2, 3 (\mathrm{mod} 4)\right\}.
\end{aligned}
\tag{1}
$$

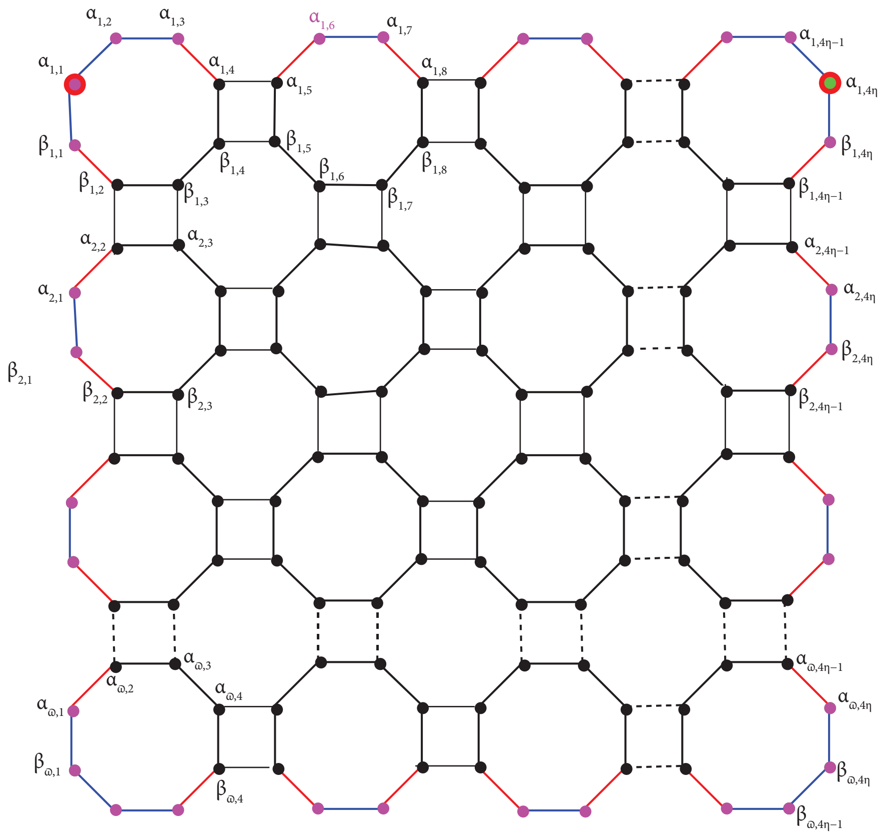

FIGURE 1: In general, the nanosheet that was formed from the octagonal grid.

Theorem 1. *Let* $NS_{\eta,\zeta}$ *octagonal grid-based nanosheet with* $\eta, \zeta \geq 1$. *Then,* $pd(NS_{\eta,\zeta}) = 3$.

Proof. Assume the partition resolving set $P = \{P_1, P_2, P_3\}$, where $P_1 = \{\alpha_{1,1}\}$, $P_2 = \{\alpha_{1,4\eta}\}$, $P_3 = V(NS_{\eta,\zeta}) \backslash \{\alpha_{1,1}, \alpha_{1,4\eta}\}$. To prove that $pd(NS_{\eta,\zeta}) \leq 3$, we will show that P is a resolving set.

The representations for $\eta, \zeta = 1$ are present in Table 1.

The unique representation of each vertex in $NS_{\eta,\zeta}$ for $\eta, \zeta > 1$ is provided as follows.

Let $\zeta(\alpha_{\diamond,\diamond}, P_1) = \blacklozenge_1, \zeta(\alpha_{\diamond,\diamond}, P_2) = \blacklozenge_2, \zeta(\alpha_{\diamond,\diamond}, P_3) = \blacklozenge_3$ and $r(\alpha_{\diamond,\diamond} \mid P) = (\blacklozenge_1, \blacklozenge_2, \blacklozenge_3)$.

$$\blacklozenge_1 = \begin{cases} \diamond + \diamond - 2, & \text{when } \diamond = 1 \,\&\, \diamond = 1, 2, \ldots, 4\eta, \\ 4\diamond - \diamond - 3, & \text{when } \diamond = 1, 2, \ldots \zeta, \diamond = 1, 2, \\ 4(\diamond - 1) + \diamond - 3 - z_1, & \text{when } \diamond \geq 3 \,\&\, 3 \leq \diamond < 4(\diamond - 2) + 2, \\ 2\diamond + \diamond - 3, & \text{when } \diamond \geq 2 \,\&\, \diamond \geq 4\diamond - 6, \end{cases} \tag{2}$$

where $z_1 = 2\lfloor \diamond - 2/4 \rfloor$.

$$\blacklozenge_2 = \begin{cases} 2\diamond - \diamond + 4\eta - 2, & \text{when } 1 \le \diamond \le \eta + 1,\ 1 \le \diamond \le 4\eta - 3x + 6, \\ 2(\diamond - 1) - \diamond + 4\eta, & \text{when } \diamond = 1,\ 1 \le \diamond \le 4\eta, \\ 4\diamond - \diamond + 2\eta - 4, & \text{when } \diamond > \eta + 1,\ \diamond = 1, 2, 3, \\ 4\diamond - 4, & \text{when } \diamond \ge 2,\ \diamond = 4\eta,\ \eta \ge 2, \\ 2\diamond - \diamond + 4\eta + z_1, & \text{when } 3 \le \diamond \le \eta + 1,\ 4(\eta - \diamond + 2) \le \diamond \le 4\eta - 5,\ \eta \ge 2, \\ 2\diamond - \diamond + 4\eta - 2 + z_2, & \text{when } \diamond \ge \eta + 1, \diamond > 4,\ \eta \ge 2, \end{cases} \tag{3}$$

where $z_1 = 2\lfloor \diamond - 4(\eta - \diamond) + 8/4 \rfloor, z_2 = 2\lfloor \diamond - 4/4 \rfloor$.

$$\blacklozenge_3 = \begin{cases} 1, & \text{when } \alpha_{1,1} \,\&\, \alpha_{1,4\eta}, \\ 0, & \text{when othervise.} \end{cases} \tag{4}$$

Let $\zeta(\beta_{\diamond,y}, P_1) = \blacklozenge_1'$, $\zeta(\beta_{\diamond,y}, P_2) = \blacklozenge_2'$, $\zeta(\beta_{\diamond,y}, P_3) = \blacklozenge_3'$ and $r(\beta_{\diamond,\diamond} \mid P) = (\blacklozenge_1', \blacklozenge_2', \blacklozenge_3')$

$$\blacklozenge_1' = \begin{cases} \diamond + \diamond - 1, & \text{when } \diamond = 1 \,\&\, \diamond = 1, 2, 3, \ldots 4\eta, \\ 4(\diamond - 1) + \diamond, & \text{when } \diamond = 1, 2, 3, \ldots \zeta, \diamond = 1, 2, 3, \\ 4(\diamond - 1) + \diamond - z_1, & \text{when } \diamond \ge 2 \,\&\, 3 < \diamond \le 4(\diamond - 1), \\ 2\diamond + \diamond - 2, & \text{when } \diamond \ge 2 \,\&\, \diamond \ge 4(\diamond - 1), \end{cases} \tag{5}$$

where $z_1 = 2\lfloor \diamond/4 \rfloor$.

$$\blacklozenge_2' = \begin{cases} 2\diamond - \diamond + 4\eta - 1, & \text{when } \diamond = 1, \diamond = 4\eta, \\ 2\diamond - \diamond + 4\eta - 1, & \text{when } \diamond \le \eta, 1 \le \diamond \le 4\eta - 3, \\ 4(\eta + \diamond) - \diamond - 3, & \text{when } 3 \le \diamond \le \eta, 4\eta - 4 \le \diamond \le 4\eta, \eta \ge 2, \\ 4(\diamond - 1) + \diamond + 2\eta - 1, & \text{when } \diamond > \eta, \diamond = 1, 2 \,\&\, \eta = 1, 2, 3, \\ 2\diamond - \diamond + 4\eta + 1 + z_1, & \text{when } 3 \le \diamond \le \eta, 4(\eta - \diamond) \le \diamond \le 4\eta - 3, \eta \ge 3, \\ 4\diamond - \diamond + 2\eta - 1 + z_2, & \text{when } \diamond > \eta + 1, \diamond \ge 3, \eta \ge 2, \end{cases} \tag{6}$$

where $z_1 = 2\lfloor \diamond - 4(\eta - \diamond) + 6/4 \rfloor, z_2 = 2\lfloor \diamond - 2/4 \rfloor$.

$$\blacklozenge_3' = \begin{cases} 1, & \text{when } \alpha_{1,1} \,\&\, \alpha_{1,4\eta}, \\ 0, & \text{when othervise.} \end{cases} \tag{7}$$

One can note from Figure 2 when $\zeta(\varepsilon_1, \alpha_{1,1}) = \zeta(\varepsilon_2, \alpha_{1,1})$, this would imply that $\zeta(\varepsilon_2, \alpha_{1,4\eta}) \ne \zeta(\varepsilon_1, \alpha_{1,4\eta})$ also $\zeta(\varepsilon_2, \alpha_{1,4\eta}) = \zeta(\varepsilon_1, \alpha_{1,4\eta})$ and this would imply that $\zeta(\varepsilon_1, \alpha_{1,1}) \ne \zeta(\varepsilon_2, \alpha_{1,1})$.

We discuss all cases where $\zeta(\varepsilon_1, \alpha_{1,1}) \ne \zeta(\varepsilon_2, \alpha_{1,1})$, while $\zeta(\varepsilon_1, \alpha_{1,4\eta}) = \zeta(\varepsilon_2, \alpha_{1,4\eta})$ or $(\varepsilon_1, \alpha_{1,1}) = \zeta(\varepsilon_2, \alpha_{1,1})$, $\zeta(\varepsilon_1, \alpha_{1,4\eta}) \ne \zeta(\varepsilon_2, \alpha_{1,4\eta})$.

From all above cases, there is no possibility in which two representations are same. This shows that $pd(G) \le 3$. On contrary, supposed that partition dimension is 2 that is not possible because $pd(G) = 2$ if and if only $G = P_n$ [24].

Hence, it is proved that $pd(NS_{\eta,\zeta}) = 3$. □

3. Partition Dimension of Nanotube Derived from Octagonal Grid

There are numerous applications for carbon nanotubes, including thin-film electronics, water filtration, automotive components, molecular electronics, catalyst supports, energy storage, boat hull construction, device modelling, biomedical applications, air and water filtration systems, sporting goods, actuators, coatings, and electromagnetic shields. There are over a thousand metric tonnes of carbon nanotubes that are manufactured every year. Over the course of the last three decades, carbon nanotubes have been instrumental in a variety of sectors, including microelectronic

Table 1: Representation of vertices of Figure 2.

Vertex	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$	$\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$	$\beta_{1,4}$
$r(. \mid P)$	(0, 3, 1)	(1, 2, 0)	(2, 1, 0)	(3, 0, 1)	(1, 4, 0)	(2, 3, 0)	(3, 2, 0)	(4, 1, 0)

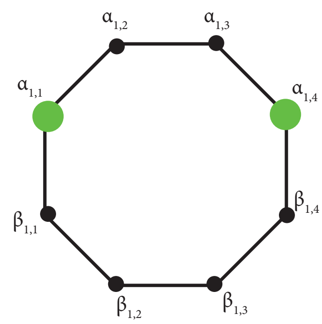

Figure 2: Vertices representation for the nanosheet.

circuits, microscopy, quantum mechanics probes, and the creation of devices that have an effect on biological systems.

The single-wall and multiwall forms of carbon nanotubes are indistinguishable from one another in terms of their expression. A wide range of qualities are shown by these nanotubes, including but not limited to the following: electrical conductivity, electron emission, aspect ratio, thermal conductivity, expansion characteristics, and strength and elasticity. Especially noteworthy is the fact that researchers have conducted a substantial study on nanotubes within the context of fluid flow dynamics. In addition, other researchers have investigated the topological indices of these particular nanotubes, especially that which pertains to graph theory. Quantifying the partition dimensions of this specific nanotube is the goal that we have set for ourselves in this section [45].

But first, we will build the octagonal-grid-derived nanotube, which we'll refer to as $\mathrm{ONS}_{\eta,\zeta}$.

The edges in Figure 3 that have end points of degrees 2 and 3 are highlighted in red. When an edge has an endpoint of degree 2, it is coloured blue, and when it has an endpoint of degree 3, it is coloured black. Vertices with degrees of 2 and 3 are coloured green and black, respectively. Employing a dual-colour scheme for the specified vertex, a dynamic rotating set is established. This configuration is influenced by the vertex's degree, set at 3, and the pivotal nature of the resolving set. Consequently, $\alpha_{1,1}$ is distinctly marked with both black and red hues. Additionally, owing to a degree of 2 and the resolving set point, $\alpha_{1,3}$ and $\beta_{\zeta,2}$ are attributed with green and red colours, respectively.

The order of $\mathrm{ONT}_{\eta,\zeta}$ is given by $|O(\mathrm{ONT}_{\eta,\zeta})| = 8\eta\zeta$, while the size of $\mathrm{NS}_{\eta,\zeta}$ is defined as $|E(\mathrm{ONT}_{\eta,\zeta})| = 12\eta\zeta + 2\eta$.

For the purpose of labelling, two parameters, η and ζ, along with two indices, are employed. The variable ζ contributes to the variation of parameter I twice, while the variable h leads to four successive shifts in index J. The vertex and edge sets of the nanotube are delineated as follows:

$$
\begin{aligned}
V(\mathrm{ONT}) &= \left\{\alpha_{\diamond,\diamond}, \beta_{\diamond,\diamond}; \diamond = 1, 2, 3 \ldots \zeta, \diamond = 1, 2, 3 \ldots 4\eta\right\},\\
E(\mathrm{ONT}) &= \left\{\alpha_{\diamond,\diamond}\alpha_{\diamond,\diamond+1}, \beta_{\diamond,\diamond}\beta_{\diamond,\diamond+1}; \diamond = 1, 2, 3 \ldots \zeta, \diamond = 1, 2, 3 \ldots 4\eta\right\},\\
&\cup \left\{\alpha_{\diamond,\diamond}\beta_{\diamond,\diamond}; \diamond = 1, 2, 3 \ldots \zeta, \diamond = 0, 1\,(\mathrm{mod}\,4)\right\},\\
&\cup \left\{\alpha_{\diamond,\diamond}\beta_{\diamond,\diamond}; \diamond = 1, 2, 3 \ldots \zeta, \diamond = 2, 3\,(\mathrm{mod}\,4)\right\},\\
&\cup \left\{\alpha_{\diamond,1}\alpha_{\diamond,4\eta}\beta_{\diamond,1}\beta_{\diamond,4\eta}; \diamond = 1, 2, 3 \ldots \zeta, \eta \geq 1\right\}.
\end{aligned} \tag{8}
$$

Theorem 2. *Let $\mathrm{ONT}_{\eta,\zeta}$ be a nanotube with $\eta, \zeta \geq 1$. Then, $2 \leq pd(\mathrm{ONT}_{\eta,\zeta}) \leq 4$.*

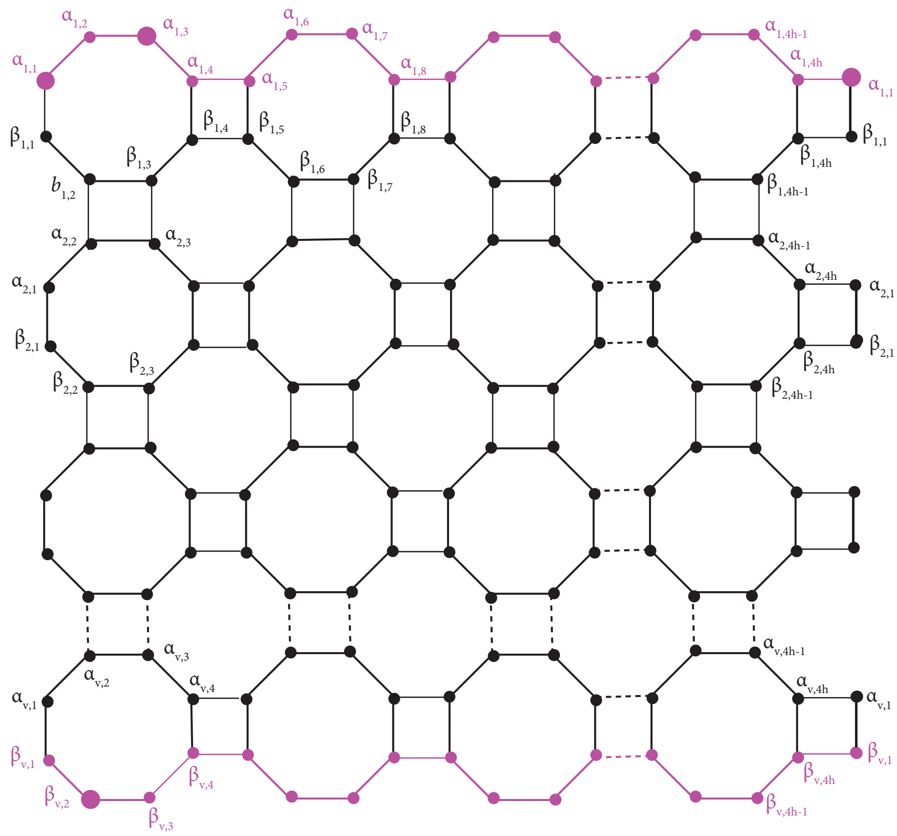

FIGURE 3: Generalize nanotube derived by the octagonal grid.

Proof. Assume the partition resolving set $P = \{P_1, P_2, P_3, P_4\}$ where $P_1 = \{\alpha_{1,1}\}$, $P_2 = \{\alpha_{1,3}\}$, $P_3 = \{\beta_{\zeta,2}\}$, $P_4 = V(\mathrm{ONT}_{\eta,\zeta})\backslash\{\alpha_{1,1}, \alpha_{1,3}, \beta_{\zeta,2}\}$. To prove that $pd(\mathrm{ONS}_{\eta,\zeta}) \leq 4$ we will show that P is a resolving set. The proof of this theorem splits into two cases. Case one for $\eta, \zeta = 1$ and case two for $\eta, \zeta \geq 1$.

The distinctive representations of all vertices within $\mathrm{ONT}_{\eta,\zeta}$ are enumerated as follows, considering the values of $\eta, \zeta \geq 1$.

The representation of $P = \{P_1, P_2, P_3, P_4\}$ for $\eta = 1 = \zeta$ is shown in Table 2.

The unique representation of each vertex in $NS_{\eta,\zeta}$ for $\eta, \zeta > 1$ is provided as follows.

Let $\zeta(\alpha_{\diamond,\diamond}, P_1) = \blacklozenge_1$, $\zeta(\alpha_{\diamond,\diamond}, P_2) = \blacklozenge_2$, $\zeta(\alpha_{\diamond,\diamond}, P_3) = \blacklozenge_3$, $\zeta(\alpha_{\diamond,\diamond}, P_4) = \blacklozenge_3$, and $r(\alpha_{\diamond,\diamond} \mid P) = (\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$.

$$\blacklozenge_1 = \begin{cases} 4(\diamond - 1) - \diamond + 1, & \text{when } \diamond \geq 1, \diamond = 1, \\ 2\diamond + \diamond - 3, & \text{when } \diamond = 1,\ 2\,\&\,2 \leq \diamond \leq 2\eta + 1, \\ 4(\diamond - 2) + \diamond + 1, & \text{when } \diamond = 3\,\&\,2 \leq \diamond \leq 5, \eta \geq 2, \\ 4(\diamond - 2) + \diamond - 1, & \text{when } \diamond = 3\,\&\,5 \leq \diamond \leq 2\eta + 1, \eta \geq 2, \\ 2(\diamond + 5) - \diamond + 13, & \text{when } \diamond = 1,\ 2\,\&\,2 \leq \diamond \leq 2\eta + 1, \\ 4(\diamond + 2) - \diamond + 13, & \text{when } \diamond = 1,\ 2,\ 3,\ 4 \ldots \zeta, \diamond = 4\eta, \\ 4(\diamond - 2) + \diamond + 1 - z_1, & \text{when } \diamond \geq 4,\ 1 \leq \diamond \leq 2\eta + 1, \eta \geq 3, \\ 4\diamond - \diamond + 3\zeta - 2 + z_2, & \text{when } \diamond \geq 4,\ 2\eta + 2 \leq \diamond \leq 4\eta, \eta \geq 3\,(\text{odd}), \\ 4\diamond - \diamond + 3\eta - 1 + z_3, & \text{when } \diamond \geq 4\,\&\,2\eta + 4 \leq \diamond \leq 4\eta, \eta \geq 2\,(\text{even}), \\ 4\diamond - \diamond + 4\eta - 9, & \text{when } \diamond = 4,\ 5, \ldots, \zeta, 2\eta + 2 \leq \diamond \leq 4\eta + 3, \eta \geq 2\,(\text{even}), \\ 4\diamond - \diamond + 4\eta - 1, & \text{when } \diamond = 4,\ 5, \ldots, \zeta, 2\eta + 2 \leq \diamond \leq 2\eta + 3, \eta = 2,\ 4, \\ 4\diamond - \diamond + 4\eta - 7, & \text{when } \diamond = 3\,\&\,4\eta - 4 \leq \diamond \leq 4\eta - 5, \eta \geq 4, \\ 4\diamond - \diamond + 4\eta - 5, & \text{when } \diamond = 3, \&\,4\eta - 4 \leq \diamond < \leq 4\eta - 1, \end{cases} \tag{9}$$

where $z_1 = 2\lfloor \diamondsuit - 2/4 \rfloor$, $z_2 = 2\lfloor \diamondsuit - 2\eta + 2/4 \rfloor$, $z_3 = 2\lfloor \diamondsuit - 2\eta + 2/4 \rfloor$.

$$\blacklozenge_2 = \begin{cases} 4\diamondsuit - \diamondsuit - 1, & \text{when } \diamondsuit = 1, 2, \ldots, \zeta, \diamondsuit = 1, 2, 3, \\ 4\diamondsuit + \diamondsuit - 7, & \text{when } \diamondsuit = 1, 2, \ldots, \zeta, \diamondsuit = 4, 5, \\ 2\diamondsuit + \diamondsuit - 5, & \text{when } \diamondsuit = 1, 2 \,\&\, 6 \le \diamondsuit \le 2\eta + 3, \\ 2\diamondsuit - \diamondsuit + 4\eta + 1, & \text{when } \diamondsuit = 1, 2 \,\&\, 2\eta + 4 \le \diamondsuit \le 4\eta - 1, \eta \ge 3, \\ 4\diamondsuit - \diamondsuit + 4\eta - 5, & \text{when } \diamondsuit = 1, 2, \ldots, \zeta, \diamondsuit = 4\eta, \eta \ge 2, \\ 4\diamondsuit + \diamondsuit - 9 - z_1, & \text{when } \diamondsuit \ge 4 \,\&\, 6 \le \diamondsuit \le 2\eta + 3, \eta \ge 2, \\ 4\diamondsuit - \diamondsuit + 4\eta - 3 - z_2, & \text{when } \diamondsuit \ge 3,\ 2\eta + 4 \le \diamondsuit \le 4\eta - 1, \eta \ge 2, \\ \diamondsuit + \diamondsuit, & \text{when } \diamondsuit = 3 \,\&\, 6 \le \diamondsuit \le 9, \eta \ge 3, \\ \diamondsuit + \diamondsuit - 2, & \text{when } \diamondsuit = 3 \,\&\, 10 \le \diamondsuit \le 4\eta + 3, \eta \ge 4, \end{cases} \tag{10}$$

where $z_1 = 2\lfloor \diamondsuit - 6/4 \rfloor$, $z_2 = 2\lfloor 4\eta - \diamondsuit - 1/4 \rfloor$

$$\blacklozenge_3 = \begin{cases} 4\eta - 4\diamondsuit + \diamondsuit + 1, & \text{when } 1 \le \diamondsuit \le \eta \,\&\, \diamondsuit = 1, 2, 3, \\ 4\eta - 4\diamondsuit + \diamondsuit - 1 - z_1, & \text{when } 1 \le \diamondsuit \ge \eta - 2,\ 4 \le \diamondsuit \le 2\eta + 2, \\ \eta \ge 3, \eta - \diamondsuit + \diamondsuit + 1, & \text{when } \diamondsuit = \eta \,\&\, 3 \le \diamondsuit \le 2\eta + 2, \\ 6\eta - 2\diamondsuit - \diamondsuit + 3, & \text{when } \eta - 1 \le \diamondsuit \le \eta, 2\eta + 3 \le \diamondsuit \le 4\eta - 3, \eta \ge 3, \\ 8\eta - 4\diamondsuit - \diamondsuit + 3, & \text{when } 1 \le \diamondsuit \le \eta, 4\eta - 3 \le \diamondsuit \le 4\eta, \eta \ge 2, \\ \eta - \diamondsuit + \diamondsuit + 2, & \text{when } 1 \le \diamondsuit \le \eta - 1,\ 4 \le \diamondsuit \le 7, \eta \ge 3, \\ \eta - \diamondsuit + \diamondsuit + 2, & \text{when } 1 \le \diamondsuit \le \eta - 1, \diamondsuit = 4{,}5 \quad if\, \eta = 2, \\ \eta - \diamondsuit + \diamondsuit, & \text{when } 1 \le \diamondsuit \le \eta - 1,\ 8 \le \diamondsuit \le 2\eta + 2, \eta \ge 3, \\ 8\eta - 4\diamondsuit - \diamondsuit + 1 - z_2 + z, & \text{when } 1 \le \diamondsuit \le \eta - 2,\ 2\eta + 3 \le \diamondsuit \le 4\eta - 3, \eta \ge 3, \end{cases} \tag{11}$$

where $z_1 = 2\left\lfloor \diamondsuit - 4/4 \right\rfloor, z_2 = 2\left\lfloor 4\eta - \diamondsuit - 3/4 \right\rfloor, z = \begin{cases} 1, & \text{if } \diamondsuit \ge 3\eta, \diamondsuit = \eta - 2 \\ 0, & \text{otherwise} \end{cases}$

$$\blacklozenge_4 = \begin{cases} 1, & \text{when } \alpha_{1,1}, \alpha_{1,3}, \beta_{v,2}, \\ 0, & \text{when othervise.} \end{cases} \tag{12}$$

Let $\zeta(\beta_{\diamondsuit,\diamondsuit}, P_1) = \blacklozenge_1'$, $\zeta(\beta_{\diamondsuit,\diamondsuit}, P_2) = \blacklozenge_2'$, $\zeta(\beta_{\diamondsuit,\diamondsuit}, P_3) = \blacklozenge_3'$, $\zeta(\beta_{\diamondsuit,\diamondsuit}, P_4) = \blacklozenge_4'$, and $r(\beta_{\diamondsuit,\diamondsuit} \mid P) = (\blacklozenge_1', \blacklozenge_2', \blacklozenge_3', \blacklozenge_4')$.

$$\blacklozenge_1' = \begin{cases} 2\diamondsuit + \diamondsuit - 2, & \text{when } \diamondsuit = 1, 2 \,\&\, 4 \le \diamondsuit \le 2\eta + 1, \\ 2\diamondsuit - \diamondsuit + 4\eta, & \text{when } \diamondsuit = 1, 2 \,\&\, 2\eta + 2 \le \diamondsuit \le 4\eta - 3, \\ 4\diamondsuit + \diamondsuit - 4, & \text{when } \diamondsuit \ge 1, \diamondsuit = 1, 2, \ldots 4\eta, \\ 4\diamondsuit - \diamondsuit + 4\eta - 2, & \text{when } \diamondsuit \ge 1,\ 4\eta - 3 \le \diamondsuit \le 4\eta, \\ \diamondsuit + \diamondsuit + 3, & \text{when } \diamondsuit = 3 \,\&\, 4 \le \diamondsuit \le 7, \\ \diamondsuit + \diamondsuit + 1, & \text{when } \diamondsuit = 3 \,\&\, 8 \le \diamondsuit \le 2\eta + 1, \eta \ge 4, \\ 4\diamondsuit + \diamondsuit - 6 - z_1, & \text{when } \diamondsuit \ge 4 \,\&\, 4 \le \diamondsuit \le 2\eta + 1, \eta \ge 2, \\ 4\diamondsuit - \diamondsuit + 3\eta + z_2, & \text{when } \diamondsuit \ge 3,\ 2\eta + 1 \le \diamondsuit \le 4\eta - 3\eta \ge 2\,(\text{even}), \\ 4\diamondsuit - \diamondsuit + 3\eta + 1 + z_3, & \text{when } \diamondsuit \ge 3,\ 2\eta + 4 \le \diamondsuit \le 4\eta - 3, \eta \ge 5\,(\text{odd}), \\ 4\diamondsuit - \diamondsuit + 3\eta - 1, & \text{when } \diamondsuit = 4, 5, \ldots, \eta, 2\eta + 1 \le \diamondsuit \le 4\eta + 3, \eta \ge 6\,(\text{even}), \end{cases} \tag{13}$$

Table 2: Unique representation of vertices of Figure 4.

Vertex	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$	$\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$	$\beta_{1,4}$
$r(.\mid P)$	(0, 2, 2, 1)	(1, 1, 3, 0)	(2, 0, 4, 1)	(1, 1, 3, 0)	(1, 3, 1, 0)	(2, 4, 0, 1)	(3, 3, 1, 0)	(2, 2, 2, 0)

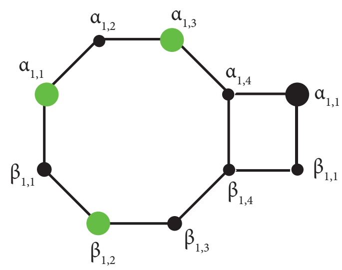

Figure 4: Vertices representation for the nanotube.

where $z_1 = 2\lfloor \diamond - 4/4\rfloor, z_2 = 2\lfloor \diamond - 2\eta + 2/4\rfloor, z_3 = 2\lfloor \diamond - 2\eta + 2/4\rfloor$.

$$\blacklozenge_2' = \begin{cases} 4\diamond - \diamond, & \text{when } \diamond = 1, 2, \ldots, \eta, \diamond = 1, \\ 4\diamond - \diamond + 2, & \text{when } \diamond = 1, 2, \ldots, \eta, \diamond = 2, 3\,\&, \\ 4\diamond + \diamond - 6 - z_1, & \text{when } \diamond \geq 3,\ 4 \leq \diamond \leq 2\eta, \\ 4\diamond + \diamond - 6, & \text{when } \diamond = 1, 2\,\&\, 4 \leq \diamond \leq 7, \\ 2\diamond + \diamond - 4, & \text{when } \diamond = 1, 2\&\, 8 \leq \diamond \leq 2\eta + 3, \eta \geq 3, \\ 2\diamond + \diamond - 4, & \text{when } \diamond = 1, 2.\,\&\diamond = 8 if\ \eta = 2, \\ 2\diamond - \diamond + 4\eta, & \text{when } \diamond = 1, 2\,\&\, 2\eta + 3 < \diamond \leq, \eta \geq 3, \\ 4\diamond - \diamond + 4\eta, & \text{when } \diamond = 1, 2.\,\&\, 4\eta - 2 \leq \diamond \leq 4\eta, \eta \geq 3, \\ 4\diamond - \diamond + 4\eta - 2 - z_2, & \text{when } \diamond \geq 3,\ 2\eta + 4 \leq \diamond \leq 4\eta - 3, \eta \geq 4. \end{cases} \tag{14}$$

where $z_1 = 2\lfloor \diamond - 4/4\rfloor, z_2 = 2\lfloor 4\eta - \diamond - 3/4\rfloor$.

$$\blacklozenge_3' = \begin{cases} 4\eta - 4\diamond + \diamond, & \text{when } 1 \leq \diamond \leq \eta, \diamond = 1, \\ 4\eta - 4\diamond + \diamond - 2 - z_1, & \text{when } 1 \leq \diamond \leq \eta - 1\,\&\, 2 \leq \diamond \leq 2\eta + 2, \eta \geq 2, \\ \eta - \diamond + \diamond - 2, & \text{when } \diamond = \eta., 3 \leq \diamond \leq 2\eta + 2, \eta \geq 2, \\ 6\eta - 2\diamond - \diamond + 2, & \text{when } \eta - 1 \leq \diamond \leq \eta, 2\eta + 2 \leq \diamond \leq 4\eta - 1, \eta \geq 2, \\ 6\eta - 2\diamond - +4, & \text{when } \diamond = \eta - 2.\,\&\, 4\eta - 4 \leq \diamond \leq 4\eta - 1, \eta \geq 3, \\ 6\eta - 2\diamond - \diamond + 2, & \text{when } \diamond = \eta - 2.\,\&\, 2\eta + 2 \leq \diamond \leq 4\eta - 5, \eta \geq 4, \\ 8\eta - 4\diamond - \diamond + 2, & \text{when } 1 \leq \diamond \leq \zeta, \diamond = 4\eta.\,\&\, \eta \geq 2, \\ 8\eta - 4\diamond - \diamond - z_2 z, & \text{when } 1 \leq \diamond \leq \zeta - 3,\ 2\eta + 2 \leq \diamond \leq 4\eta - 1, \eta \geq 4, \end{cases} \tag{15}$$

where $z_1 = 2\lfloor \diamond - 2/4 \rfloor, z_2 = 2\lfloor 4\eta - \diamond - 1/4 \rfloor$, $z = \begin{cases} 1, & \text{if } \diamond \geq 3\eta, \diamond = \eta - 2, \\ 0, & \text{otherwise.} \end{cases}$

$$\blacklozenge_4' = \begin{cases} 1, & \text{when } \alpha_{1,1}, \alpha_{1,3}, \beta_{\zeta,2}, \\ 0, & \text{when othervise.} \end{cases} \tag{16}$$

According to the concept that was presented earlier in the form of representation, it is important to note that the unique representation is provided by each vertex and that it meets the resolution set criterion. This demonstrates that $2 \leq pd(\mathrm{ONT}_{\eta,\zeta}) \leq 4$.

This shows that $pd(\mathrm{ONT}_{\eta,\zeta}) \leq 4$. On contrary, we suppose that the partition dimension is 2 that is not possible because $pd(G) = 2$ if and if only $G = P_n$ [24].

Hence, it is proved that $2 \leq pd(\mathrm{ONT}_{\eta,\zeta}) \leq 4$. □

4. Conclusion

The topic of computational and applied mathematics known as chemical graph theory is exceedingly broad. Because of this, graph theory is widely utilized in the field of applied sciences. Another relevant subject in chemical graph theory is the partition dimension. A graph's partition dimension may be constant, although certain graphs have bounded partition dimensions. In this paper, we consider two different chemical structures' nanosheet and a nanotube derived from an octagonal grid. We computed the partition dimension of the nanosheet derived from the octagonal grid as a constant that is 3 and the partition dimension of the nanotube derived from the octagonal grid is bounded from 4. For the future research direction, one can consider the exact partition dimension of the given graphs and also one can study the newly developed version of the chosen graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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