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Research Article

Enveloping Dual Banach Algebras and Approximate Properties

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Suppose that A is a Banach algebra and $F(A)$ is its enveloping dual Banach algebra, we show that $F(A)$ is approximately contractible (approximately amenable) if A has the same property. Also, we study the relation between the pseudoamenability of $F(A)$ and the pseudoamenability of the second dual A^{**} and we also characterize approximate biflatness and approximate biprojectivity of $F(A)$ associated with approximate biflatness and approximate biprojectivity of the second dual A^{**}.

1. Introduction and Preliminaries

Approximate contractibility and approximate amenability for Banach algebras were defined and studied by Ghahramani and Loy by [1]. They proved that a Banach algebra A is approximately amenable, if the second dual A^{**} is approximately amenable by [1], Theorem 2.3. They denoted that the measure algebra $M(G)$ is approximately amenable if and only if G is discrete and amenable by [1], Theorem 3.1, and also they showed that the group algebra $L^1(G)$ is approximately amenable if and only if G is amenable by [1], Theorem 3.2. As shown in [1], Theorem 3.3, $L^1(G)^{**}$ is approximately amenable if and only if G is finite.

The basic properties of biprojectivity and biflatness are investigated in [2]. In 1999, Zhang introduced the notion of approximate biprojectivity for Banach algebras [3]. It is well known that a Banach algebra A is pseudocontractible if and only if A is approximately biprojective and has a central approximate identity ([4], Proposition 3.8).

The concept of approximate biflatness for Banach algebra was introduced by Samei et al. [5]. They showed that a Banach algebra A is pseudoamenable whenever it is approximately biflat and has an approximate identity. Also, they studied approximate biflatness for various classes of Segal algebras in both group algebra, $L^1(G)$, and the Fourier algebra, $A(G)$, of a locally compact group G [5].

The module cohomological properties for Banach algebras, namely, module (approximate) biprojectivity and module (approximate) biflatness for Banach algebras which are generalizations of the classical cases, were introduced in [6, 7]. In these articles, the authors found necessary and sufficient conditions for $\ell^1(S)$ to be approximately module biprojective and module biflat, where S is an inverse semigroup.

For a Banach algebra A and a Banach A-bimodule X, the collection of all elements $x \in X$ is such that the module maps $A \longrightarrow X$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weakly compact denoted by $WAP_A(X)$ and the A-bimodule $WAP_A(X^*)$ is denoted by $F_A(X)_*$ [5]. Hence, we can write $F_A(X) = (F_A(X)_*)^* = (WAP_A(X^*))^*$. Now, if $X = A$, then we write $F(A) = (WAP(A^*))^*$ and it is well known that $F(A)$ is a universal canonical dual Banach algebra [5]. $F(A)$ is called the enveloping dual Banach algebra associated with A by [8].

Choi et al. defined the concept of WAP-virtual diagonal for a Banach algebra A [8]. They showed that A has a WAP-virtual diagonal if and only if the dual Banach algebra $F(A)$ is Connes-amenable. They also proved that for the group algebra $L^1(G)$, the existence of a virtual diagonal is equivalent to the existence of a WAP-virtual diagonal [8].

Motivated by the results mentioned above, in this paper, we study the approximate amenability, approximate contractibility, and pseudoamenability of the enveloping dual Banach algebra. We also investigate the cohomological properties of the enveloping dual Banach algebra $F(L^1(G))$ and $F(M(G))$, where G is a locally compact group. We study the relation between the pseudoamenability of $F(A)$ and the pseudoamenability of the second dual A^{**} and we also characterize approximate biflatness and approximate biprojectivity of $F(A)$ associated with the approximate biflatness and approximate biprojectivity of the second dual A^{**}. By giving examples of some Banach algebras, we investigate the cohomological properties of these Banach algebras with respect to their enveloping dual Banach algebras.

For a Banach algebra A and a Banach A-bimodule X, a derivation $D: A \longrightarrow X$ is a linear map such that

$$D(ab) = D(a)\cdot b + a\cdot D(b), \tag{1}$$

for every $a, b \in A$. A bounded derivation $D: A \longrightarrow X$ is called approximately inner, if there exists a net $(x_\alpha) \subseteq X$ such that $D(a) = \lim a\cdot x_\alpha - x_\alpha \cdot a$ for all $a \in A$.

Let A be a Banach algebra. We remind that the projective tensor product $A\otimes_p A$ is a Banach A-bimodule via the following actions:

$$c\cdot(a\otimes b) = ca\otimes b,\ (a\otimes b)\cdot c = a\otimes bc, \tag{2}$$

for every $a, b, c \in A$. Also, the map, $\pi_A: A\otimes_p A \longrightarrow A$ denotes the product morphism which is defined by $\pi_A(a\otimes b) = ab$.

A Banach algebra A is approximately contractible (approximately amenable), if for all A-bimodule X, every bounded derivation $D: A \longrightarrow X$ ($D: A \longrightarrow X^*$) is approximately inner by [1]. We remind that A is pseudoamenable, if there is a net $(u_\alpha) \subseteq A\otimes_p A$, such that $\lim_\alpha (au_\alpha - u_\alpha a) = 0$ and $\lim_\alpha \pi_A(u_\alpha)a = a$ for all $a \in A$ [4] and also A is called approximately biflat, if there exists a net $\rho_\alpha: (A\otimes_p A)^* \longrightarrow A^*$ of bounded A-bimodule morphisms such that $W^*OT \lim_\alpha \rho_\alpha \circ \pi_A^* = id_{A^*}$, where W^*OT is the weak* operator topology on $B(A^*)$. We recall that the weak* operator topology on $B(A^*)$ is a locally convex topology determined by the seminorms $\{p_{a,f}: a \in A, f \in A^*\}$, where $p_{a,f}(T) = |\langle a, T(f)\rangle|$ [5]. A Banach algebra A is called approximately biprojective if there is a net of bounded A-bimodule morphism $\rho_\alpha: A \longrightarrow A\otimes_p A$ such that $\lim \pi_A \circ \rho_\alpha(a) = a$ for all $a \in A$ (for more details see [3]).

2. Cohomological Properties of the Dual Banach Algebras

Throughout this section, $F(A)$ is the enveloping dual Banach algebra associated with A and we will study the cohomological properties of $F(A)$ such as approximate amenability and pseudoamenability.

Theorem 1. *Let A be a Banach algebra. If A is approximately contractible, then $F(A)$ is approximately contractible.*

Proof. Suppose that $D: F(A) \longrightarrow X$ is a bounded derivation, where X is a Banach $F(A)$-bimodule, then we can extend $D: F(A) \longrightarrow X$ to a bounded derivation $D^{**}: F(A)^{**} \longrightarrow X^{**}$ by [9], Lemma 2.2. Suppose that $k_{F(A)}$ is the canonical inclusion of $F(A) \longrightarrow F(A)^{**}$ and k_A is the canonical inclusion of $A \longrightarrow A^{**}$ and q_A is the adjoint of the inclusion map $WAP(A^*) \longrightarrow A^*$, then, we define a map $\overline{D}: A \longrightarrow X^{**}$ by $\overline{D} = D^{**}\circ k_{F(A)}\circ q_A \circ k_A$. However, $k_{F(A)}\circ q_A\circ k_A$ are the homomorphism algebra, so $\overline{D}$ is a derivation. Since A is approximately contractible, there exists a net $(x_\alpha) \subseteq X^{**}$ such that

$$\overline{D}(a) = D^{**}(a) = \lim_\alpha (a\cdot x_\alpha - x_\alpha\cdot a), \tag{3}$$

for all $a \in A$. Since the module action on X^{**} is weak$*$-continuous by [9], Lemma 1.1 and by Goldstein's theorem, we can imply that

$$D^{**}(h) = wk^* - \lim_\alpha (h\cdot x_\alpha - x_\alpha\cdot h), \tag{4}$$

for $h \in A^{**}$. In particular, we have

$$D(t) = D^{**}(t) = wk^* - \lim_\alpha (t\cdot x_\alpha - x_\alpha\cdot t), \tag{5}$$

for $t \in F(A)$. By Goldstein's theorem, there exists a bounded net $(y^\alpha_\beta) \subseteq X$ such that $wk^* - \lim_\beta y^\alpha_\beta = x_\alpha$ and so

$$D(t) = wk^* - \lim_\alpha\ wk^* - \lim_\beta \left(t\cdot y^\alpha_\beta - y^\alpha_\beta\cdot t\right), \tag{6}$$

for $t \in F(A)$.

Suppose that $E = J\times\Lambda^J$ is a directed set with product ordering which is defined by

$$(\alpha,\beta) \le_E (\alpha',\beta') \Leftrightarrow \alpha \le_J \alpha', \beta \le_{\Lambda^J} \beta', \tag{7}$$

where $\alpha, \alpha' \in J, \beta, \beta' \in \Lambda^J$ and Λ^J are the set of all maps $J \longrightarrow \Lambda$. We recall that $\beta \le_{\Lambda^J} \beta'$ means that $\beta(j) \le_\Lambda \beta'(j)$ for all $j \in J$.

We set $\gamma = (\alpha,\beta) \in E$ and $\zeta_\gamma = y^\alpha_\beta$. By iterated limit theorem ([10], page 69) and by the equation (6), we see that

$$D(t) = wk^* - \lim_\gamma \left(t\cdot\zeta_\gamma - \zeta_\gamma\cdot t\right), \tag{8}$$

for $t \in F(A)$. On the other hand, $(t\cdot\zeta_\gamma - \zeta_\gamma\cdot t) \in X$ and $D(t) \in X$. So,

$$D(t) = wk - \lim_\gamma \left(t\cdot\zeta_\gamma - \zeta_\gamma\cdot t\right), \tag{9}$$

for $t \in F(A)$. This follows that D is weakly approximately inner and equivalently, approximately inner. Thus, $F(A)$ is contractible. □

Corollary 2. *Let A be an approximately amenable Banach algebra. Then, $F(A)$ is approximately amenable.*

Proof. We know that a Banach algebra A is approximately amenable if and only if A is approximately contractible ([4], Theorem 3.1). Since A is approximately amenable, A is approximately contractible. So, by the previous theorem,

$F(A)$ is approximately contractible, and hence $F(A)$ is approximately amenable.

The following example shows us that the converse of Corollary 2 is not true. □

Example 1. Let G be an amenable locally compact group which is not discrete ([11], Chapter 2). If the converse of Corollary 2 is true, then, we have that $L^1(G)$ is approximately amenable if and only if $F(L^1(G))$ is approximately amenable. Since $L^1(G)$ has a bounded approximate identity, $F(L^1(G))$ is unital ([12], Lemma 2.9) and so the authors in [4], Proposition 3.2 show that $L^1(G)$ is pseudoamenable if and only if $F(L^1(G))$ is pseudoamenable. However, the measure algebra $M(G)$ is the quotient of $F(L^1(G))$ [13]. Hence, $M(G)$ is pseudoamenable ([4], Proposition 2.2). On the other hand, the authors in [4], Proposition 4.2 show that $M(G)$ is pseudoamenable if and only if G is amenable and discrete. So, we can imply that G must be amenable and discrete which is a contradiction. Therefore, the converse of Corollary 2 is not true.

Theorem 3. *Let A be a Banach algebra with a bounded approximate identity and let A be approximately amenable. Then, $F(A)$ is pseudoamenable.*

Proof. Since A has a bounded approximate identity, $F(A)$ is unital by ([12], Lemma 2.9). However, a unital Banach algebra is pseudoamenable if and only if it is approximately amenable ([4], Proposition 3.2). Since A is approximately amenable, Corollary 2 follows that $F(A)$ is approximately amenable. However, $F(A)$ is a unital Banach algebra. So, $F(A)$ is pseudoamenable. □

Theorem 4. *Let A be a unital Banach algebra and let A be pseudoamenable. Then, $F(A)$ is pseudoamenable.*

Proof. Since A has an identity, $F(A)$ is unital ([12], Lemma 2.9). However, a unital Banach algebra is pseudoamenable if and only if it is approximately amenable ([4], Proposition 3.2). Since A is unital and pseudoamenable, A is approximately amenable. So, by Corollary 2, we imply that $F(A)$ is approximately amenable. Hence, $F(A)$ is pseudoamenable. □

Theorem 5. *Let A be a Banach algebra and let the second dual A^{**} be pseudoamenable. Then, $F(A)$ is pseudoamenable.*

Proof. Let $q_A\colon A^{**} \longrightarrow F(A)$ be the adjoint of the inclusion map $WAP(A^*) \longrightarrow A^*$, which is a continuous epimorphism. Then, by hypothesis, $F(A)$ is pseudoamenable ([4], proposition 2.2).

In the following example, we show that there is a Banach algebra A such that $F(A)$ is pseudoamenable but A^{**} is not pseudoamenable. □

Example 2. Note that, the second dual $L^1(G)^{**}$ is pseudoamenable if and only if G is finite ([4], proposition 4.2). Let G be an infinite amenable locally compact group. Then, Theorem 3 follows that $F(L^1(G))$ is pseud amenable, but $L^1(G)^{**}$ is not pseudoamenable.

3. Algebras Related to Locally Compact Groups

In this section, we study pseudoamenability and approximate amenability of the enveloping dual Banach algebra $F(L^1(G))$ and $F(M(G))$, where G is a locally compact group.

Proposition 6. *Let G be a locally compact group. Then, $F(L^1(G))$ is pseudoamenable if G is amenable.*

Proof. Let G be amenable. Then, $L^1(G)$ is approximately amenable. By Corollary 2, we imply that $F(L^1(G))$ is approximately amenable. Since $L^1(G)$ has a bounded approximate identity, $F(L^1(G))$ is unial ([12], Lemma 2.9). So, $F(L^1(G))$ is pseudoamenable if and only if $F(L^1(G))$ is approximately amenable ([4], Proposition 3.2). Hence, $F(L^1(G))$ is pseudoamenable. □

Proposition 7. *Let G be a locally compact group. Then, $F(M(G))$ is pseudoamenable if G is discrete and amenable.*

Proof. Let G be amenable and discrete. Then, $M(G)$ is approximately amenable by ([1], Theorem 3.1). By Corollary 2, $F(M(G))$ is approximately amenable. Since $M(G)$ has an identity, $F(M(G))$ is unital ([12], Lemma 2.9). Hence, $F(M(G))$ is pseudoamenable if and only if $F(M(G))$ is approximately amenable ([4], Proposition 3.2). Hence, $F(M(G))$ is pseudoamenable. □

Proposition 8. *Let G be a locally compact group. Then, $F(L^1(G))$ is approximately amenable if G is amenable.*

Proof. Let G be an amenable group. Then, $L^1(G)$ is approximately amenable by ([1], Theorem 3.2). So, by Corollary 2, one can show that $F(L^1(G))$ is approximately amenable. □

Proposition 9. *Let G be a locally compact group. Then, $F(M(G))$ is approximately amenable if G is amenable and discrete.*

Proof. We know that $M(G)$ is approximately amenable if and only if G is amenable and discrete ([1], Theorem 3.1). Hence, Corollary 2 follows that $F(M(G))$ is approximately amenable. □

Corollary 10. *Let G be a locally compact group. Then, $F(L^1(G)^{**})$ is approximately amenable if G is finite.*

Proof. We know that $L^1(G)^{**}$ is approximately amenable if and only if G is finite ([1], Theorem 3.3). So, we can imply that $F(L^1(G)^{**})$ is approximately amenable by Corollary 2.

In the following example, we show that there is a Banach algebra A such that A is pseudoamenable but A^{**} is not pseudoamenable. □

Example 3. Let G be an infinite amenable locally compact group. Then, $L^1(G)$ is pseudoamenable by [4], Proposition 4.1. However, since G is infinite, $L^1(G)^{**}$ is not pseudoamenable ([4], Proposition 4.2).

4. Approximate Biflat and Approximate Biprojective

Now, we study approximate biflatness and approximate biprojectivity of the enveloping dual Banach algebra $F(A)$, associated with the second dual A^{**}.

Remark 11. By [14], we suppose that A is a Banach algebra and X is a subspace of A^*. Then, X is called a A-left invariant if $X\cdot a\subseteq X$ for every $a\in A$. We suppose that $X\subseteq A^*$ is a left invariant subspace. If $X^*\cdot X\subseteq X$, then the subspace X is called A-left introverted. The notation A-will often be omitted to simplify.

Remark 12. We consider the left introverted subspaces X and Y of A^* such that $Y\subseteq X$ and define the map $H\colon X^*\longrightarrow Y^*$ by $H(g)=f|_Y$, for every $g\in X^*$, which is a continuous homomorphism from X^* onto Y^* and its kernel is weak*-closed ideal $Y^{\perp}$ of X^* by [14], Lemma 1.1, where $Y^{\perp}=\{x^*\in X^*;\ <x^*,y>=0 \text{ for every } y\in Y\}$. Indeed, we have the hollowing direct sum decomposition $X^*=Y^*\oplus Y^{\perp}$.

Since $WAP(A^*)$ is an introverted subspace of A^* [14], there is a direct sum decomposition $A^{**}=(WAP(A^*))^*\oplus WAP(A^*)^{\perp}$, where $(WAP(A^*))^*=F(A)$ is the enveloping dual Banach algebra associated to A by [15], Remark 3.4.

The following result is given in [5], Proposition 2.8.

Proposition 13. *Let $\{A_i\colon i\in I\}$ be a family of (quantized) Banach algebras.*

(i) If each A_i is (operator) approximately biflat, then for $1\le p<\infty$, $\oplus_{i\in I}^{p}A_i$ is (operator) approximately biflat.

(ii) If A_1 and A_2 are (operators) approximately biflat quantized Banach algebras and $A=A_1\oplus A_2$ has an operator space structure such that the projection maps $A\longrightarrow A_i$ are completely bounded, then A is also an operator approximately biflat.

(iii) If each A_i is (operator) biflat and $\sup_i BF_{A_i}<\infty$ (respectively, $\sup_i BF_{A_i}^{op}<\infty$), then $\oplus_{i\in I}^{1}A_i$ is (operator) biflat.

Proposition 14. *Let A be a Banach algebra. If $F(A)$ and $WAP(A^*)^{\perp}$ are approximately biflat, then $A^{**}=F(A)\oplus WAP(A^*)^{\perp}$ is approximately biflat.*

Proof. It is clear from the previous proposition. □

Theorem 15. *Let $A^{**}=F(A)\oplus WAP(A^*)^{\perp}$ be approximately biflat. Then, $F(A)$ is approximately biflat.*

Proof. Since A^{**} is approximately biflat, there is a net $\lambda_\alpha\colon ((A^{**})\otimes_p(A^{**}))^*\longrightarrow (A^{**})^*$ of bounded A^{**}-bimodule morphisms such that $W^*OT-\lim_\alpha \lambda_\alpha\circ\pi_{A^{**}}^*=id_{(A^{**})^*}$.

Let $P_{F(A)}\colon F(A)\oplus WAP(A^*)^{\perp}\longrightarrow F(A)$ be the projection map and let $\sigma_{F(A)}\colon F(A)\longrightarrow F(A)\oplus WAP(A^*)^{\perp}$ be defined by $\sigma_{F(A)}(a)=(a,0)$ for every $a\in F(A)$. Now, we define $\widetilde{\lambda}_\alpha\colon (F(A)\otimes_p F(A))^*\longrightarrow F(A)^*$ by $\widetilde{\lambda}_\alpha=\sigma_{F(A)}^*\circ\lambda_\alpha\circ(P_{F(A)}\otimes P_{F(A)})^*$, which is a net of bounded $F(A)$-bimodule morphisms such that $W^*OT-\lim_\alpha\widetilde{\lambda}_\alpha\circ\pi_{F(A)}^*=id_{F(A)^*}$. To see this, suppose that $\psi\in F(A)^*$ and $b\in F(A)$, then, we have $(P_{F(A)}\otimes P_{F(A)})^*\circ\pi_{F(A)}^*(\psi)=\pi_{A^{**}}^*\circ P_{F(A)}^*(\psi)$. Since $W^*OT-\lim_\alpha\lambda_\alpha\circ\pi_{A^{**}}^*=id_{(A^{**})^*}$, we have

$$\begin{aligned}\lim_\alpha\left\langle\widetilde{\lambda}_\alpha\circ\pi_{F(A)}^*(\psi),b\right\rangle &=\lim_\alpha\left\langle\sigma_{F(A)}^*\circ\lambda_\alpha\circ\left(P_{F(A)}\otimes P_{F(A)}\right)^*\circ\pi_{F(A)}^*(\psi),b\right\rangle\\ &=\lim_\alpha\left\langle\sigma_{F(A)}^*\circ\lambda_\alpha\circ\pi_{A^{**}}^*\circ P_{F(A)}^*(\psi),b\right\rangle\\ &=\lim_\alpha\left\langle\sigma_{F(A)}^*\circ P_{F(A)}^*(\psi),b\right\rangle=\langle\psi,b\rangle.\end{aligned}\tag{10}$$

Hence, $F(A)$ is approximately biflat. □

Theorem 16. *If $F(A)$ and $WAP(A^*)^\perp$ are approximately biprojective, then $A^{**} = F(A) \oplus WAP(A^*)^\perp$ is approximately biprojective.*

Proof. Since $F(A)$ and $WAP(A^*)^\perp$ are approximately biprojective, there exists a net of bounded $F(A)$-bimodule morphisms $\rho_\lambda: F(A) \longrightarrow F(A) \otimes_p F(A)$ and a net of bounded $WAP(A^*)^\perp$-bimodule morphisms $\mu_\gamma: WAP(A^*)^\perp \longrightarrow WAP(A^*)^\perp \otimes_p WAP(A^*)^\perp$ such that $\lim_\lambda \pi_{F(A)} \circ \rho_\lambda = id_{F(A)}$ and $\lim_\gamma \pi_{WAP(A^*)^\perp} \circ \mu_\gamma = id_{WAP(A^*)^\perp}$, respectively.

Suppose that $P_{F(A)}: A^{**} = F(A) \oplus WAP(A^*)^\perp \longrightarrow F(A)$ and $P_{WAP(A^*)^\perp}: A^{**} = F(A) \oplus WAP(A^*)^\perp \longrightarrow WAP(A^*)^\perp$ are the projection map and $\sigma_{F(A)}: F(A) \longrightarrow A^{**} = F(A) \oplus WAP(A^*)^\perp$ and $\sigma_{WAP(A^*)^\perp}: WAP(A^*)^\perp \longrightarrow A^{**} = F(A) \oplus WAP(A^*)^\perp$ are defined by $\sigma_{F(A)}(a) = (a, 0)$ and $\sigma_{WAP(A^*)^\perp}(b) = (0, b)$, respectively, for every $a \in F(A)$ and $b \in WAP(A^*)^\perp$, then, we have

$$\begin{aligned} \sigma_{F(A)} \circ \pi_{F(A)} &= \pi_{A^{**}} \circ \left(\sigma_{F(A)} \otimes \sigma_{F(A)}\right), \\ \sigma_{WAP(A^*)^\perp} \circ \pi_{WAP(A^*)^\perp} &= \pi_{A^{**}} \circ \left(\sigma_{WAP(A^*)^\perp} \otimes \sigma_{WAP(A^*)^\perp}\right). \end{aligned} \tag{11}$$

So,

$$\begin{aligned} &\lim_\gamma \lim_\lambda \pi_{A^{**}} \circ \left(\left(\sigma_{F(A)} \otimes \sigma_{F(A)}\right) \circ \rho_\lambda \circ P_{F(A)} + \left(\sigma_{WAP(A^*)^\perp} \otimes \sigma_{WAP(A^*)^\perp}\right) \circ \mu_\gamma \circ P_{WAP(A^*)^\perp}\right) \\ &\quad = \lim_\gamma \lim_\lambda \pi_{A^{**}} \circ \left(\sigma_{F(A)} \otimes \sigma_{F(A)}\right) \circ \rho_\lambda \circ P_{F(A)} \\ &\qquad + \lim_\gamma \lim_\lambda \pi_{A^{**}} \circ \left(\sigma_{WAP(A^*)^\perp} \otimes \sigma_{WAP(A^*)^\perp}\right) \circ \mu_\gamma \circ P_{WAP(A^*)^\perp} \\ &\quad = \lim_\gamma \lim_\lambda \sigma_{F(A)} \circ \pi_{F(A)} \circ \rho_\lambda \circ P_{F(A)} + \lim_\gamma \lim_\lambda \sigma_{WAP(A^*)^\perp} \circ \pi_{WAP(A^*)^\perp} \circ \mu_\gamma \circ P_{WAP(A^*)^\perp} \\ &\quad = \sigma_{F(A)} \circ id_{F(A)} \circ P_{F(A)} + \sigma_{WAP(A^*)^\perp} \circ id_{WAP(A^*)^\perp} \circ P_{WAP(A^*)^\perp} \\ &\quad = id_{A^{**}}. \end{aligned} \tag{12}$$

Let $Z = \Lambda \times \Gamma^\Lambda$ be directed by the product ordering. For every $\beta = (\lambda, (\gamma_{\lambda'})_{\lambda' \in \Lambda}) \in Z$, we define $\psi_\beta = ((\sigma_{F(A)} \otimes \sigma_{F(A)}) \circ \rho_\lambda \circ P_{F(A)} + (\sigma_{WAP(A^*)^\perp} \otimes \sigma_{WAP(A^*)^\perp}) \circ \mu_{\gamma_\lambda} \circ P_{WAP(A^*)^\perp})$. By iterated limit theorem ([10], P. 69), equation (12), implies the following:

$$\lim_\beta \pi_{A^{**}} \circ \psi_\beta = id_{A^{**}} \tag{13}$$

and certainly $\psi_\beta: A^{**} \longrightarrow A^{**} \otimes_p A^{**}$ is a net of bounded A^{**}-bimodule morphisms. Hence, A^{**} is approximately biprojective. □

Theorem 17. *Let $A^{**} = F(A) \oplus WAP(A^*)^\perp$ be approximately biprojective, then, $F(A)$ is approximately biprojective.*

Proof. Since A^{**} is approximately biprojective, there is a net of bounded A^{**}-bimodule morphism $\lambda_\alpha: A^{**} \longrightarrow A^{**} \otimes_p A^{**}$ such that $\lim_\alpha \pi_{A^{**}} \circ \lambda_\alpha = id_{A^{**}}$. We define $\mu_\alpha: F(A) \longrightarrow F(A) \otimes_p F(A)$ by $\mu_\alpha = (P_{F(A)} \otimes P_{F(A)}) \circ \lambda_\alpha \circ \sigma_{F(A)}$, where σ_A and P_A are defined by Theorem 15. Certainly, μ_α is a net of bounded $F(A)$-bimodule morphisms. Now, we prove that $\lim_\alpha \pi_{F(A)} \circ \mu_\alpha = id_{F(A)}$.

We have $\pi_{F(A)} \circ (P_{F(A)} \otimes P_{F(A)}) = P_{F(A)} \circ \pi_{A^{**}}$. Hence,

$$\begin{aligned} \lim_\alpha \pi_{F(A)} \circ \mu_\alpha &= \lim_\alpha \pi_{F(A)} \circ \left(P_{F(A)} \otimes P_{F(A)}\right) \circ \lambda_\alpha \circ \sigma_{F(A)} \\ &= \lim_\alpha P_{F(A)} \circ \pi_{A^{**}} \circ \lambda_\alpha \circ \sigma_{F(A)} \\ &= P_{F(A)} \circ id_{A^{**}} \circ \sigma_{F(A)} \\ &= id_{F(A)}. \end{aligned} \tag{14}$$

So, $F(A)$ is approximately biprojective. □

Example 4. Let G be a commutative compact group. Then, $L^2(G)$ with convolution multiplication is an approximately biprojective Banach algebra by [3] and so it is approximately biflat. Since $L^2(G)$ is a Hilbert space, we have $L^2(G)^{**} = L^2(G)$. Hence, Theorems 15 and 17 imply that $F(L^2(G))$ is approximately biflat and approximately biprojective, respectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] F. Ghahramani and R. J. Loy, "Generalized notions of amenability," *Journal of Functional Analysis*, vol. 208, pp. 229–260, 2004.

[2] A. Y. Helemskii, *The Homology of Banach and Topological Algebras*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.

[3] Y. Zhang, "Nilpotent ideals in a class of Banach algebras," *Proceedings of the American Mathematical Society*, vol. 127, no. 11, pp. 3237–3242, 1999.
[4] F. Ghahramani and Y. Zhang, "Pseudo-amenable and pseudo-contractible Banach algebras," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 142, no. 1, pp. 111–123, 2007.
[5] E. Samei, N. Spronk, and R. Stokke, "Biflatness and pseudo-amenability of Segal algebras," *Canadian Journal of Mathematics*, vol. 62, no. 4, pp. 845–869, 2010.
[6] A. Bodaghi and M. Amini, "Module biprojective and module biflat Banach algebras," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 75, no. 3, pp. 25–36, 2013.
[7] A. Bodaghi and S. Grailoo Tanha, "Module approximate biprojectivity and module approximate biflatness of Banach algebras," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 70, no. 1, pp. 409–425, 2021.
[8] Y. Choi, E. Samei, and R. Stokke, "Extension of derivations, and Connes-amenability of the enveloping dual Banach algebra," *Mathematica Scandinavica*, vol. 117, no. 2, pp. 258–303, 2015.
[9] F. Gourdeau, "Amenability and the second dual of a Banach algebra," *Studia Mathematica*, vol. 125, no. 1, pp. 75–81, 1997.
[10] J. L. Kelly, *General Topology*, D. Van Nostrand Company, Inc, New York, NY, USA, 1955.
[11] V. Runde, "Lectures on amenability," *Lecture Notes in Mathematics*, Vol. 1774, Springer-Verlag, Berlin, Germany, 2002.
[12] S. F. Shariati, A. Pourabbas, and A. Sahami, "WAP-biprojectivity of the enveloping dual Banach algebras," *Bollettino dell'Unione Matematica Italiana*, vol. 13, no. 1, pp. 91–101, 2020.
[13] V. Runde, "Dual Banach algebras: connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule," *Mathematica Scandinavica*, vol. 95, no. 1, pp. 124–144, 2004.
[14] A. T.-M. Lau and R. J. Loy, "Weak amenability of Banach algebras on locally compact groups," *Journal of Functional Analysis*, vol. 145, no. 1, pp. 175–204, 1997.
[15] N. Razi and A. pourabbas, "Some homological properties of the enveloping dual Banach algebra," *Boletín de la Sociedad Matemática Mexicana*, vol. 28, no. 3, pp. 59–10, 2022.