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Research Article

Higher Derivations Satisfying Certain Identities in Rings

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Let n and m be fixed positive integers. In this paper, we establish some structural properties of prime rings equipped with higher derivations. Motivated by the works of Herstein and Bell-Daif, we characterize rings with higher derivations $D = (d_i)_{i\in\mathbb{N}}$ satisfying (i) $[d_n(x), d_m(y)] \in Z(\mathscr{R})$ for all $x, y \in \mathscr{R}$ and (ii) $d_n([x, y]) \in Z(\mathscr{R})$ for all $x, y \in \mathscr{R}$.

1. Introduction

Knowing the structure of a ring has always been an interest to various authors, be it its number of ideals, commutativity, and behavior for certain special identities. In this pursuit, derivations were often used. Like Posner, Herstein, Bell, Daif, and a number of other enthusiastic mathematicians brought the idea of differential identities to study the structure of rings. There are some other types of derivations such as Jordan derivations, (θ, ϕ) derivations, generalized derivations, and skew derivations which can be proved helpful for the same. Indeed, most of these forms of derivations have already proven fruitful. The usage of the idea of higher derivations in this direction is relatively a new one. To explore this further, we first revisit the idea of higher derivations on rings. An additive map $d\colon \mathscr{R} \longrightarrow \mathscr{R}$ is said to form a derivation if $d(\mathrm{xy}) = d(x)y + \mathrm{x}d(y)$ for all $x, y \in \mathscr{R}$ where $\mathscr{R}$ is any ring. Let $\mathbb{N}$ be the set of all nonnegative integers. Following [1], a family of additive mappings $(d_i)_{i\in\mathbb{N}}$ from $\mathscr{R}$ to itself with d_0 defined as the identity map on $\mathscr{R}$ is termed as a higher derivation if $d_n(\mathrm{xy}) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathscr{R}$ and $n \geq 1$. On similar lines, a higher derivation of rank $s \geq 1$ is defined as a family of additive mappings (say) $(d_0, d_1, \ldots, d_s)$, where d_0 is the identity map on $\mathscr{R}$ and $d_n(\mathrm{xy}) = \sum_{i+j=n} d_i(x)d_j(y)$ for all $x, y \in \mathscr{R}$ and $1 \leq n \leq s$ (see [2], page 540 for more details). A ring $\mathscr{R}$ is prime if $x\mathscr{R}y = 0$ implies either $x = 0$ or $y = 0$ and semiprime if $x\mathscr{R}x = 0$ implies $x = 0$. The center of $\mathscr{R}$ is denoted by $Z(\mathscr{R})$. $\mathscr{Q}$ will denote the Martindale ring of quotients of $\mathscr{R}$ and $\mathscr{C}$ will represent its extended centroid. Recall that a derivation $d\colon \mathscr{R} \longrightarrow \mathscr{R}$ is said to be inner if there exists $a \in \mathscr{R}$ such that $d(x) = \mathrm{xa} - \mathrm{ax}$, for every $x \in \mathscr{R}$. This derivation is usually called the inner derivation associated with 'a' and will be denoted here by δ_a. Also, if $d\colon \mathscr{R} \longrightarrow \mathscr{R}$ is a derivation of $\mathscr{R}$ and there exists $q \in \mathscr{Q}$ such that $d(x) = \delta_q(x)$, for every $x \in \mathscr{R}$, then d is said to be X-inner [3]. In this case, we will denote d again by δ_q.

The main aim of this paper is to describe the structure of prime rings equipped with higher derivations.

2. Background

Let us mention some known results which will be used frequently while proving our results:

Lemma 1 (see [4]). *Let $\mathscr{R}$ be a prime ring with 1 and let $\mathscr{T} = \mathscr{R}\mathscr{C}$. Suppose $\sum_{i=1}^{n} a_i x b_i = 0$ for all $x \in \mathscr{T}, a_i, b_i \in \mathscr{R}$. Then, either $\{a_i\}$ is $\mathscr{C}$-dependent or $\{b_i\}$ is $\mathscr{C}$-dependent.*

Lemma 2 (see [1]). *Assume that $\mathscr{R}$ is a prime ring and $D = (d_i)_{i\in\mathbb{N}\cup\{0\}}$ is a higher derivation of $\mathscr{R}$ which satisfies an $\mathscr{R}$-linear relation of minimal length n on $\mathscr{R}$.*

Then there exist $q_0 = 1, q_1, \ldots, q_{n-1} \in \mathcal{Q}$ such that $\sum_{i=1}^{n} q_{n-i} d_i = 0$. In particular, D satisfies a monic $\mathcal{Q}$–linear relation of length n on $\mathcal{R}$. Moreover, $d_1 = \delta_{q_1}$ and $d_m(x) = \delta_{q_m}(x) - \sum_{i=1}^{m-1} q_i d_{m-i}(x)$, for every $x \in \mathcal{R}$ and $2 \le m \le n$.

Lemma 3 (see [5]). *Let $\mathcal{R}$ be a ring without any nonzero nilpotent ideals. Then any element of $\mathcal{R}$ which commutes with all elements of $[\mathcal{R}, \mathcal{R}]$ must lie in the center of $\mathcal{R}$.*

Lemma 4 (see [1]). *(Proposition 1.1) Let $\mathcal{R}$ be a semiprime ring and $D = (d_i)_{i \in \mathbb{N} \cup \{0\}}$ a higher derivation of $\mathcal{R}$. Then there exists a unique higher derivation $D^* = (d_i^*)_{i \in \mathbb{N} \cup \{0\}}$ of $\mathcal{Q}$ such that $d_n^*|_{\mathcal{R}} = d_n$, for every $n \in \mathbb{N} \cup \{0\}$.*

Fact 5 (see [6]). $\mathcal{R}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identity (GPI).

Lemma 6. *Let $\mathcal{R}$ be any ring and $(d_i)_{i \in \mathbb{N} \cup \{0\}}$ be a higher derivation defined on $\mathcal{R}$, where $d_0 = id_{\mathcal{R}}$. Then $d_i(z) \in Z(\mathcal{R})$ whenever $z \in Z(\mathcal{R})$.*

Proof. Clearly, the result holds for d_0. For $i = 1$, we have

$$d_1(\mathrm{xz}) = d_1(\mathrm{zx}) \text{ for all } \quad x \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{1}$$

This gives

$$x d_1(z) = d_1(z) x \text{ for all } \quad x \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{2}$$

Therefore, $d_1(z) \in Z(\mathcal{R})$. Thus, the result is true for $i = 1$. Let us suppose the result is true for $i = n$, that is, $d_n(z) \in Z(\mathcal{R})$ for all $z \in Z(\mathcal{R})$. For $i = n + 1$, we have

$$d_{n+1}(xz) = d_{n+1}(\mathrm{zx}) \text{ for all } \quad x \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{3}$$

On computing the above equation, we obtain

$$x d_{n+1}(z) = d_{n+1}(z) x \text{ for all } \quad x \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{4}$$

Using the principle of mathematical induction, we conclude that $d_i(z) \in Z(\mathcal{R})$ for all $i \in \mathbb{N}$. Hence, this is the desired result. □

3. Herstein's Result for Higher Derivations

Herstein [7] in the year 1978, proved that for a prime ring $\mathcal{R}$ with a nonzero derivation d satisfying $[d(x), d(y)] = 0$ for all $x, y \in \mathcal{R}$, is a commutative integral domain if $\mathrm{char}(\mathcal{R}) \neq 2$, and for $\mathrm{char}(\mathcal{R}) = 2$, $\mathcal{R}$ is commutative or is an order in a simple algebra which is 4-dimensional over its center. This result was extended to semiprime rings in 1998 by Daif [8], and furthermore, it was extended for any two-sided ideal of $\mathcal{R}$ by Bell and Daif [9]. Bell and Rehman [10] applied this identity for generalized derivations and were able to obtain some interesting results. Motivated by these ideas, we wish to explore similar criteria for the case of higher derivations and describe the structure of prime rings. In fact, we prove the following result:

Theorem 7. *Let $\mathcal{R}$ be a prime ring and n, m be fixed positive integers. Next, let $D = (d_i)_{i \in \mathbb{N}}$ be a nonzero higher derivation of $\mathcal{R}$, where $d_0 = id_{\mathcal{R}}$ satisfying $[d_n(x), d_m(y)] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Then, either some linear combination of $(d_i)_{i \in \mathbb{N}}$ maps center of $\mathcal{R}$ to 0 or $\mathcal{R}$ is commutative.*

Proof. For a fixed pair $n, m \in \mathbb{N}$, we have $[d_n(x), d_m(y)] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Replace y by zy, where $z \in Z(\mathcal{R})$ to obtain

$$[d_n(x), d_m(zy)] \in Z(\mathcal{R}). \tag{5}$$

On solving, we obtain

$$\begin{aligned}&[d_n(x), d_m(z)y + d_{m-1}(z)d_1(y) + \cdots + d_1(z)d_{m-1}(y)\\&\quad + z d_m(y)] \in Z(\mathcal{R}),\end{aligned} \tag{6}$$

for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$ which gives

$$\begin{aligned}&d_m(z)[d_n(x), y] + d_{m-1}(z)[d_n(x), d_1(y)]\\&\quad + \cdots + d_1(z)[d_n(x), d_{m-1}(y)] \in Z(\mathcal{R}).\end{aligned} \tag{7}$$

On commuting with $t \in \mathcal{R}$, we obtain

$$\begin{aligned}&[d_m(z)[d_n(x), y] + d_{m-1}(z)[d_n(x), d_1(y)]\\&\quad + \cdots + d_1(z)[d_n(x), d_{m-1}(y)], t] = 0,\end{aligned} \tag{8}$$

for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$. Replace t by tr, $r \in \mathcal{R}$, to obtain

$$\begin{aligned}&[d_m(z)[d_n(x), y] + d_{m-1}(z)[d_n(x), d_1(y)] + \cdots +\\&\quad d_1(z)[d_n(x), d_{m-1}(y)], \mathrm{tr}] = 0,\end{aligned} \tag{9}$$

for all $x, y, t \in \mathcal{R}, z \in Z(\mathcal{R})$ which gives

$$\begin{aligned}&d_m(z)[[d_n(x), y], \mathrm{tr}] + d_{m-1}(z)[[d_n(x), d_1(y)], \mathrm{tr}]\\&\quad + \cdots + d_1(z)[[d_n(x), d_{m-1}(y)], \mathrm{tr}] = 0,\end{aligned} \tag{10}$$

for all $x, y, t \in \mathcal{R}, z \in Z(\mathcal{R})$. On computation, we get

$$\begin{aligned}&d_m(z)t[[d_n(x), y], r] + d_{m-1}(z)t[[d_n(x), d_1(y)], r]\\&\quad + \cdots + d_1(z)t[[d_n(x), d_{m-1}(y)], r] = 0,\end{aligned} \tag{11}$$

for all $x, y, t \in \mathcal{R}, z \in Z(\mathcal{R})$. By Fact 5, $\mathcal{R}$ and $\mathcal{Q}$, its Martindale ring of quotients, satisfy the same generalized polynomial identity, and every higher derivation on $\mathcal{R}$ can be uniquely extended to a higher derivation on $\mathcal{Q}$ by Lemma 4. Therefore, we conclude that

$$\begin{aligned}&d_m(z)t[[d_n(x), y], r] + d_{m-1}(z)t[[d_n(x), d_1(y)], r]\\&\quad + \cdots + d_1(z)t[[d_n(x), d_{m-1}(y)], r] = 0,\end{aligned} \tag{12}$$

for all $x, y, t, r \in \mathcal{Q}, z \in \mathcal{C}$. The application of Lemma 1 helps us deduce that, either the set $\{d_1(z), d_2(z), \ldots, d_m(z)\}$ is linearly dependent over $\mathcal{C}$ or the set $\{[[d_n(x), d_0(y)], r], [[d_n(x), d_1(y)], r], \ldots, [[d_n(x), d_{m-1}(y)], r]\}$ is linearly dependent over $\mathcal{C}$. Suppose the set $\{d_1(z), d_2(z), \ldots, d_m(z)\}$ is linearly dependent over $\mathcal{C}$. So, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ from $\mathscr{C}$ not all zeros such that,

$$\sum_{i=1}^{m} \alpha_i d_i(z) = 0 \text{ for all } \quad z \in \mathscr{C}. \tag{13}$$

Hence, we have, $\sum_{i=1}^{m} \alpha_i d_i(Z(\mathscr{R})) = 0$, proving our claim that some linear combination of $(d_i)_{i\in\mathbb{N}}$'s maps center of $\mathscr{R}$ to zero. Next, we suppose the set $\{[[d_n(x), d_0(y)], r], [[d_n(x), d_1(y)], r], \ldots, [[d_n(x), d_{m-1}(y)], r]\}$ is linearly dependent over $\mathscr{C}$. There exist scalars $\beta_0, \beta_1, \ldots, \beta_{m-1}$ from $\mathscr{C}$ not all zeros, such that

$$\sum_{i=0}^{m-1} \beta_i [[d_n(x), d_i(y)], r] = 0 \text{ for all } \quad x, y, r \in \mathscr{Q}. \tag{14}$$

Let j be the highest index such that $\beta_j \neq 0$, making the sum

$$\sum_{i=0}^{j} \beta_i [[d_n(x), d_i(y)], r] = 0 \text{ for all } \quad x, y, r \in \mathscr{Q}. \tag{15}$$

Now, set the inner derivations $[[d_n(x), d_i(y)], r]$ as

$$\delta_{[d_n(x), d_i(y)]}(r) = [[d_n(x), d_i(y)], r] \text{ for all } \quad x, y, r \in \mathscr{Q}. \tag{16}$$

Therefore, the set $\{\delta_{[d_n(x),d_0(y)]}, \delta_{[d_n(x),d_1(y)]}, \ldots, \delta_{[d_n(x),d_j(y)]}\}$ satisfies a $\mathscr{C}$-linear relation over $\mathscr{Q}$ of length $j+1$. The application of Lemma 2 yields that there exist elements $q_0 = 1, q_1, q_2, \ldots, q_j \in \mathscr{Q}$ such that

$$\sum_{i=0}^{j} q_{j-i} \delta_{[d_n(x), d_i(y)]} = 0. \tag{17}$$

In addition to this, $\delta_{[d_n(x),d_0(y)]} = \delta_{q_1}$ and

$$\delta_{[d_n(x), d_k(y)]}(r) = \delta_{q_k}(r) - \sum_{i=1}^{k-1} q_i \delta_{k-i}(r) \text{ for all } r \in \mathscr{Q}, 2 \leq k \leq j. \tag{18}$$

Using $\delta_{[d_n(x),d_0(y)]} = \delta_{q_1}$, we obtain

$$[[d_n(x), y], r] = [q_1, r] \text{ for all } \quad x, y, r \in \mathscr{Q}. \tag{19}$$

Replace y by $z \in \mathscr{C}$ to obtain

$$[q_1, r] = 0 \text{ for all } \quad r \in \mathscr{Q}. \tag{20}$$

Hence, we obtain, $q_1 \in \mathscr{C}$. Next, by using (18), we have

$$[[d_n(x), d_k(y)], r] = [q_k, r] - \sum_{i=1}^{k-1} q_i [[d_n(x), d_{k-i}(y)], r], \tag{21}$$

for all $x, y, r \in \mathscr{Q}$, $2 \leq k \leq j$. Replacing y by z, where $z \in \mathscr{C}$,

$$[[d_n(x), d_k(z)], r] = [q_k, r] - \sum_{i=1}^{k-1} q_i [[d_n(x), d_{k-i}(z)], r]. \tag{22}$$

Using the fact that $d_i(z) \in \mathscr{C}$ for all $z \in \mathscr{C}$, we obtain

$$[q_k, r] = 0 \text{ for all } \quad r \in \mathscr{Q}, 2 \leq k \leq j. \tag{23}$$

Therefore, $q_k \in \mathscr{C}$ for all $2 \leq k \leq j$. Hence, (18) becomes

$$[[d_n(x), d_k(y)], r] + \sum_{i=1}^{k-1} q_i [[d_n(x), d_{k-i}(y)], r] = 0, \tag{24}$$

for all $x, y, r \in \mathscr{Q}$ and $2 \leq k \leq j$. For $k = j$, the above equation becomes

$$\delta_{[d_n(x), d_j(y)]}(r) + \sum_{i=1}^{j-1} q_i \delta_{[d_n(x), d_{j-i}(y)]}(r) = 0, \tag{25}$$

for all $r \in \mathscr{Q}$. Subtracting (25) from (17), we obtain

$$q_j \delta_{[d_n(x), d_0(y)]}(r) = 0, \tag{26}$$

that is,

$$q_j [[d_n(x), y], r] = 0 \text{ for all } \quad x, y, r \in \mathscr{Q}. \tag{27}$$

Since $q_j \in \mathscr{C}$, we have $[[d_n(x), y], r] = 0$ for all $x, y, r \in \mathscr{Q}$. So, we can conclude $[d_n(x), y] \in \mathscr{C}$ for all $x, y \in \mathscr{Q}$. Thus, we have

$$t[d_n(x), y] = [d_n(x), y]t \text{ for all } \quad x, y, t \in \mathscr{Q}, \tag{28}$$

which implies

$$td_n(x)(d_n(x)y - yd_n(x)) = d_n(x)t(d_n(x)y - yd_n(x)). \tag{29}$$

Therefore,

$$(d_n(x)y - yd_n(x))(d_n(x)t - td_n(x)) = 0, \tag{30}$$

for all $x, y, t \in \mathscr{Q}$. Replacing y by yt, we obtain

$$(d_n(x)yt - ytd_n(x))(d_n(x)t - td_n(x)) = 0, \tag{31}$$

for all $x, y, t \in \mathscr{Q}$. Since

$$\begin{aligned} d_n(x)yt - ytd_n(x) = (d_n(x)y - yd_n(x))t \\ + y(d_n(x)t - td_n(x)), \end{aligned} \tag{32}$$

we obtain

$$\begin{aligned} (d_n(x)t - td_n(x))\{(d_n(x)y - yd_n(x))t \\ + y(d_n(x)t - td_n(x))\} = 0, \end{aligned} \tag{33}$$

which implies

$$\begin{aligned} (d_n(x)t - td_n(x))(d_n(x)y - yd_n(x))t \\ + (d_n(x)t - td_n(x))y(d_n(x)t - td_n(x)) = 0, \end{aligned} \tag{34}$$

or

$$(d_n(x)t - td_n(x))y(d_n(x)t - td_n(x)) = 0, \tag{35}$$

for all $x, y, t \in \mathscr{Q}$. Since $\mathscr{Q}$ is prime, therefore

$$d_n(x)t - td_n(x) = 0 \text{ for all } \quad x, t \in \mathcal{Q}. \quad (36)$$

Thus, $d_n(x) \in \mathcal{C}$ for all $x \in \mathcal{Q}$. Since $\mathcal{R}$ forms a subring of $\mathcal{Q}$, we conclude $d_n(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Hence, $[d_n(x), x] = 0$ for all $x \in \mathcal{R}$. The application of ([12], Theorem 3.1) implies either $\mathcal{R}$ is commutative or some linear combination of (d_i)'s maps center to zero. Hence, this is the required result. □

Corollary 8. *Let $\mathcal{R}$ be a prime ring. Next, let d be a derivation on $\mathcal{R}$ such that $[d(x), d(y)] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then either $d(Z(\mathcal{R})) = 0$ or $\mathcal{R}$ is a commutative ring.*

Proof. Take rank of the higher derivation one, i.e., $s = 1$ in Theorem 7 and put $d_1 = d$. So, we have a family of $\{d_0, d_1\}$. The application of Theorem 7 gives either $\mathcal{R}$ is commutative or some linear combination of d maps center of $\mathcal{R}$ to zero, i.e., $\alpha d(z) = 0$ for all $z \in Z(\mathcal{R})$. This implies $d(z) = 0$ for all $z \in Z(\mathcal{R})$. Since $z \in Z(\mathcal{R})$ was arbitrary, we obtain $d(Z(\mathcal{R})) = 0$. □

4. Bell's Result for Higher Derivations

In the year 1998, Bell and Daif [9] while extending Herstein's results proved a very interesting idea. With a derivation d defined on a prime ring $\mathcal{R}$, he showed that if $d([x, y]) = 0$ for all x, y in a two-sided ideal of $\mathcal{R}$, then the ring $\mathcal{R}$ is commutative. It was further extended to generalized derivations by Ali et al. in [11]. Motivated by this, we plan to venture this identity in case of higher derivations.

Theorem 9. *Let $n \in \mathbb{N}$ be a fixed positive integer and $\mathcal{R}$ be a prime ring. Next, let $D = (d_i)_{i \in \mathbb{N}}$ be a nonzero higher derivation of $\mathcal{R}$, where $d_0 = id_{\mathcal{R}}$ satisfying $d_n([x, y]) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Then, either $\mathcal{R}$ is commutative or some linear combination of $(d_i)_{i \in \mathbb{N}}$ maps center of $\mathcal{R}$ to 0.*

Proof. We have

$$d_n([x, y]) \in Z(\mathcal{R}) \text{ for all } \quad x, y \in \mathcal{R}, \quad (37)$$

which implies

$$\begin{aligned} &d_n(x)y + d_{n-1}(x)d_1(y) + \cdots + xd_n(y) - d_n(y)x \\ &\quad - d_{n-1}(y)d_1(x) - \cdots - yd_n(x) \in Z(\mathcal{R}), \end{aligned} \quad (38)$$

for all $x, y \in \mathcal{R}$. On further solving, we obtain

$$\begin{aligned} &[d_n(x), y] + [d_{n-1}(x), d_1(y)] \\ &\quad + \cdots + [d_1(x), d_{n-1}(y)] + [x, d_n(y)] \in Z(\mathcal{R}), \end{aligned} \quad (39)$$

for all $x, y \in \mathcal{R}$. Now, replacing y by zy, where $z \in Z(\mathcal{R})$, we obtain

$$\begin{aligned} &[d_n(x), zy] + [d_{n-1}(x), d_1(zy)] \\ &\quad + \cdots + [d_1(x), d_{n-1}(zy)] + [x, d_n(zy)] \\ &\quad \in Z(\mathcal{R}) \text{ for all } \quad x, y \in \mathcal{R}. \end{aligned} \quad (40)$$

Since $z \in Z(\mathcal{R})$, the above equation becomes

$$\begin{aligned} &z([d_n(x), y] + [d_{n-1}(x), d_1(y)] + \cdots + [x, d_n(y)]) \\ &\quad + d_1(z)([d_{n-1}(x), y] + \cdots + [d_1(x), d_{n-2}(y)] \\ &\quad + [x, d_{n-1}(y)]) + \cdots + d_{n-1}(z)([d_1(x), y] + [x, d_1(y)]) \\ &\quad + d_n(z)[x, y] \in Z(\mathcal{R}) \text{ for all } \quad x, y \in \mathcal{R}. \end{aligned} \quad (41)$$

In view of (39), we obtain

$$\begin{aligned} &d_1(z)([d_{n-1}(x), y] + \cdots + [d_1(x), d_{n-2}(y)] \\ &\quad + [x, d_{n-1}(y)]) + \cdots + d_{n-1}(z)([d_1(x), y] \\ &\quad + [x, d_1(y)]) + d_n(z)[x, y] \in Z(\mathcal{R}), \end{aligned} \quad (42)$$

for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$. On solving this further, we obtain

$$\begin{aligned} &d_1(z)d_{n-1}([x, y]) + d_2(z)d_{n-2}([x, y]) \\ &\quad + \cdots + d_{n-1}(z)d_1([x, y]) + d_n(z)[x, y] \in Z(\mathcal{R}), \end{aligned} \quad (43)$$

for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$. For any $t \in \mathcal{R}$, we have

$$\begin{aligned} &[d_1(z)d_{n-1}([x, y]) + d_2(z)d_{n-2}([x, y]) \\ &\quad + \cdots + d_{n-1}(z)d_1([x, y]) + d_n(z)[x, y], t] = 0, \end{aligned} \quad (44)$$

for all $x, y, t \in \mathcal{R}$, $z \in Z(\mathcal{R})$. Replacing t by tr in the above equation, we obtain

$$\begin{aligned} &[d_1(z)d_{n-1}([x, y]) + d_2(z)d_{n-2}([x, y]) \\ &\quad + \cdots + d_{n-1}(z)d_1([x, y]) + d_n(z)[x, y], \mathrm{tr}] = 0, \end{aligned} \quad (45)$$

for all $x, y, t, r \in \mathcal{R}$, $z \in Z(\mathcal{R})$. This implies that

$$\begin{aligned} &d_1(z)t[d_{n-1}([x, y]), r] + d_2(z)t[d_{n-2}([x, y]), r] \\ &\quad + \cdots + d_n(z)t[[x, y], r] = 0, \end{aligned} \quad (46)$$

for all $x, y, t, r \in \mathcal{R}$, $z \in Z(\mathcal{R})$. By Fact 5, $\mathcal{R}$ and $\mathcal{Q}$, its Martindale ring of quotients, satisfy the same generalized polynomial identity and every higher derivation on $\mathcal{R}$ can be uniquely extended to a higher derivation on $\mathcal{Q}$ by Lemma 4. Therefore, we deduce that

$$\begin{aligned} &d_1(z)t[d_{n-1}([x, y]), r] + d_2(z)t[d_{n-2}([x, y]), r] \\ &\quad + \cdots + d_n(z)t[[x, y], r] = 0 \text{ for all } \quad x, y, t, r \in \mathcal{Q}, z \in \mathcal{C}. \end{aligned} \quad (47)$$

As $\mathcal{Q}$, for a prime ring $\mathcal{R}$ is prime and $1 \in \mathcal{Q}$, the application of Lemma 1 gives either the set $\{d_1(z), d_2(z), \ldots, d_n(z)\}$ is linearly dependent over $\mathcal{C}$ or the set $\{[d_0([x, y]), r], [d_1([x, y]), r], \ldots, [d_{n-1}([x, y]), r]\}$ is linearly dependent over $\mathcal{C}$. Suppose first, that the set $\{d_1(z), d_2(z), \ldots, d_n(z)\}$ is linearly dependent over $\mathcal{C}$. So, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ from $\mathcal{C}$ not all zeros, such that

$$\sum_{i=1}^{n} \alpha_i d_i(z) = 0 \text{ for all } \quad z \in \mathcal{C}. \quad (48)$$

Since $\mathcal{R}$ is a subring of $\mathcal{Q}$, therefore we can rewrite the above equation as

$$\sum_{i=1}^{n} \alpha_i d_i(z) = 0 \text{ for all } \quad z \in Z(\mathscr{R}); \tag{49}$$

thus proving our claim that some linear combination of $(d_i)_{i\in\mathbb{N}}$ maps center of $\mathscr{R}$ to 0.

Next, suppose the set $\{[d_0([x,y]),r], [d_1([x,y]),r], \ldots, [d_{n-1}([x,y]),r]\}$ is linearly dependent over $\mathscr{C}$. Again, we will have n scalars $\beta_0, \beta_1, \ldots, \beta_{n-1}$ not all zeros from $\mathscr{C}$, such that

$$\sum_{i=0}^{n-1} \beta_i [d_i([x,y]),r] = 0 \text{ for all } \quad x,y,r \in \mathscr{Q}. \tag{50}$$

Suppose j is the highest index such that $\beta_j \neq 0$. Thus, the sum becomes

$$\sum_{i=0}^{j} \beta_i [d_i([x,y]),r] = 0 \text{ for all } \quad x,y,r \in \mathscr{Q}. \tag{51}$$

Now, set the inner derivations $[d_i([x,y]),r]$ as

$$\delta_{d_i([x,y])}(r) = [d_i([x,y]),r] \text{ for all } \quad r \in \mathscr{Q}. \tag{52}$$

So, the set of derivations $\{\delta_{d_0([x,y])}, \delta_{d_1([x,y])}, \ldots, \delta_{d_j([x,y])}\}$ satisfies a $\mathscr{C}$-linear relation over $\mathscr{Q}$ of length $j+1$. In view of Lemma 2, we have the existence of $q_0 = 1, q_1, \ldots, q_j \in \mathscr{Q}$, such that

$$\sum_{i=0}^{j} q_{j-i}\delta_{d_i([x,y])} = 0 \text{ for all } \quad x,y \in \mathscr{Q}. \tag{53}$$

In addition to this, $\delta_{d_0([x,y])} = \delta_{q_1}$ and

$$\delta_{d_k([x,y])}(t) = \delta_{q_k}(t) - \sum_{i=1}^{k-1} q_i \delta_{d_{k-i}([x,y])}(t), \tag{54}$$

for all $t \in \mathscr{Q}$, $2 \le k \le j$. With $\delta_{d_0([x,y])} = \delta_{q_1}$, we obtain

$$[d_0([x,y]),t] = [q_1,t] \text{ for all } \quad x,y,t \in \mathscr{Q}, \tag{55}$$

giving us

$$[[x,y],t] = [q_1,t] \text{ for all } x,y,t \in \mathscr{Q}. \tag{56}$$

Replacing x by y, we obtain

$$[q_1,t] = 0 \text{ for all } t \in \mathscr{Q}. \tag{57}$$

Therefore, $q_1 \in \mathscr{C}$. Also,

$$\delta_{d_k([x,y])}(t) = \delta_{q_k}(t) - \sum_{i=1}^{k-1} q_i \delta_{d_{k-i}([x,y])}(t), \tag{58}$$

for all $t \in \mathscr{Q}$, $2 \le k \le j$. That is

$$[d_k([x,y]),t] = [q_k,t] - \sum_{i=1}^{k-1} q_i [d_{k-i}([x,y]),t], \tag{59}$$

for all $x,y,t \in \mathscr{Q}, 2 \le k \le j$. Replacing x by y in (59) and using the fact that $d_j(0) = 0$, for all $j \ge 1$, the above equation gives

$$[q_k,t] = 0 \text{ for all } \quad t \in \mathscr{Q}, 2 \le k \le j. \tag{60}$$

Thus, $q_k \in \mathscr{C}$ for all $1 \le k \le j$. So, for $k = j$, (59) reduces to

$$\delta_{d_j([x,y])}(t) + \sum_{i=1}^{j-1} q_i \delta_{d_{j-i}([x,y])}(t) = 0 \text{ for all } \quad t \in \mathscr{Q}. \tag{61}$$

Subtracting (61) from (53), we obtain

$$q_j \delta_{d_0([x,y])}(t) = 0 \text{ for all } \quad t \in \mathscr{Q}. \tag{62}$$

Therefore,

$$q_j[[x,y],t] = 0 \text{ for all } \quad x,y,t \in \mathscr{Q}. \tag{63}$$

Since $q_j \in \mathscr{C}$, we have

$$[[x,y],t] = 0 \text{ for all } \quad x,y,t \in \mathscr{Q}. \tag{64}$$

The application of Lemma 3 gives $t \in \mathscr{C}$. As t was chosen arbitrarily from $\mathscr{Q}$, we conclude that $\mathscr{Q}$ is commutative. $\mathscr{R}$ being a subring of $\mathscr{Q}$ must be commutative as well. This completes the proof of the theorem. □

Corollary 10. *Let $\mathscr{R}$ be a prime ring. Next, let d be a derivation on $\mathscr{R}$ such that $d([x,y]) \in Z(\mathscr{R})$ for all $x,y \in \mathscr{R}$, then either $d(Z(\mathscr{R})) = 0$ or $\mathscr{R}$ is a commutative ring.*

Proof. Take rank of the higher derivation one, i.e., $s = 1$ in Theorem 9 and put $d_1 = d$. So, we have a family of $\{d_0, d_1\}$. The application of Theorem 9 gives either $\mathscr{R}$ is commutative or some linear combination of d maps center of $\mathscr{R}$ to zero, i.e., $\alpha d(z) = 0$ for all $z \in Z(\mathscr{R})$. This implies $d(z) = 0$ for all $z \in Z(\mathscr{R})$. Since $z \in Z(\mathscr{R})$ was arbitrary, we obtain $d(Z(\mathscr{R})) = 0$. □

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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