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Research Article

Characterization of Fractional Mixed Domination Number of Paths and Cycles

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Let G' be a simple, connected, and undirected (UD) graph with the vertex set $M(G')$ and an edge set $N(G')$. In this article, we define a function $f\colon M \cup N \longrightarrow [0, 1]$ as a fractional mixed dominating function (FMXDF) if it satisfies $f(R_m[x]) = \sum_{y \in R_m[x]} f(y) \geq 1$ for all $x \in M(G') \cup N(G')$, where $R_m[x]$ indicates the closed mixed neighbourhood of x, that is the set of all $y \in M(G') \cup N(G')$ such that y is adjacent to x and y is incident with x and also x itself. Here, $p(f) = \sum_{x \in M \cup N} f(x)$ is the poundage (or weight) of f. The fractional mixed domination number (FMXDN) is denoted by $\gamma^*_{fm}(G')$ and is designated as the lowest poundage among all FMXDFs of G'. We compute the FMXDN of some common graphs such as paths, cycles, and star graphs, the middle graph of paths and cycles, and shadow graphs. Furthermore, we compute upper bounds for the sum of the two fractional dominating parameters, resulting in the inequality $\gamma'_{f_1}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T}) \leq r + p - \mathrm{rad}(\mathrm{T}) - \alpha$, where γ'_{f_1} and γ^*_{fm} are the fractional edge domination number and FMXDN, respectively. Finally, we compare γ^*_{fm} to other resolvability-related parameters such as metric and fault-tolerant metric dimensions on some families of graphs.

1. Introduction

Throughout this article, we use G' as a graph with vertex set $M(G')$ and an edge set $N(G')$ and T is referred to as a tree. The maximum degree is indicated by α, which is the degree of the vertex with the greatest incident edges. The vertex with the lowest incidence edges is said to have the smallest degree β. A vertex of degree one is known as a pendant vertex. In the case of trees, a pendant vertex is known as leaf because it has only 1 degree and it is denoted by (l_1), while its neighbour is known as a support vertex (s_1). The greatest distance between a vertex m_1 and any other node in G' is known as the eccentricity of m_1 (see [1]). The smallest eccentricity is represented by the radius rad (G') of G'. For any $m_1 \in M(G')$, the open neighbourhood of m_1 is represented as the set of all vertices adjacent to $m_1 \in M(G')$, and it is denoted by $R(m_1)$, and the closed neighbourhood of m_1 is denoted by $R[m_1] = R(m_1) \cup \{m_1\}$ (see [2]). Similarly, for any edge $n_1 \in N(G')$, the open and closed neighbourhoods of n_1 are defined as $R(n_1) = \{n \in N$ such that n is adjacent to $n_1\}$ and $R[n_1] = R(n_1) \cup \{n_1\}$, respectively. The set of all edges incident to m as well as the vertices adjacent to m is the open mixed neighbourhood of a vertex m. The collection of incident vertices m_1 and m_2 as well as all the edges adjacent to $n = m_1m_2$ is described as the open mixed neighbourhood of an edge n.

Throughout this paper, we choose an element $x \in M(G') \cup N(G')$, and the open mixed neighbourhood of x is indicated by $R_m(x)$, which is the set of all $y \in M(G') \cup N(G')$ such that y is adjacent to x as well as y is incident with x, and the closed mixed neighbourhood of x is denoted by $R_m[x] = R_m(x) \cup \{x\}$. Recall that a set $B \subseteq M$ is called a dominating set of G' if every vertex of $M - B$ is adjacent to at least one vertex in B. The minimum cardinality of all possible dominating sets in G' is given by the domination number $\gamma(G')$. Haynes et al. [3] presented the concept of multiple types of graph domination and also gave some information on fractional dominating function and mixed domination number. A set $R \subseteq M \cup N$ is said to be a mixed dominating (MXD) set of G' if every element not in R has at least one mixed element in R. The mixed domination number $\gamma_m(G')$ of G' is the minimum cardinality of all possible MXD sets of G'. In 1992, Sampathkumar and Kamath [4] worked on the concept of mixed domination number, in which vertices dominate edges in certain ways and vice versa.

Define a dominating function $f_0: M(G') \longrightarrow [0, 1]$ which is known as a fractional dominating function (FDF) if $f_0(R[m]) = \sum_{m_1 \in R[m]} f_0(m_1) \geq 1$ for every $m \in M(G')$. The fractional domination number $\gamma_{f_0}(G')$ is the minimum poundage (weight) among all fractional dominating functions of G', where the poundage of f_0 is $p(f_0) = \sum_{m \in M(G')} f_0(m)$. Scheinerman and Ullman worked on the concept of fractional domination number for the minimum weight among all FDFs of G' in [5]. Similarly, we define a dominating function $f_1: N(G') \longrightarrow [0, 1]$ which is said to be fractional edge dominating function if $f_1(R[n]) = \sum_{n_1 \in R[n]} f_1(n_1) \geq 1$ for every $n \in N(G')$. The fractional edge domination number $\gamma'_{f_1}(G')$ is the minimum poundage of all fractional edge dominating functions of G', where the poundage of f_1 is equal to $\sum_{n \in N(G')} f_1(n)$ (see [6]). We define a fractional mixed dominating function (FMXDF) is a function $f: M \cup N \longrightarrow [0, 1]$ if $f(R_m[x]) = \sum_{y \in R_m[x]} f(y) \geq 1$ for all $x \in M(G') \cup N(G')$. The poundage of f is given by $p(f) = \sum_{x \in M \cup N} f(x)$. The fractional mixed domination number is used by the symbol of $\gamma^*_{fm}(G')$, which represents the function with the least poundage among all FMXDFs of G'. For convenience, we take f_0, f_1, and f as the fractional dominating function, fractional edge dominating function, and FMXDF of G'. For every FMXDFs, we write every closed mixed neighbourhood of an edge of G' as $R_m[m_1 m_2] = f(m_1) + S_{m_1} + f(m_2) + S_{m_2} - f(m_1 m_2)$ for all $m_1 m_2 \in N(G')$, where star of m_1 and m_2 is the set of all incident edges of the vertex m_1 and m_2, and it is denoted by S_{m_1} and S_{m_2}, and we write every closed mixed neighbourhood of a vertex of G' as $R_m[m] = R[m] + S_m$ for every $m \in M(G')$.

In [7–13], the authors studied the resolvability-related parameters like the metric dimension and fault-tolerant metric dimension of certain families of graphs. The classification of total dominating and fractional dominating parameters is by removing a vertex from graph G, and its related concepts are provided in [14, 15]. The concept of reinforcement number with respect to half domination number is given in [16], and Sridharan et al. [17] introduced the parameter $\gamma_\lambda(G)$, where $0 \leq \lambda \leq 1$, with the help of $\{0, (1/2), 1\}$.

In recent years, various fractional dominating parameters that define the function of the domain set as either the vertex set or an edge set independently have been researched. Here, we combined the concept of a FDF and MXD set to create a function with the domain set as both a vertex and an edge set $(V \cup E)$ for the minimum weight with the condition of the summation of the closed mixed neighbourhood of any element $x \in V \cup E$, which is at least one. In this paper, we introduce the concept of fractional mixed domination number and develop some of the results derived from this parameter. The fractional mixed domination number is greater than or equal to the fractional domination number of graph, which is the significance of this article.

In Section 2, we enumerate the FMXDN of some standard graphs, such as paths, cycles, star graphs, the middle graph of the path graphs, and the shadow graphs. In Section 3, provides the upper bound on the sum of the two parameters where fractional edge domination and FMXDN, whose resultant graph gives $\gamma'_{f_1}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T}) \leq r + p - \mathrm{rad}\ (\mathrm{T}) - \alpha$.

2. Some Standard Graphs

This section discusses the FMXDN of some definitive graphs such as paths, cycles, star graphs, the middle graphs of paths and cycles, and shadow graphs.

Theorem 1. *If P_r is a nontrivial path with $r \geq 2$, then $\gamma^*_{fm}(P_r) = \lfloor 2r + 3/5 \rfloor$.*

Proof. Let G' be a path with $r \geq 2$ vertices, where $m_1, m_2, \ldots, m_r$ represent vertices of P_r and $n_1, n_2, \ldots, n_{r-1}$ represent edges of P_r. Here, $|M(P_r)| = r$, and $|N(P_r)| = r - 1$. Let x_i, $(1 \leq i \leq 2r - 1)$, represent elements of P_r that are both either vertices or edges. For P_2, obviously $\gamma^*_{fm}(P_2) = 1 = \lfloor 2r + 3/5 \rfloor$. For P_4.

In Figure 1, first we assign the value 1 to $f(m_2)$ and $f(n_3)$ and 0 to the rest of the elements. In another way, we assign s/t to $f(n_1)$ and $f(m_4)$ and $(1 - s/t)$ to $f(m_2)$ and $f(n_3)$ and 0 to the remaining elements, where $s < t$, $s = 1, 2, \ldots t - 1$, and $t \geq 2$. For example, we assign the fractional value $f(n_1)$, $f(m_4) = (1/2)$ and $f(n_3)$, $f(m_2) = (1 - (1/2))$, and the remaining elements are assigned the value of zero. Next, we set the fractional value $f(n_1)$, $f(m_4) = (1/3)$ or $(2/3)$ and $f(n_3)$, $f(m_2) = (1 - (1/3))$ or $(1 - (2/3))$ and the remaining elements are assigned the value of zero. Again, we set the fractional values $f(n_1)$, $f(m_4) = (1/4)$ or $(2/4)$ or $(3/4)$ and $f(n_3)$, $f(m_2) = (1 - (1/4))$ or $(1 - (2/4))$ or $(1 - (3/4))$ and the remaining elements are assigned the value of zero. We assign the values in this way. The only condition is when to give (s/t) to $f(n_1)$ and $f(m_4)$, and other $f(m_2)$ and $f(n_3)$ must have $(1 - (s/t))$. For the possibilities listed above, we have $\gamma^*_{fm}(P_4) = 2 = \lfloor 2r + 3/5 \rfloor$.

Case (i): For $r = 5k + 3$, where $k \in \mathbb{Z}^+$ and $\mathbb{Z}^+$ is the positive integer. Let x be any element that is either a vertex or an edge. In this case, we define a function $f\colon M \cup N \longrightarrow [0, 1]$ by $f(x) = \begin{cases} 1 & \text{if } m_{5q+2} \text{ and } n_{5q+4} \\ 0 & \text{otherwise} \end{cases}$.

For an integer q, $(0 \le q < \lceil r/2 \rceil - \lfloor r - 2/4 \rfloor)$. Thus, the poundage of f is $p(f) = \gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = \lfloor 2r + 3/5 \rfloor$.

Case (ii): for $r = 5k, 5k + 1$, where $k \in \mathbb{N}$, where $\mathbb{N}$ as a natural number.

Subcase (i): when $r = 5k$.

Similarly, we assign the value 1 to $f(m_{5q+2})$ and $f(n_{5q+4})$ and we give 0 to the remaining elements. This means that $p(f) = \gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = \lfloor 2r + 3/5 \rfloor$ is the poundage of f.

Another way: Alternately, to provide the function values for P_5, we assign (s/t) to $f(n_1)$ and $f(m_4)$ and $(1 - (s/t))$ to $f(m_2)$ and $f(n_4)$ and 0 to the remaining elements, where $s < t, s = 1, 2, \ldots, t - 1$, and $t \ge 2$. In this way, we proceed with the paths $P_{10}, P_{15}, \ldots$ to get the poundage of P_r, where $r = 5k$. Generally, we assign the values of the paths $P_{10}, P_{15}, \ldots$ as (s/t) to the elements m_{5q+2} and n_{5q+4}, and assign $1 - (s/t)$ to the elements m_{5q+4} and n_{5q+1}, and 0 to the remaining elements, where $q \in \mathbb{Z}$, and $(0 \le q < \lceil r/2 \rceil - \lfloor r - 2/4 \rfloor)$. This implies that poundage of f is $p(f) = \gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = \lfloor 2r + 3/5 \rfloor$.

Subcase (ii): when $r = 5k + 1$.

We assign the value 1 to $f(m_{5q+2})$, $f(n_{5q+4})$, and $f(m_r)$ and we give 0 to the remaining elements. Another way: Alternately, to provide the function values for P_6, we assign $f(n_1)$, $f(m_4) = (s/t)$ and $f(n_4), f(m_2) = (1 - (s/t))$, $f(m_6) = 1$ and the remaining elements are assigned a value of zero. In this way, we proceed with the paths $P_{11}, P_{16}, \ldots$ to get the poundage of P_r, where $r = 5k + 1$. In general, we assign the values of the paths $P_{11}, P_{16}, \ldots$ as (s/t) to the elements m_{5q+2} and n_{5q+4}, and assign $1 - (s/t)$ to the elements m_{5q+4} and n_{5q+1}, $f(m_r) = 1$, and 0 to the remaining elements. This implies that poundage of f is $p(f) = \gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = \lfloor 2r + 3/5 \rfloor$.

Case (iii): for $r = 5k + 2, 5k + 4$, where $k \in \mathbb{N}$.

Subcase (i): In the similar process of the above cases, for $r = 5k + 2$, we assign 1 to m_{5q+2} and n_{5q+4}, and 0 to the rest of the elements. For $r = 5k + 4$, we assign 1 to m_{5q+2}, n_{5q+4}, and n_{r-1} and 0 to the rest of the elements. Thus, the poundage of f is $p(f) = \gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = \lfloor 2r + 3/5 \rfloor$.

Another way: For $r = 5k + 2$, the values (s/t) are assigned to m_{5q+2} and n_{5q+4}, and $1 - (s/t)$ is assigned to m_{5q+4} and n_{5q+1}, and 0 is assigned to the remaining elements. For $r = 5k + 4$, the values (s/t) are assigned to m_{5q+2}, n_{5q+4}, and n_{r-1} and $1 - (s/t)$ is assigned to m_{5q+4} and n_{5q+1}, and 0 is assigned to the remaining elements, where $s < t, s = 1, 2, \ldots, t - 1$, and $t \ge 2$, and $(0 \le q < \lceil r/2 \rceil - \lfloor r - 2/4 \rfloor)$. This implies that the poundage of f is $p(f) = \gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = \lfloor 2r + 3/5 \rfloor$. □

Theorem 2. *For any cycle C_r with $r \ge 3$, we have $\gamma^*_{fm}(C_r) = (2r/5)$.*

Proof. Let $G' = C_r$ be a cycle graph with $r \ge 3$ vertices. Let $m_1, m_2, \ldots, m_r$ be the vertices of C_r and $n_1, n_2, \ldots, n_r$ be the edges of C_r. Here, $|M(C_r)| = r = |N(C_r)|$. Let x be an element of C_r, which is either a vertex or edge. We define a function $f\colon M(C_r) \cup N(C_r) \longrightarrow [0.1]$ by $f(x_i) = (1/5)$ for all i, where $x \in M \cup N$. In this graph, we have $f(R[x_i]) = 1$ for every i, and its poundage is the minimum among all the FMXDFs of C_r. Thus, the poundage of f is $p(f) = \gamma^*_{fm}(C_r) = \sum_{x\in M\cup N} f(x) = (2r/5)$. □

Theorem 3. *For $r \ge 2$, $\gamma^*_{fm}\ (K_{1,r}) = 1$.*

Proof. Let G' be a star graph and $m, m_i\ (1 \le i \le \mathrm{r})$ be vertices of $G' = K_{1,r}$, where m is the central vertex of $K_{1,r}$. Now, we define a function $f_1\colon M \cup N \longrightarrow [0, 1]$ by $f_1(m) = 1$, $f_1(m_i) = 0$ and $f_1(mm_i) = 0$. This implies that the poundage of f_1 is $p(f_1) = \sum_{x\in M\cup N} f_1(x) = 1$. Suppose if we define another function $f_2\colon M \cup N \longrightarrow [0, 1]$ by $f_2(m) = 0, f_2(mm_i) = 0$ and $f_2(m_i) = 1$ for all i; then, obviously $p(f_2) = f_2(m) + f_2(m_i) + f_2(mm_i) > 1 = p(f_1)$. As a result, $p(f_1)$ is the smallest when compared to (f_2). Hence, $\gamma^*_{fm}\ (K_{1,r}) = 1$. □

Theorem 4. *For any complete graph K_r with $r \ge 2$, then $\gamma^*_{fm}(K_r) = (r + s/r + t)$, where s and r are size and order, respectively, and $t \ge 1$.*

Proof. Let $G' = K_r$ be a complete graph with $r \ge 2$. Consider that r and s represent the order and size of G', and $t = 2, 3, 4, \ldots$ Here, we define a function $f\colon M \cup N \longrightarrow [0, 1]$ by $f(x_i) = (1/|R_m[x]|)$ for all i, where $\mathrm{x} \in M \cup N$. Then, the poundage of f is $\gamma^*_{fm}\ (G') = \sum_{x\in M\cup N} f(x) = (r + s/r + t)$. Thus, $\gamma^*_{fm}\ (G') = (r + s/r + t)$. □

Observation 5 (see [6]). For any graph G, we have $\gamma'_f(G) = \gamma(L(G))$. Hence, it follows that $\gamma'_f(C_n) = \gamma_f(C_n) = n/3$ and $\gamma'_f(P_n) = \gamma_f(P_{n-1}) = \lceil n - 1/3 \rceil$.

Corollary 6. *If $\gamma_{f_0}(P_r) = \lceil r/3 \rceil$ and $\gamma^*_{fm}(P_r) = \lfloor 2r + 3/5 \rfloor$ for any path graph P_r, then $\gamma_{f_0}(P_r) \le \gamma^*_{fm}(P_r)$.*

Proof. From Observation 5, we have $\gamma_f(P_{n-1}) = \lceil n - 1/3 \rceil$ and in this paper, for our convenience, we take the notation of FDF as γ_{f_0}. Then, $\gamma_{f_0}(P_r) = \lceil r/3 \rceil$. By Theorem 1, we have $\gamma^*_{fm}(P_r) = \lfloor 2r + 3/5 \rfloor$. Therefore, we get $\gamma_{f_0}(P_r) \le \gamma^*_{fm}(P_r)$. □

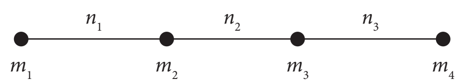

Figure 1: Path P_4.

Corollary 7. *For any cycle C_r with $r \geq 3$, we have $\gamma_{f_0}(C_r) = r/3$, which is always less than $(2r/5) = \gamma^*_{fm}(C_r)$.*

Proof. From Observation 5, we have $\gamma_{f_0}(C_r) = r/3$ and by Theorem 2, we have $\gamma^*_{fm}(C_r) = 2r/5$. Thus, $\gamma_{f_0}(C_r) = r/3 < (2r/5) = \gamma^*_{fm}(C_r)$. □

Definition 8 (see [18]). The shadow graph of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex of u' of G' to the neighbours of the corresponding vertex u'' of G''. The shadow graph of G is denoted by $D_2(G)$.

Definition 9 (see [18]). The middle graph of a connected graph G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent i*f*

(i) They are adjacent edges of G

(ii) One is a vertex of G and the other is an edge incident with it

Theorem 10. *For $r \geq 3$, the shadow graph of P_r is $\gamma^*_{fm}(D_2(P_r)) = r - 1$.*

Proof. Consider two copies of P_r and assume that $G' = D_2(P_r)$ is a shadow graph with $r \geq 3$. Let $m_1, m_2, \ldots, m_r$ represent the vertices of the first copy of P_r and $m_1', 'm_2, \ldots, m_r'$ represent the vertices of a second copy of P_r. Let $n_1, n_2, \ldots, n_{r-1}$ be the edges of the first copy of P_r and $n_1'', n_2, \ldots, n_{r-1}'$ be the edges of a second copy of P_r, with n_i connecting m_i' and m_{i+1}' and n_i' connecting m_i and m_{i+1}. Here, $|M(G')| = 2r$ and $|N(G')| = 4(r-1)$. Assume $f(q) = 1/5$ for all $q \in M \cup N$ when $r = 2$. Then, the poundage of f is $p(f) = 8/5$. For $r \geq 3$, the function $f\colon M \cup N \longrightarrow [0, 1]$ is defined by the following types of possibilities:

(i) If $r = 3k$, where $k \in \mathbb{N}$, then we assign a value of 1 to $f(n_{3s+1})$, $f(m'_{3s+2})$, and $f(n'_{3s+3})$, and a value of 0 to the remaining elements of G'.

(ii) If $r = 3k + 1$, where $k \in \mathbb{N}$, then we assign a value of 1 to $f(n_{3s+2})$, $f(m'_{3s+2})$, and $f(m'_{3s+3})$, and a value of 0 to the remaining elements of G'.

(iii) If $r = 3k + 2$, where $k \in \mathbb{N}$, then we assign a value of 1 to $f(n_{3s+1})$, $f(m'_{3s+2})$, $f(n'_{3s+3})$, and $f(m_{r-1})$, and a value of 0 to the remaining elements of G', where $s \in \mathbb{Z}$, $(0 \leq s \leq \lceil r - 3/2 \rceil)$. Then, the poundage of f is $\gamma^*_{fm}(G') = \sum_{x \in M \cup N} f(x) = r - 1$. Hence, $\gamma^*_{fm}(D_2(P_r)) = r - 1$. □

Theorem 11. *The middle graph of path P_r for $r \geq 2$ is $\gamma_{f_0}(M'(P_r)) = \lceil r/2 \rceil$.*

Proof. Let $m_1, m_2, \ldots, m_r$ be the vertices of the path and $n_1, n_2, \ldots, n_{r-1}$ be the edges of P_r.

In this proof, we build the middle graph of path P_r, that is, $G' = M'(P_r)$, with the vertex set $M(G') = \{q_s = (m_i \cup n_j)\colon (1 \leq s \leq 2r - 1)\}$ and an edge set $N(G') = \{m_1 n_1, n_1 m_2, m_2 n_2, n_2 m_3, \ldots, n_{r-1} m_r\} \cup \{n_1 n_2, n_2 n_3, \ldots, n_{r-2} n_{r-1}\}$. Here, $r = |m_i|$. In Figure 2, for $M'(P_4)$, we set $f_0(n_1) = 1$ and $f_0(n_3) = 1$, and all other remaining vertices are set to 0. In $M'(P_5)$ has $f_0(n_1) = 1$, $f_0(n_3) = 1$ and $f_0(m_5) = 1$, with the remaining vertices all being 0. Now, we define a function $f_0\colon M(M'(P_r)) \longrightarrow [0, 1]$ by

$$f_0(q_s) = \begin{cases} 1, & \text{for } n_{3t+1} \text{ or } n_{3t+1} \text{ and } m_r, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$, $(0 \leq t < \lceil r/2 \rceil - 1)$. Then, the poundage of f_0 is $p(f_0) = \sum_{m \in M(G')} f_0(m) = \lceil r/2 \rceil$. Hence, $\gamma_{f_0}(M'(P_r)) = \lceil r/2 \rceil$. □

Theorem 12. *For $r \geq 3$, $\gamma_{f_0}(M'(C_r)) = \lceil r/2 \rceil$.*

Proof. Consider the graph $G' = M'(C_r)$ with $r \geq 3$. A fractional dominating function f_0 is defined as $f_0\colon M(G') \longrightarrow [0, 1]$ by $f_0(m_i) = 1/5$ for all i, where $1 \leq i \leq 2$r. Thus, the poundage of f_0 is $p(f_0) = \sum_{m \in M(G')} f_0(m) = \lceil r/2 \rceil$. Hence, $\gamma_{f_0}(M'(C_r)) = \lceil r/2 \rceil$. □

Corollary 13. *For any graph G', the following holds true:*

(i) $\gamma_{f_0}(G') \leq \gamma^*_{fm}(G')$.

(ii) $\gamma'_{f_1}(G') \leq \gamma^*_{fm}(G')$, where f_0 and f_1 are fractional dominating and fractional edge dominating functions of G'.

Proof. Assume G' as connected graph and f as FMXDF of G'. The fractional domination number $\gamma_{f_0}(G') = \sum_{m \in M} f_0(m) \leq \sum_{x \in M \cup N} f(x) = \gamma^*_{fm}(G')$.

Similarly, the fractional edge domination number $\gamma'_{f_1}(G') = \sum_{n \in N} f_1'(n) \leq \sum_{x \in M \cup N} f(x) = \gamma^*_{fm}(G')$. □

Corollary 14. *For any path graph, $\gamma_{f_0}(M'(P_r)) \leq \gamma^*_{fm}(M'(P_r))$.*

Proof. From Theorem 11 and Corollary 13, we get $\gamma_{f_0}(M'(P_r)) \leq \gamma^*_{fm}(M'(P_r))$. □

Corollary 15. *For $r \geq 3$, we have $\gamma_{f_0}(M'(C_r)) < \gamma^*_{fm}(M'(C_r))$.*

Proof. From Theorem 12 and Corollary 13, we get $\gamma_{f_0}(M'(C_r)) < \gamma^*_{fm}(M'(C_r))$. □

3. Bounds

In this section, we provide some bounds on gamma $\gamma^*_{fm}(G')$ in terms of fractional domination, fractional edge domination, maximum degree α, minimum degree β, support s', and leaves l' and compare γ^*_{fm} to other resolvability-related parameters.

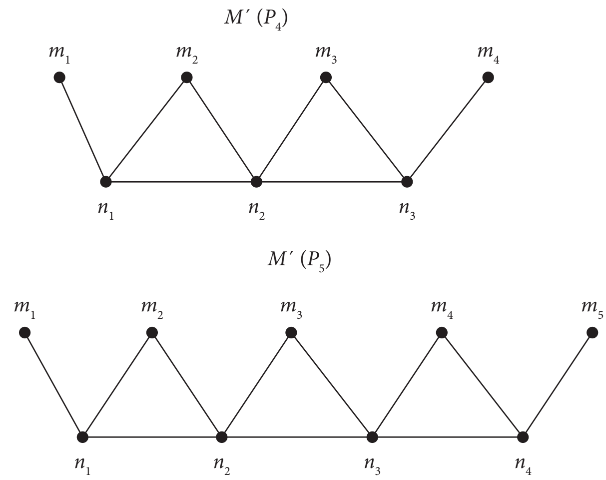

Figure 2: Middle graph of P_4 and P_5.

Theorem 16. *For each tree, we have* $3\left\{\gamma_{f_0}(\mathrm{T}) + \gamma^{*}_{fm}(\mathrm{T})\right\} + 2 \leq r + 2(l' - s') + 4 + 2p$ *and this bound is sharp.*

Proof. Let T be a tree with the number of vertices r, the number of edges p, the number of leaves l', and the number of supporting vertices s'. For our convenience, we take the notation as $|M(G')| = r$ and $r - 1 = p = |N(G')|$. In this theorem, f is the least FMXDF of G'. A set of support vertices and leaves in a tree T is represented by $s'(\mathrm{T})$ and $l'(\mathrm{T})$. Here, α (β) is the maximum (minimum) degree of the vertex. The vertex set can be divided into the following sets: $Ms_1 = \left\{m \in M\colon f(m) = 1, m \in s'(\mathrm{T}), \text{and } \deg(m) \leq \alpha(\mathrm{T})\right\}$, $Ms_0 = \left\{m \in M\colon f(m) = 0, m \in s'(\mathrm{T}), \text{and } \deg(m) \geq \beta(\mathrm{T})\right\}$, $Ml_1 = \left\{m \in M\colon f(m) = 1, \text{and } m \in l'(\mathrm{T})\right\}$, $Ml_0 = \left\{m \in M\colon f(m) = 0, \text{and } m \in l'(\mathrm{T})\right\}$, $Mt_1 = \{m \in M\colon f(m) = 1, \text{and } m \text{ is neither leaf nor support vertex}\}$, and $Mt_0 = \{m \in M\colon f(m) = 0, \text{and } m \text{ is neither leaf nor support vertex}\}$. Also, the edge set can be partitioned into two sets, where $N_0 = \{n \in N\colon f(n) = 0\}$ and $N_1 = \{n \in N\colon f(n) = 1\}$. Then, the poundage of f is

$$\begin{aligned}
\gamma^{*}_{fm}(\mathrm{T}) &= \sum_{x \in M \cup N} f(x) \\
&= \sum_{m \in M} f(m) + \sum_{n \in N} f(n) \\
&= \sum_{m \in Ml_1 \cup Ml_0} f(m) + \sum_{m \in Ms_1 \cup Ms_0} f(m) + \sum_{m \in Mt_1 \cup Mt_0} f(m) + \sum_{N_1 \in \cup N_0} f(n) \\
&< \left\lceil \frac{\left|M\left(G'\right)\right|}{2} \right\rceil + \left|N_1\right| = \left\lceil \frac{r}{2} \right\rceil + \left|N_1\right|.
\end{aligned} \tag{2}$$

By Corollary 13, we have $\gamma_{f_0}(G') \leq \gamma^{*}_{fm}(G')$. Therefore, we have two cases here.

Case (i): if $\gamma_{f_0} < \gamma^*_{fm}$, then

$$\begin{aligned}
\gamma_{f_0} + \gamma^*_{fm} &= 2\left[\sum_{m\in M} f(m)\right] + \sum_{n\in N} f(n) \\
&< 2\left(\left\lfloor \frac{r}{2} \right\rfloor\right) + |N_1| \\
&< 2\left(\left\lfloor \frac{r}{2} \right\rfloor\right) + \left(l' - s'\right) \\
&\leq \frac{3r + 2(l' - s')}{3} \\
&= \frac{r + 2(r - 1 + 1) + 2(l' - s')}{3} \\
&= \frac{r + 2(p + 1) + 2(l' - s')}{3}.
\end{aligned} \tag{3}$$

Case (ii): if $\gamma_{f_0} = \gamma^*_{fm}$, then

$$\begin{aligned}
\gamma_{f_0} + \gamma^*_{fm} &= 2\left[\sum_{m\in M} f(m)\right] + \sum_{n\in N} f(n) \\
&< 2\left(\left\lfloor \frac{r}{2} \right\rfloor\right) + 0 \\
&< 2\left(\left\lfloor \frac{r}{2} \right\rfloor\right) + (l' - s') \\
&\leq \frac{3r + 2(l' - s')}{3} \\
&= \frac{r + 2(r - 1 + 1) + 2(l' - s')}{3} \\
&= \frac{r + 2(p + 1) + 2\left(l' - s'\right)}{3}.
\end{aligned} \tag{4}$$

From (3) and (4), we get $\gamma_{f_0}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T}) \leq (r + 2(p + 1) + 2(l' - s')/3)$. It follows that

$$\begin{aligned}
&3\left\{\gamma_{f_0}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T})\right\} \leq r + 2p + 2 + 2(l' - s') + 2 - 2 \\
&\Longrightarrow 3\left\{\gamma_{f_0}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T})\right\} + 2 \leq r + 2p + 4 + 2(l' - s').
\end{aligned} \tag{5}$$

For P_2 and P_4, Theorem 16 gives the exact bound value. This completes the proof. □

Theorem 17. *For every tree with $\alpha(\mathrm{T}) \geq 2$, then $\gamma'_{f_1}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T}) \leq r + p - rad(\mathrm{T}) - \alpha$ and the bound is sharp.*

Proof. Let T be a graph with no cycles, with the vertex as M, the edge as N, the maximum degree as α, and the radius as rad(T). Take f to be the minimal FMXDF of T. The degree of an edge is described as $d(n) = d(m_1) + d(m_2) - 2$. The vertex set can be divided into the following sets: $M_{\alpha 1} = \{m \in M\colon f(m) = 1 \text{ and } m \in \alpha(T)\}$, $M_{\alpha 0} = \{m \in M\colon f(m) = 0 \text{ and } m \in \alpha(T)\}$, $M_{\beta 1} = \{m \in M\colon f(m) = 1 \text{ and } m \in \beta(T)\}$, $M_{\beta 0} = \{m \in M\colon f(m) = 0 \text{ and } m \in \beta(T)\}$, $M_{q0} = \{m \in M\colon f(m) = 0 \text{ and } m \notin \beta(d(m)) \cap \alpha(d(m))\}$, and $M_{q1} = \{m \in M\colon f(m) = 1 \text{ and } m \notin \beta(d(m)) \cap \alpha(d(m))\}$. The edge set can be divided as $N_{\alpha_1} = \{n \in N : f(n) = 1 \text{ and } n \in \alpha\}$, $N_{\alpha_0} = \{n \in N\colon f(n) = 0 \text{ and } n \in \alpha\}$, $N_{\beta 1} = \{n \in N\colon f(n) = 1 \text{ and } n \in \beta\}$, $N_{\beta 0} = \{n \in N\colon f(n) = 0 \text{ and } n \in \beta\}$, $N_{q0} = \{n \in N\colon f(n) = 0 \text{ and } n \notin \beta(d(n)) \cap \alpha(d(n))\}$, and $N_{q1} = \{n \in N\colon f(n) = 1 \text{ and } n \notin \beta(d(n)) \cap \alpha(d(n))\}$. Then, the poundage of f is

$$\begin{aligned}
\gamma^*_{fm}(\mathrm{T}) &= \sum_{x\in M\cup N} f(x) = \sum_{m\in M} f(m) + \sum_{n\in N} f(n) \\
&= \sum_{m\in M\alpha_1\cup M\alpha_0} f(m) + \sum_{m\in M\beta_1\cup M\beta_0} f(m) + \sum_{m\in Mq_1\cup Mq_0} f(m) + \sum_{n\in N\alpha_1\cup N\alpha_0} f(n) + \sum_{n\in N\beta_1\cup N\beta_0} f(n) + \sum_{n\in Nq_1\cup Nq_0} f(n) \\
&= \left|M\beta_1 \cup M\alpha_1 \cup Mq_1 \cup N\beta_1 \cup N\alpha_1 \cup Nq_1\right| \\
&< \left|M\beta_0 \cup M\beta_1 \cup M\alpha_1 \cup M\alpha_0 \cup Mq_1 \cup Mq_0\right| + \left|N\beta_0 \cup N\beta_1 \cup N\alpha_1 \cup N\alpha_0 \cup Nq_1 \cup Nq_0\right| \\
&= r + p - \mathrm{rad}(\mathrm{T}) - \alpha(\mathrm{T}).
\end{aligned} \tag{6}$$

By Corollary 13, we have $\gamma'_{f_1}(G') \leq \gamma^*_{fm}(G')$. Therefore, we have two cases here.

Case (i): if $\gamma'_{f_1} < \gamma^*_{fm}$, then

$$\begin{aligned}\gamma'_{f_1} + \gamma^*_{fm} &= 2\left[\sum_{n\in N} f(n)\right] + \sum_{m\in M} f(m)\\ &= \left|M\beta_1 \cup M\alpha_1 \cup Mq_1 \cup N\beta_1 \cup N\alpha_1 \cup Nq_1\right|\\ &\le r + p - \mathrm{rad}(\mathrm{T}) - \alpha(\mathrm{T}).\end{aligned} \tag{7}$$

Case (ii): if $\gamma'_{f_1} = \gamma^*_{fm}$, then

$$\begin{aligned}\gamma'_{f_1} + \gamma^*_{fm} &= 2\left[\sum_{n\in N} f(n)\right] + \sum_{m\in M} f(m)\\ &= \left|M\beta_1 \cup M\alpha_1 \cup Mq_1 \cup N\beta_1 \cup N\alpha_1 \cup Nq_1\right|\\ &\le r + p - \mathrm{rad}(\mathrm{T}) - \alpha(\mathrm{T}).\end{aligned} \tag{8}$$

From Case (i) and Case (ii), we get $\gamma'_{f_1}(\mathrm{T}) + \gamma^*_{fm}(\mathrm{T}) \le r + p - \mathrm{rad}(\mathrm{T}) - \alpha$. For P_3 and P_4, Theorem 17 gives the exact bound value. □

Definition 18 (see [10]). The length of the shortest path between a pair $m_1, m_2 \in M(G')$ is said to be the distance $d(m_1, m_2)$ between them. Let $C = \{m_s\}$ where $s = 1, 2, \ldots, l$ and $C \subseteq M(G')$ and let $y \in M(G')$. The distance representation $rs_C\,(y)$ is the vector $(d(y, m_1), d(y, m_2), \ldots, d(y, m_l))$ of distances from y to $m_s\,(1 \le s \le l)$. Such a set is called a resolving set in G'. A resolving set of minimum cardinality is said to be a metric dimension of G', and it is denoted by $\dim(G')$ *or* $\beta(G')$. If $C' = C \setminus m$ is also a resolving set for any $m \in C$, then C' is called a fault-tolerant resolving set of G'. The smallest cardinality of the fault-tolerant resolving set is said to be the fault-tolerant metric dimension, and it is denoted as $\dim_f\,(G')$ *or* $\beta'(G')$. For our convenience, $\dim(G')(\dim_f\,(G'))$ is the metric dimension (fault-tolerant metric dimension) instead of $\beta(G')(\beta'(G'))$, respectively.

Definition 19 (see [7]). The graph honeycomb rectangular torus, denoted by $\mathrm{HRT}_{r,c}$, is a graph constructed with $r = 4$ where r is the number of rows and $c \ge 4\,(c = \text{even})$ is the number of columns. The order and size of this graph are $4 \times c$ and $4 \times (c + \wp + 1)$, respectively, where $\wp = 1, 2, 3, \ldots, c$. The graph $\mathrm{HRT}_{r,c}$ is 3-regular graph. The vertex and edge sets are given, respectively, as $V(\mathrm{HRT}_{r,c}) = \{a_{i,\wp};\ i = 1, 2, \ldots, 4, \wp = 1, 2, 3, \ldots, c\}$, $E(\mathrm{HRT}_{r,c}) = \{a_{i,\wp}a_{i,\wp+1};\ i = 1, 2, \ldots, 4, \wp = 1, 2, 3, \ldots, c - 1\} \cup \{a_{i,1}a_{i,\wp};\ i = 1, 2, \ldots, 4, \wp = c\} \cup \{a_{1,\wp}\ a_{4,\wp};\ \wp = 1, 2, 3, \ldots, c - 1\} \cup \{a_{i,\wp}a_{i+1,\wp};\ i = 1,3;\ \wp = 2, 4, 6\ldots, c\} \cup \{a_{2,\wp}\ a_{3,\wp};\ \wp = 1, 2, 3, \ldots, c - 1\}$.

Theorem 20 (see [7]). *Let $HRT_{4,c}$ be a rectangular honeycomb structure with $c \ge 4\,(c = even)$. Then,* $\dim(HRT_{4,c}) = 4$.

Theorem 21 (see [7]). *Let $HRT_{4,c}$ be a rectangular honeycomb structure with $c \ge 4\,(c = even)$. Then,* $\dim_f\,(HRT_{4,c}) = 8$.

Theorem 22. *If $HRT_{4,c}$ is a rectangular honeycomb structure with $c \ge 4\,(c = even)$, then $\gamma^*_{fm}(HRT_{4,c}) = ([4 \times c] + [4 \times (c + \wp + 1)]/7)$.*

Proof. Let $\mathrm{HRT}_{4,c}$ be a honeycomb rectangular torus with $c \ge 4\,(c \text{ is even})$ and also assume f as a minimal FMXDF of $\mathrm{HRT}_{4,c}$. In this theorem, we put the function value 1/7 for each and every element $(x \in M \cup N)$ of $\mathrm{HRT}_{4,c}$. Therefore, the poundage of f is $p(f) = \gamma^*_{fm}(\mathrm{HRT}_{4,c}) = \sum_{x\in M\cup N} f(x) = (|M| + |N|/7) = ([4 \times c] + [4 \times (c + \wp + 1)]/7)$. □

Theorem 23 (see [9]). *A connected graph has metric dimension 1 if and only if it is the path graph.*

Theorem 24 (see [11]). *A graph has $\beta'(\Gamma) = 2$ if and only if it is the path graph.*

Theorem 25 (see [13]). *For integer $n \ge 3$, let C_n be the n-dimensional ring network. Then, $\beta'(C_n) = 3$.*

Theorem 26 (see [8]). *Let $K_{t_1,t_2,\ldots,t_r}$ be the complete r-partite graph with $1 \le t_1 \le t_2 \le \ldots \le t_r$ and $\sum_{i=1}^{r} t_i = n$. Then, $\beta(K_{t_1,t_2,\ldots,t_r}) = \sum_{i=1}^{r} t_i - r = n - r$.*

Theorem 27 (see [8]). *Let $K_{t_1,t_2,\ldots,t_r}$ be the complete r-partite graph with $1 \le t_1 \le t_2 \le \ldots \le t_r$ and $\sum_{i=1}^{r} t_i = n$. Then, $\beta'(K_{t_1,t_2,\ldots,t_r}) = \sum_{i=1}^{r} t_i = n$.*

Theorem 28. *If $K_{t_1,t_2,\ldots,t_r}$ is the complete r-partite graph with $1 \le t_1 \le t_2 \le \ldots \le t_r$, then*

$$\gamma^*_{fm}\left(K_{t_1,t_2,\ldots,t_r}\right) = \begin{cases} 1, & \text{for } 1 = t_1 \le t_2 \le \ldots \le t_i,\ (i = 2, 3, \ldots, r),\\ 2, & \text{for } 2 = t_1 \le t_2 \le \ldots \le t_i,\ (i = 2, 3, \ldots, r),\\ \dfrac{|M| + |N|}{|N_m[x]|}, & \text{otherwise}, \end{cases} \tag{9}$$

where $|M|$ and $|N|$ represent the cardinality of the vertex set and edge set of $K_{t_1,t_2,\ldots,t_r}$.

Proof. Consider $K_{t_1,t_2,\ldots,t_r}$ as a complete r-partite graph with $1 \le t_1 \le t_2 \le \ldots \le t_r$, and also assume f as a minimal FMXDF of $K_{t_1,t_2,\ldots,t_r}$. The cardinality of the vertex and edge sets can be represented as $|M|$ and $|N|$, respectively.

Case (i): if $r = 2$, then we have to find the FMXDN of K_{t_1,t_2} with $1 \le t_1 \le t_2$.

Subcase (i): Suppose $1 = t_1 \le t_2$. Thus, we assign the value 1 for a maximum degree vertex and 0 to the rest of the elements of K_{t_1,t_2}. Then, the poundage is $p(f) = \gamma^*_{fm}(K_{t_1,t_2}) = 1$.

Subcase (ii): Suppose $2 = t_1 \le t_2$. Then, we assume the value 1 for all maximum degree vertices and 0 for the rest of the elements of K_{t_1,t_2}. Then, $p(f) = \gamma^*_{fm}(K_{t_1,t_2}) = 2$.

Subcase (iii): Take $3 \le t_1 \le t_2$. In this case, we assign the value $1/|N_m[x = v]|$ for all $x \in M(G')$, where v is the minimum degree vertex of K_{t_1,t_2}. Then, $p(f) = \gamma^*_{fm}(K_{t_1,t_2}) = \{1/|N_m[v]| + 1/|N_m[v]| + \ldots + (|M| + |N|)\text{times}\} = |M| + |N|/|N_m[x]|$.

Case (ii): if $r = 3$, then we have to find the FMXDN of K_{t_1,t_2,t_3} with $1 \le t_1 \le t_2 \le t_3$.

Subcase (i): Let $1 = t_1 \le t_2 \le t_3$. Here, assign the value 1 for a maximum degree vertex and 0 for the rest of the elements of K_{t_1,t_2,t_3}. Then, the poundage is $p(f) = \gamma^*_{fm}(K_{t_1,t_2,t_3}) = 1$.

Subcase (ii): Let $2 = t_1 \le t_2 \le t_3$. Now, assign the value 1 for all maximum degree vertices and 0 for the rest of the elements of K_{t_1,t_2,t_3}. Then, $p(f) = \gamma^*_{fm}(K_{t_1,t_2,t_3}) = 2$.

Subcase (iii): Let $3 \le t_1 \le t_2 \le t_3$. In this case, we assign the value $1/|N_m[x = v]|$ for all $v \in M(G')$, where v is the minimum degree vertex of K_{t_1,t_2,t_3}. Then, $p(f) = \gamma^*_{fm}(K_{t_1,t_2,t_3}) = \{1/|N_m[v]| + 1/|N_m[v]| + \ldots + (|M| + |N|)\text{times}\} = |M| + |N|/|N_m[x]|$.

Case (iii): in general, continuing the above process for $r \ge 4$, we have to find the FMXDN of $K_{t_1,t_2,\ldots,t_r}$ with $1 \le t_1 \le t_2 \le \ldots \le t_r$.

Subcase (i): Take $1 = t_1 \le t_2 \le \ldots \le t_r$. Then, assign the value 1 for a maximum degree vertex and 0 for the rest of the elements of $K_{t_1,t_2,\ldots,t_r}$. Then, the poundage is $p(f) = \gamma^*_{fm}(K_{t_1,t_2,\ldots,t_r}) = 1$.

Subcase (ii): Take $2 = t_1 \le t_2 \le \ldots \le t_r$. Here, we put the value 1 for all maximum degree vertices and 0 for the rest of the elements of $K_{t_1,t_2,\ldots,t_r}$. Then, $p(f) = \gamma^*_{fm}(K_{t_1,t_2,\ldots,t_r}) = 2$.

Subcase (iii): Take $3 \le t_1 \le t_2 \le \ldots \le t_r$. In this case, we assign the value $1/|N_m[x = v]|$ for all $x \in M(G')$, where v is the minimum degree vertex of $K_{t_1,t_2,\ldots,t_r}$. Then, $p(f) = \gamma^*_{fm}(K_{t_1,t_2,\ldots,t_r}) = \{1/|N_m[v]| + 1/|N_m[v]| + \ldots + (|M| + |N|)\text{times}\} = |M| + |N|/|N_m[x]|$. □

Theorem 29. *For any tree, $\gamma^*_{fm}(T) \le \dim(T) + \lfloor r - 1/2 \rfloor$ and this bound is sharp for P_r, where $r = 2, 4,$ and 6.*

Proof. Let T be a tree with r vertices and $r - 1$ edges. Consider f as a minimal FMXDF of T, and by Definition 18, define a resolving set $rs_C(m) = (d(m, m_1), d(m, m_2), \ldots, d(m, m_l))$ represents the distances from m to m_s ($1 \le s \le l$) forming the resolving set of m with respect to C where $C = \{m_s/1 \le s \le l\}$. Now we have to compare the value of FMXDN and the resolvability parameter like metric dimension (the resolving set with minimum cardinality). By using Theorems 1 and 23, we obtain the inequality as $\gamma^*_{fm}(P_r) \le \dim(P_r) + \lfloor r - 1/2 \rfloor$. Suppose T is a star graph with m, m_i, where $1 \le i \le r$. By Theorem 3, we obtain $\gamma^*_{fm}(K_{1,r}) = 1$. Now we have to find the resolving set for $K_{1,r}$. When $r = 1$, the resolving set $rs = m_1$. Next, the resolving set for $K_{1,2}$ is $rs = \{m_1\}$. For $K_{1,3}$, the resolving set is $rs = \{m_1, m_2\}$. Continuing this way we obtain the minimum resolving set for $K_{1,1}$ and $K_{1,2}$ is 1, and $k_{1,r} = r - 1$ for $r \ge 3$. Therefore, $\gamma^*_{fm}(K_{1,r}) \le \dim(K_{1,r}) + \lfloor r - 1/2 \rfloor$. In this way, we create a resolving set for any tree and obtain the inequality as $\gamma^*_{fm}(T) \le \dim(T) + \lfloor r - 1/2 \rfloor$. Therefore, Theorem 29 is sharp for the path P_r, where $r = 2, 4,$ and 6.

Table 1 provides a comparison between the FMXDN and some resolvability-related parameters of certain graphs, such as paths, cycles, honeycomb rectangular toruses, and complete multipartite graphs. □

Theorem 30. *For any graph G', we have $\lfloor \gamma^*_{fm}(G') + 1 \rfloor \ge -r + 2\max\{\lceil \alpha + 2/2 \rceil, \lceil \beta + 2\gamma_{f_0}(G')/2 \rceil\}$ and this bound is sharp.*

Proof. Consider G' as a graph with a cardinality of vertex set M as r, α as its maximum degree, and β as its minimum degree. Let f be the smallest FMXDF of G'. Here, we have two claims.

Claim 1: $\lfloor \gamma^*_{fm}(G') + 1 \rfloor \ge -r + 2\lceil \alpha + 2/2 \rceil$. For all FMXDFs, we write every closed region of an edge as $R_m[m_1 m_2] = f(m_1) + S_{m_1} + f(m_2) + S_{m_2} - f(m_1 m_2)$ for all $n_1 = m_1 m_2 \in N(G')$, where S_{m_1} and S_{m_2} denote the star of m_1 and m_2, and we write every closed mixed neighbourhood of a vertex as $f[m] = R[m] + S_m$ for every $m \in M(G')$. Then, the poundage of f is $p(f) = \gamma^*_{fm}(G') = \sum_{n_1 = m_1 m_2 \in N(G')} f[m_1 m_2] + \sum_{m \in MG'} f[m] \ge 1 \ge \alpha$. It follows that $\gamma^*_{fm}(G') \ge \alpha \Longrightarrow \gamma^*_{fm}(G') \ge \lceil \alpha/2 \rceil$. Then, obviously,

$$\left\lfloor \gamma^*_{fm}\left(G'\right) + 1 \right\rfloor \ge -r + 2\left\lceil \frac{\alpha + 2}{2} \right\rceil. \tag{10}$$

Claim 2: $\lfloor \gamma^*_{fm}(G') + 1 \rfloor \ge -r + 2\lceil \beta + 2\gamma_{f_0}(G')/2 \rceil$. In this claim, the poundage of f is, $\gamma^*_{fm}(G') = \sum_{x \in M \cup N} f(x) = \sum_{n_1 = m_1 m_2 \in N(G')} f[m_1 m_2] + \sum_{m_1 \in MG'} f[m_1] \ge \sum_{m_1 \in MG'} f[m_1] = r \ge \gamma_{f_0}(G')$. It follows that $\lfloor \gamma^*_{fm}(G') + 1 \rfloor > \gamma_{f_0}(G') > -r + \gamma_{f_0}(G')$. Here $\beta \ge 1$; then obviously

Table 1: The comparison of other resolvability-related parameters.

Graphs	Metric dimension $(\dim(G'))$	The fault-tolerant metric dimension $(\dim_f(G'))$	FMXDN $\gamma_{\mathrm{fm}}^{*}(G')$
Path (P_r)	1	2	$\lfloor (2r+3)/5 \rfloor$
Cycle (C_r)	2	3	$2r/5$
$\mathrm{HRT}_{4,c}$	4	8	$([4\times c]+[4\times (c+\wp+1)])/7$
$K_{t_1,t_2,\ldots,t_r}$	$n-r$	n	$\begin{cases} 1 & \text{for } 1 = t_1 \leq t_2 \leq \ldots \leq t_i \, (i = 2,3,\ldots,r) \\ 2 & \text{for } 2 = t_1 \leq t_2 \leq \ldots \leq t_i \, (i = 2,3,\ldots,r) \\ (\lvert M\rvert + \lvert N\rvert)/(\lvert N_m[x]\rvert) & \text{otherwise} \end{cases}$

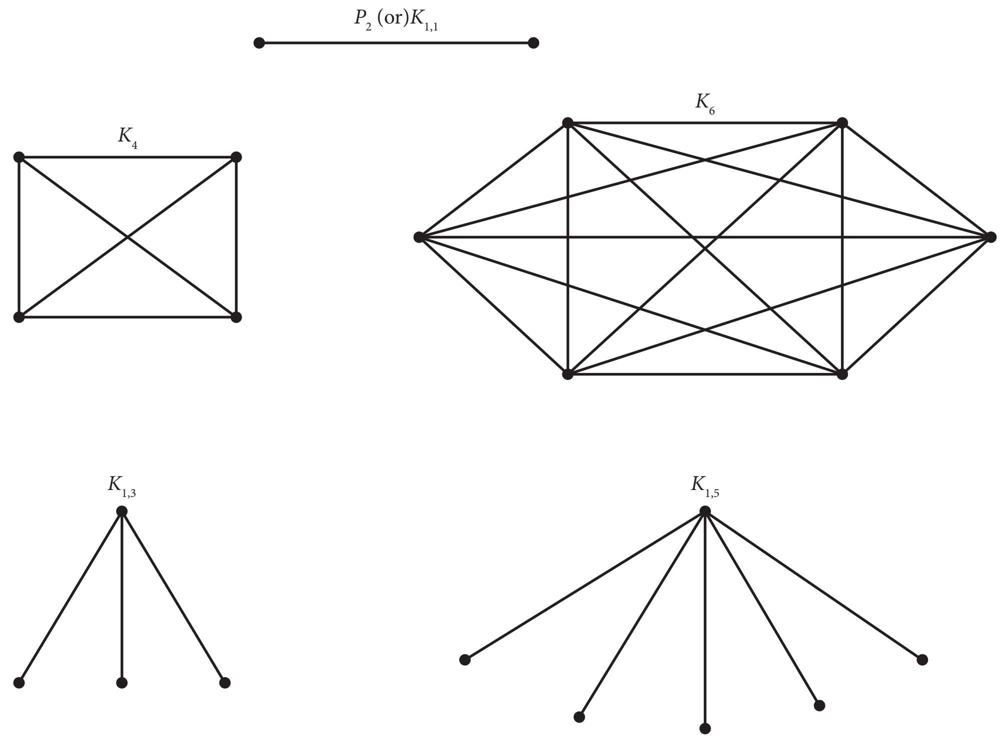

FIGURE 3: Sharpness with P_2, K_4, K_6, and $K_{1,r}$, where r is odd.

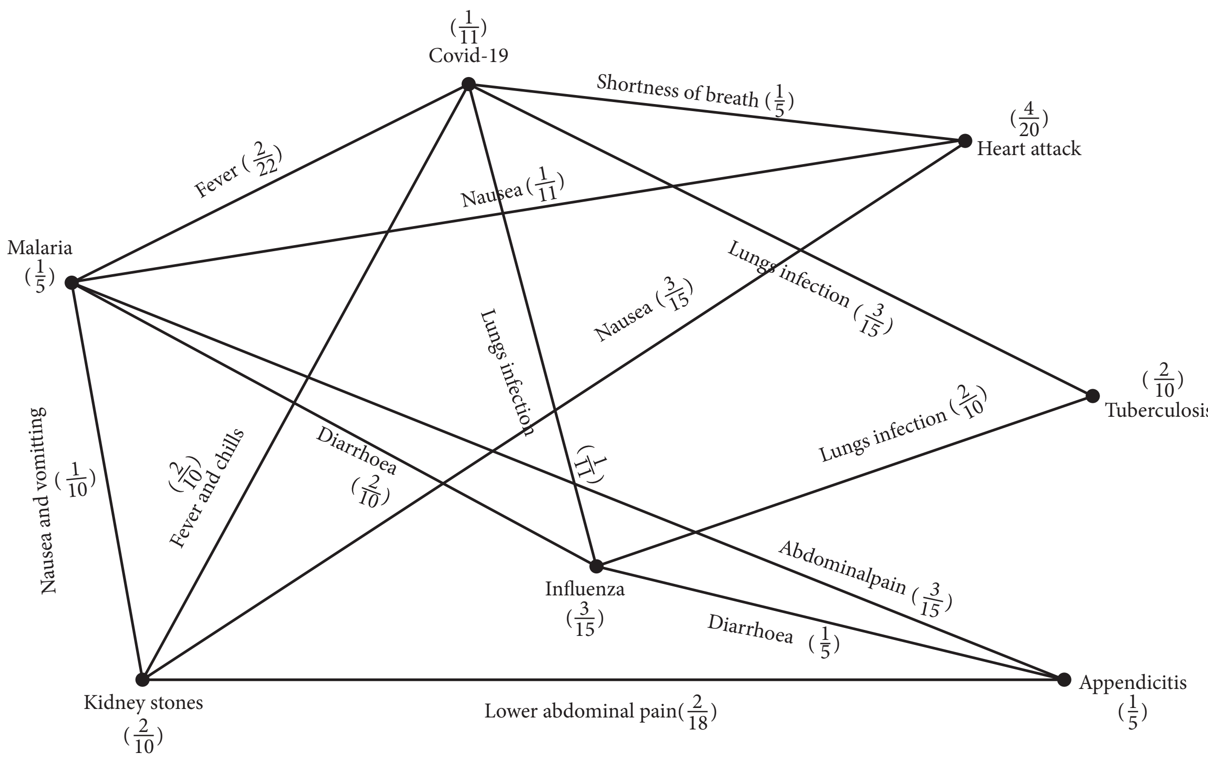

FIGURE 4: G'.

$$\lfloor \gamma_{fm}^{*}\left(G^{'}\right)+1\rfloor \geq -r+2\left\lceil\frac{\beta+2\gamma_{f_0}\left(G^{'}\right)}{2}\right\rceil. \tag{11}$$

From (10) and (11), we get $\lfloor \gamma_{fm}^{*}(G^{'})+1\rfloor \geq -r+2\max\left\{\lceil\alpha+2/2\rceil, \lceil\beta+2\gamma_{f_0}(G^{'})/2\rceil\right\}$. Furthermore, this bound is sharp for P_2, K_4, K_6, and $K_{1,r}$ when r is odd. Figure 3 gives the sharpness of Theorem 30. □

4. Application

The fractional mixed domination is used in hospitals. In this situation, the vertex set can be as various types of seven patients. The edge set is the connection between those patients having any one of the common symptoms (like shortness of breath, fever, fever and chills, lung infection, nausea and vomiting, abdominal pain, and diarrhea (see Figure 4)).

Doctors' aim is to reduce the patients' disease symptoms in a minimum number of hours. Here, the fractional mixed domination number illustrates that the minimum number of hours to reduce the disease of the patients. Here, in Figure 4, $\gamma_{fm}^{*}(G^{'})=3$.

5. Conclusion

We found the exact value of the FMXDN of some standard graphs, such as paths, cycles, the middle graph of the paths and cycles, and shadow graphs. Also, some upper bounds on the sum of the two fractional dominating parameters, whose resultant graph gives this inequality $\gamma_{f_1}^{'}(\mathrm{T})+\gamma_{fm}^{*}(\mathrm{T})\leq r+p-\mathrm{rad}(\mathrm{T})-\alpha$, were obtained in terms of fractional edge domination and fractional mixed domination. Finally, we provided a comparison result of γ_{fm}^{*} and other resolvability parameters such as metric and fault-tolerant metric dimension. This new parameter will be applicable for the optimization problems in our future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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