New Solutions of Time- and Space-Fractional Black–Scholes European Option Pricing Model via Fractional Extension of He-Aboodh Algorithm

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Received 6 April 2023; Revised 6 February 2024; Accepted 14 February 2024; Published 24 February 2024

Academic Editor: Ammar Alsinai

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The current study explores the space and time-fractional Black–Scholes European option pricing model that primarily occurs in the financial market. To tackle the complexities associated with solving models in a fractional environment, the Aboodh transform is hybridized with He’s algorithm. This facilitates in improving the efficiency and applicability of the classical homotopy perturbation method (HPM) by ensuring the rapid convergence of the series form solution. Three cases that are time-fractional scenario, space-fractional scenario, and time-space-fractional scenario are observed through graphs and tables. 2D graphical analysis is performed to depict the behaviour of a given option pricing model for varying time, stock price, and fractional parameters. Solutions of the European option pricing model at various fractional orders are also presented as 3D plots. The results obtained through these graphs unfold the interchange between time- and space-fractional derivatives, presenting a comprehensive study of option pricing under fractional dynamics. The competency of the proposed scheme is illustrated via solutions and errors throughout the fractional domain in tabular form. The validity of the He-Aboodh results is exhibited by comparison with existing errors. Analysis shows that the proposed methodology (He-Aboodh algorithm) is a valuable scheme for solving time-space-fractional models arising in business and economics.

1. Introduction

One of the most crucial theories in contemporary finance is the Black–Scholes model. This mathematical formula calculates the potential value of derivatives based on other financial instruments while accounting for the effects of time and other risk factors [1]. It is frequently used in contracts for option pricing [2, 3]. Fischer Black and Myron Scholes developed the first Black–Scholes model in 1973 [4]. Later, Robert Merton published an article [5] to expand the model’s applications. The classical Black–Scholes equation was originated to determine the theoretical value of an option contract using current stock prices, the option’s strike price, expected dividends, time of expiration, expected interest rates, and volatility. Some other modifications of Black–Scholes models have been suggested that are the jump-diffusion model [6], transaction cost models [7, 8], stochastic interest model [9], and stochastic volatility model [10]. After the fractal structures for the financial market [11] were discovered, the standard Brownian motion of the classical Black–Scholes equation was replaced by fractional Brownian motion to obtain the fractional Black–Scholes model. Some such models are the fractional Black–Scholes pricing model on arbitrage and replication [12], tempered fractional Black–Scholes equation for European double barrier option [13], pricing financial options model in fractal transmission system [14], fractional Black–Scholes model with stochastic volatility [15], pricing double barrier options in a time-fractional Black–Scholes model [16], fractional Black–Scholes equation under the constant elasticity of variance (CEV) model [17], fractional Black–Scholes model with European option [18], and two-dimensional fractional
are price, respectively. The final and boundary conditions of (1) can be observed. Introducing fractional derivatives in the above model, more complex phenomena such as the long-term memory effect can be observed. The results obtained from this study indicate the European option price. By introducing fractional derivatives in the above model, more complex phenomena such as the long-term memory effect can be observed.

Several techniques have been introduced in the literature to solve ordinary and partial differential equations. Galerkin method [21], implicit finite difference scheme [22], Adomian decomposition method [23], Crank–Nicolson scheme [24], homotopy perturbation method [25], backward Euler method [26], differential transform method [27], and stabilized meshless technique [28] are some of them. Many of these approaches have been utilized for the numerical solution of fractional differential equations including the Navier–Stokes equation [29, 30], Schrödinger equation [31], COVID-19 model [32], Kundu–Mukherjee–Naskar equation [33], and Black–Scholes model [34]. Chen et al. [35] employed a Legendre neural network for generalized Black–Scholes models. Chebyshev collocation method is applied by Mesgarani et al. [36] to analyze time-fractional Black–Scholes models. The Crank–Nicolson scheme is employed by Roul and Goura [37] to solve generalized Black–Scholes with the European call option. An et al. [38] proposed a space-time spectral method for the solution of Black–Scholes equations. Time-fractional Black–Scholes European option pricing equations are solved through the residual power series method by Dubey et al. [39]. Roul and Goura [40] introduced a finite difference scheme for the fractional Black–Scholes equation. The homotopy analysis method is utilized by Fadugba [41] for European call options with the time-fractional Black–Scholes model.

\[
\frac{\partial^\alpha \mathcal{W}(S, T)}{\partial T^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \mathcal{W}(S, T)}{\partial S^2} + (r - D)\mathcal{W}(S, T) - r\mathcal{W}(S, T) = 0, \tag{1}
\]

where \((S, T) \in (0, \infty) \times (0, T), \rho, \eta\) are the Caputo fractional derivatives with respect to \(T\) and \(S\), respectively. \(\alpha (\geq 0), \beta > 0, \delta, \eta, \kappa\) represent the volatility of the return, risk-free rate, dividend rate, expiry time, and stock price, respectively. The final and boundary conditions of (1) are

\[
\mathcal{W}(0, T) = \mathcal{C}_1, \quad \mathcal{W}(\infty, T) = \mathcal{C}_2, \quad \mathcal{W}(x, T) = \mathcal{C}_3. \tag{2}
\]

Suppose \(T = \bar{T} - t\) and \(x = \ln S\). Then, by defining \(\mathcal{B}(x, t) = \mathcal{W}(e^x, \bar{T} - t)\), the model in equation (1) can be rewritten in dimensionless form as

\[
\frac{\partial^\gamma \mathcal{B}(x, t)}{\partial t^\gamma} + \gamma_1 \frac{\partial^\gamma \mathcal{B}(x, t)}{\partial x^\gamma} + \gamma_2 \frac{\partial^\gamma \mathcal{B}(x, t)}{\partial x^\gamma} + \gamma_3 \mathcal{B}(x, t) - g(x, t) = 0, \tag{3}
\]

where \(\gamma_1 = \alpha/2, \gamma_2 = \beta - \gamma_1, \gamma_3 = \beta, \) and \(g(x, t)\) are the source terms. Moreover, \((\bar{I}_d, \bar{I}_u)\) is the finite domain, and the function \(\mathcal{B}\) indicates the European option price. By introducing fractional derivatives in the above model, more complex phenomena such as the long-term memory effect can be observed.

The homotopy perturbation method (HPM) provides a semi-analytical algorithm for solving both linear and nonlinear ordinary/partial differential equations [42]. It is also applied to differential system of equations [43]. In order to solve differential equations in fractional form more accurately, many modifications of HPM have been introduced. Baleanu and Jassim [44] extended the modified fractional homotopy perturbation technique on Helmholtz and coupled Helmholtz equations. Qayyum et al. [45, 46] utilized the He–Laplace transform method to solve generalized third- and fifth-order time-fractional KdV models. Fractional Navier–Stokes equations are investigated by Jena and Chakraverty [47] through homotopy perturbation Elzaki transform. Another modification is the He–Aboudh algorithm [48] which combines Aboudh transform and HPM. Manimegalai et al. [49] studied strongly nonlinear oscillators by applying the Aboudh transform and the homotopy perturbation method. An iterative scheme and Aboudh transform are employed by Ghenga and Mahmudov [50] to analyze the fractional spatial diffusion of a biological population model. Compared to classical HPM, He-Aboudh has broader applicability and improved convergence and accuracy. It eliminates integral terms which give an efficient procedure when dealing with fractional derivatives. Thus, in this paper, we have adapted the He-Aboudh algorithm for the solution and analysis of time-space-fractional Black–Scholes model (3). The scenarios involving time-fractional, space-fractional, and time-space-fractional derivatives are taken in the Caputo sense. The results obtained from this study indicate improvement in the predictive accuracy of option pricing particularly the systems involving noninteger-order derivatives. It also enhanced the comprehension of risks associated with option pricing in financial markets.
The format of this research article is as follows: Section 2 contains some basic definitions of Aboodh transform, Caputo fractional derivatives, and their Aboodh transform. A general methodology of the He-Aboodh algorithm is in Section 3 whereas Section 4 is centered on the application and solutions of the time-space-fractional Black–Scholes model. Results and discussion is given in Section 5, and the conclusion of the paper is in Section 6.

2. Preliminaries

Definition 1 (see [51]). The Aboodh transform $\mathfrak{A}$ of the function $\mathcal{B}(x,t)$ for $t \geq 0$ is

$$\mathfrak{A}[\mathcal{B}(x,t)] = \mathcal{K}(x,s) = \frac{1}{s} \int_0^\infty \mathcal{B}(x,t)e^{-st}dt, \quad s \in (k_1, k_2),$$

(4)

$k_1, k_2 > 0$ may be finite or infinite.

Definition 2 (see [52]). The inverse Aboodh transform $\mathfrak{A}^{-1}$ of the function $\mathcal{K}(x,s)$ is given as

$$\mathfrak{A}^{-1}[\mathcal{K}(x,s)] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} s \mathcal{K}(x,s)e^{st}ds, \quad s \in (k_1, k_2).$$

(5)

Definition 3 (see [53]). The Caputo time- and space-fractional derivatives $\text{CD}_t^{\alpha,\rho}$ and $\text{CD}_x^{\beta,\eta}$ of a function $\mathcal{B}(x,t)$ is, respectively, defined by

$$\text{CD}_t^{\alpha,\rho}\mathcal{B}(x,t) = \frac{1}{\Gamma(\alpha-\rho)} \int_0^t (t-A)^{\alpha-\rho-1}\mathcal{B}^{(\rho)}(x,A)dA, \quad \alpha-1 < \rho \leq \alpha,$$

(6)

$$\text{CD}_x^{\beta,\eta}\mathcal{B}(x,t) = \frac{1}{\Gamma(\beta-\eta)} \int_0^x (x-A)^{\beta-\eta-1}\mathcal{B}^{(\eta)}(A,t)dA, \quad \beta-1 < \eta \leq \beta,$$

$\rho$ and $\eta$ are fractional parameters.

Definition 4 (see [54]). The Aboodh transform $\mathfrak{A}$ of Caputo time- and space-fractional derivatives is, respectively, given as

$$\mathfrak{A}\left[\text{CD}_t^{\alpha,\rho}\mathcal{B}(x,t)\right] = s^\alpha \mathfrak{A}[\mathcal{B}(x,t)] - \sum_{q=0}^{\alpha-1} s^{\alpha-q-2}\mathfrak{A}[\mathcal{B}^{(q)}(x,0)], \quad \alpha-1 < \rho \leq \alpha,$$

(7)

$$\mathfrak{A}\left[\text{CD}_x^{\beta,\eta}\mathcal{B}(x,t)\right] = s^\beta \mathfrak{A}[\mathcal{B}(x,t)] - \sum_{q=0}^{\beta-1} s^{\beta-q-2}\mathfrak{A}[\mathcal{B}^{(q)}(0,t)], \quad \beta-1 < \eta \leq \beta.$$

Definition 5 (see [55]). The double Aboodh transform w.r.t. $t$ and $x$ on Caputo time- and space-fractional derivatives ($\text{CD}_t^{\alpha,\rho}$ & $\text{CD}_x^{\beta,\eta}$) can be described as

$$\mathfrak{A}_t \mathfrak{A}_x\left[\text{CD}_t^{\alpha,\rho}\mathcal{B}(x,t)\right] = s^\alpha s^\rho \mathfrak{A}_t \mathfrak{A}_x[\mathcal{B}(x,t)] - \sum_{q=0}^{\alpha-1} s^{\alpha-q-2} \mathfrak{A}_t \mathfrak{A}_x[\mathcal{B}^{(q)}(x,0)], \quad \alpha-1 < \rho \leq \alpha,$$

(8)

$$\mathfrak{A}_t \mathfrak{A}_x\left[\text{CD}_x^{\beta,\eta}\mathcal{B}(x,t)\right] = s^\beta s^\rho \mathfrak{A}_t \mathfrak{A}_x[\mathcal{B}(x,t)] - \sum_{q=0}^{\beta-1} s^{\beta-q-2} \mathfrak{A}_t \mathfrak{A}_x[\mathcal{B}^{(q)}(0,t)], \quad \beta-1 < \eta \leq \beta.$$
3. General Methodology of He-Aboodh Algorithm for Space-Time Fractional Models

Consider a general nonlinear time-space-fractional model

\[
\mathcal{D}_t^\rho \mathcal{B}(x,t) + \mathcal{D}_x^\eta \mathcal{B}(x,t) + \mathcal{L}[\mathcal{B}(x,t)] + \mathcal{N}[\mathcal{B}(x,t)] = 0, \quad \alpha - 1 < \rho \leq \alpha,
\]

\[
\beta - 1 < \eta \leq \beta,
\]

where \(\mathcal{D}_t^\rho\) and \(\mathcal{D}_x^\eta\) represent the time- and space-fractional derivatives of unknown function \(\mathcal{B}\), respectively. \(\mathcal{L}\) and \(\mathcal{N}\) are the linear and nonlinear operators of \(\mathcal{B}\).

3.1. Time-Fractional Scenario. The process will begin by applying the Aboodh transform with respect to time and taking into account the space derivative in integer order

\[
\mathcal{A}_t \mathcal{D}_t^\rho \mathcal{B}(x,t) + \mathcal{A}_t \mathcal{D}_x^\eta \mathcal{B}(x,t) + \mathcal{L}[\mathcal{B}(x,t)] + \mathcal{N}[\mathcal{B}(x,t)] = 0.
\]

Application of Aboodh transform on Caputo fractional derivative gives

\[
\mathcal{A}_t \mathcal{B}(x,t) - \frac{1}{\Gamma(\rho)} \sum_{q=0}^{\rho-1} s^{\rho-q-2} \mathcal{B}^{(q)}(x,0) + \frac{1}{\Gamma(\rho)} \sum_{q=0}^{\rho-1} s^{\rho-q-2} \mathcal{B}^{(q)}(x,0) + \frac{1}{\Gamma(\rho-1)} \mathcal{A}_t \mathcal{D}_x \mathcal{B}(x,t) + \mathcal{L}[\mathcal{B}(x,t)] + \mathcal{N}[\mathcal{B}(x,t)] = 0.
\]

The general homotopy is

\[
\text{Hom: } (1 - \lambda)(\mathcal{A}_t \mathcal{B}(x,t) - \mathcal{B}_0(x,t)) + \lambda \left( \mathcal{A}_t \mathcal{B}(x,t) - \frac{1}{\Gamma(\rho)} \sum_{q=0}^{\rho-1} s^{\rho-q-2} \mathcal{B}^{(q)}(x,0) + \frac{1}{\Gamma(\rho)} \sum_{q=0}^{\rho-1} s^{\rho-q-2} \mathcal{B}^{(q)}(x,0) + \frac{1}{\Gamma(\rho-1)} \mathcal{A}_t \mathcal{D}_x \mathcal{B}(x,t) + \mathcal{L}[\mathcal{B}(x,t)] + \mathcal{N}[\mathcal{B}(x,t)] \right) = 0,
\]

where \(\mathcal{B}_0\) represents the initial guess and \(0 \leq \lambda \leq 1\). Expansion of \(\mathcal{B}(x,t)\) in power series w.r.t. \(\lambda\) leads to

\[
\mathcal{B}(x,t) = \sum_{i=0}^{j} \lambda^i \mathcal{B}_i(x,t).
\]

Substituting equation (14) in equation (13) and then comparing identical coefficients of \(\lambda\) gives the following equations:

At 1st order,

\[
\mathcal{A}_t \mathcal{B}_1(x,t) + \mathcal{B}_0(x,t) - \frac{1}{\Gamma(\rho)} \sum_{q=0}^{\rho-1} s^{\rho-q-2} \mathcal{B}^{(q)}(x,0) + \frac{1}{\Gamma(\rho)} \sum_{q=0}^{\rho-1} s^{\rho-q-2} \mathcal{B}^{(q)}(x,0) + \mathcal{L}[\mathcal{B}_0(x,t)] + \mathcal{N}[\mathcal{B}_0(x,t)] = 0,
\]

\(\mathcal{B}_1(x,0) = 0\).
In general, at $k^{th}$ order,

$$\mathfrak{U}_t[A_{k-1}(x, t)] + \left(\frac{1}{s}\right)\mathfrak{U}_t[D_x A_{k-1}(x, t) + \mathcal{L}[A_{k-1}(x, t)] + \mathcal{N}[A_{k-1}(x, t)]] = 0,$$

$$B_k(x, 0) = 0.$$  \hfill (16)

A solution can be obtained by taking the inverse Aboodh transform.

At $\lambda^1$,

$$B_1(x, t) + \mathfrak{U}_t^{-1}\left\{ B_0 - \left(\frac{1}{s}\right) \frac{\beta^{-1}}{\eta^q-2} B^{(q)} (x, 0) + \mathfrak{U}_t[D_x B_0(x, t) + \mathcal{L}[B_0] + \mathcal{N}[B_0]] \right\} = 0.$$  \hfill (17)

At $\lambda^p$,

$$B_p(x, t) + \mathfrak{U}_t^{-1}\left\{ \left(\frac{1}{s}\right) \mathfrak{U}_t[D_x B_p(x, t) + \mathcal{L}[B_p] + \mathcal{N}[B_p]] \right\} = 0, \quad p = 2, \ldots, k.$$  \hfill (18)

The approximate series solution of equation (11) is

$$\tilde{B} = \sum_{i=0}^{j} B_i(x, t).$$  \hfill (19)

For $\eta=1$, we may obtain the residual function by inserting equation (19) in equation (9)

$$\mathfrak{U}_x[D_x^n B(x, t)] + \mathfrak{U}_x[D_t B(x, t)] + \mathfrak{U}_x[\mathcal{L}[B] + \mathcal{N}[B]] = 0.$$  \hfill (21)

Aboodh transform of space derivative gives

$$\mathfrak{U}_x[B(x, t)] - \left(\frac{1}{s}\right) \sum_{q=0}^{\beta-1} s^{\eta-q-2} B^{(q)} (0, t) + \left(\frac{1}{s}\right) \mathfrak{U}_x[D_x B(x, t) + \mathcal{L}[B] + \mathcal{N}[B]] = 0.$$  \hfill (22)

Homotopy of equation is

\begin{align*}
\text{Hom: } & (1 - \lambda)\left( \mathfrak{U}_x[B(x, t)] - B_0(x, t) \right) + \lambda \left( \mathfrak{U}_x[B(x, t)] - \left(\frac{1}{s}\right) \sum_{q=0}^{\beta-1} s^{\eta-q-2} B^{(q)} (0, t) \\
& + \left(\frac{1}{s}\right) \mathfrak{U}_x[D_x B(x, t) + \mathcal{L}[B] + \mathcal{N}[B]] \right) = 0.
\end{align*}  \hfill (23)
The same process as in Case 6 results in the following equation:

\[
\mathcal{A}_x \{ \mathcal{B}_1 (x, t) \} + \mathcal{B}_0 - \left( \frac{1}{\eta} \right) \sum_{q=0}^{\beta-1} s^{\eta-q-2} \mathcal{B}^{(q)} (0, t) + \left( \frac{1}{\eta} \right) \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B}_0 (x, t) + \mathcal{L} [ \mathcal{B}_0 ] + \mathcal{N} [ \mathcal{B}_0 ] \} = 0.
\] (24)

At \( \lambda^1 \),

\[
\mathcal{A}_x \{ \mathcal{B}_p (x, t) \} + \left( \frac{1}{\eta} \right) \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B}_p (x, t) + \mathcal{L} [ \mathcal{B}_p ] + \mathcal{N} [ \mathcal{B}_p ] \} = 0, \quad p = 2, \ldots, k.
\] (25)

Solutions can be found by using the inverse Aboodh transform.

At \( \lambda^1 \),

\[
\mathcal{B}_1 (x, t) + \mathcal{A}_x^{-1} \left\{ \mathcal{B}_0 - \left( \frac{1}{\eta} \right) \sum_{q=0}^{\beta-1} s^{\eta-q-2} \mathcal{B}^{(q)} (0, t) + \left( \frac{1}{\eta} \right) \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B}_0 (x, t) + \mathcal{L} [ \mathcal{B}_0 ] + \mathcal{N} [ \mathcal{B}_0 ] \} \right\} = 0.
\] (26)

At \( \lambda^p \),

\[
\mathcal{B}_p (x, t) + \mathcal{A}_x^{-1} \left\{ \left( \frac{1}{\eta} \right) \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B}_p (x, t) + \mathcal{L} [ \mathcal{B}_p ] + \mathcal{N} [ \mathcal{B}_p ] \} \right\} = 0, \quad p = 2, \ldots, k.
\] (27)

Adding these terms gives the approximate series solution. The residual function is produced by substituting the obtained approximate solution in equation (9) at \( \rho = 1 \).

\[
\mathcal{R} = \mathcal{D}_x \mathcal{B} + \mathcal{L} [ \mathcal{B} ] + \mathcal{N} [ \mathcal{B} ].
\] (28)

3.3. Time-Space-Fractional Scenario. For both time- and space-fractional model, we will take double Aboodh transform w.r.t. time and space.

\[
\mathcal{A}_x \{ \mathcal{D}_x \mathcal{B} (x, t) \} + \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B} (x, t) \} + \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B} (x, t) \} + \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B} (x, t) \} = 0.
\] (29)

Aboodh transform of time-space derivative gives

\[
s^{\eta} \mathcal{A}_x \{ \mathcal{B} (x, t) \} - \sum_{q=0}^{\alpha-1} s^{\eta-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (x, 0) \} + s^{\eta} \mathcal{A}_x \{ \mathcal{B} (x, t) \} - \sum_{q=0}^{\beta-1} s^{\eta-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (0, t) \}
\]

\[
+ \mathcal{A}_x \{ \mathcal{D}_x \mathcal{B} + \mathcal{N} [ \mathcal{B} ] \} = 0.
\] (30)
Homotopy equation of equation (30) is

\[
\text{Hom: } (1 - \lambda) \left( \mathcal{A}_\lambda \mathcal{A}_x \{ \mathcal{B} (x, t) \} - \mathcal{B}_0 (x, t) \right) + \lambda \left( \mathcal{A}_\lambda \mathcal{A}_x \{ \mathcal{B} (x, t) \} + \frac{1}{s^p} \left( - \sum_{q=0}^{n-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (x, 0) \} + s^q \mathcal{A}_x \{ \mathcal{B} (x, t) \} \right) - \sum_{q=0}^{n-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (0, t) \} + \mathcal{A}_x \{ \mathcal{B} [0] + \mathcal{N} [\mathcal{B}] \} \right) = 0.
\]

(31)

Mapping the same process as in Case 6 gives the following equation:

At \( \lambda^1 \),

\[
\mathcal{A}_x \{ \mathcal{B} \} \{ x, t \} + \mathcal{A}_x \{ \mathcal{B} \{ 0 \} \} + \frac{1}{s^p} \left( - \sum_{q=0}^{n-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (x, 0) \} + s^q \mathcal{A}_x \{ \mathcal{B} (x, t) \} \right) - \sum_{q=0}^{n-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (0, t) \} + \mathcal{A}_x \{ \mathcal{B} \{ 0 \} + \mathcal{N} [\mathcal{B}] \} = 0.
\]

(32)

At \( \lambda^p \),

\[
\mathcal{A}_x \{ \mathcal{B} \} \{ x, t \} + \frac{1}{s^p} \left( - \sum_{q=0}^{n-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (x, 0) \} + s^q \mathcal{A}_x \{ \mathcal{B} (x, t) \} \right) - \sum_{q=0}^{n-q-2} \mathcal{A}_x \{ \mathcal{B}^{(q)} (0, t) \} + \mathcal{A}_x \{ \mathcal{B} \{ 0 \} + \mathcal{N} [\mathcal{B}] \} = 0, \quad p = 2, \ldots, k.
\]

(33)

Inverse Aboudh transform and then the sum of obtained terms lead towards an approximate series solution. The residual function is given by

\[
\frac{\partial^\rho \mathcal{B}}{\partial t^\rho} - \gamma_1 \frac{\partial^{2q} \mathcal{B}}{\partial x^{2q}} - \gamma_2 \frac{\partial^{q} \mathcal{B}}{\partial x^{q}} + \gamma_3 \mathcal{B} - g (x, t) = 0, \quad 0 < \rho \leq 1, 0 < \eta \leq 1, t > 0.
\]

(35)

\[
\mathcal{B} (0, t) = 0, \mathcal{B} (1, t) = 0,
\]

\[
\mathcal{B} (x, 0) = x^2 (1 - x),
\]

(36)

where \( g (x, t) = \frac{1}{(2t^2 - \rho \Gamma (3 - \rho)) + (2t^2 - \rho \Gamma (2 - \rho))} x^2 (1 - x) - (t + 1)^2 \gamma_1 (2 - 6x) + \gamma_2 (2x - 3x^2) - \gamma_3 (x^2 (1 - x)). \)

At \( \rho = \eta = 1 \), the exact solution of equation (35) is [56]

\[
\mathcal{B} (x, t) = x^2 (1 - x)(t + 1)^2.
\]

(37)

### 4. Application and Solution of Time-Space-Fractional Black–Scholes European Option Pricing Model

**Example 1** (see [20]).

4.1. Solution

4.1.1. Case 1: Time Fractional. Consider \( \eta = 1 \) in equation (35). Taking Aboodh transform w.r.t. time (\( \mathcal{A}_x \)) and then utilizing differential property of the Aboodh transform give a homotopy equation

Hom: \( (1 - \lambda) \left( \mathcal{A}_\lambda \{ \mathcal{B} - x^2 (1 - x) \} + \lambda \left( \mathcal{A}_\lambda \{ \mathcal{B} - \left( \frac{1}{s^1} \right) x^2 (1 - x) \right) + \left( \frac{1}{s^1} \right) \mathcal{A}_\lambda \left\{ - \gamma_1 \frac{\partial^2 \mathcal{B}}{\partial x^2} - \gamma_2 \frac{\partial \mathcal{B}}{\partial x} + \gamma_3 \mathcal{B} - g (x, t) \right\} \right) = 0. \)

(37)
Substituting equation (14) in equation (37) and comparing alike coefficients of $\lambda$ leads to the following equation:

$$\mathcal{A}_i[\mathcal{B}_1] + x^2(1-x) - \left( \frac{1}{s^{1+\rho}} \right) x^2(1-x) + \left( \frac{1}{s^{1+\rho}} \right) \mathcal{A}_i \left\{ -y_1 \frac{\partial^2 \mathcal{B}}{\partial x^2} - y_2 \frac{\partial \mathcal{B}}{\partial x} + y_3 \mathcal{B} - g(x,t) \right\} = 0. \quad (38)$$

Application of Aboodh transform inverse generates the following equation:

$$\mathcal{B}_1 = -2t^\rho \left( \frac{t (y_1 (2-6x) + x (y_2 (2-3x) + (y_1 + 1) (x-1)x))}{\Gamma (\rho + 2)} + \frac{(x-1)x^2}{\Gamma (\rho + 1)} \right)$$

$$- 2t^\rho \left( \frac{t^2 (y_1 (2-6x) + x (y_2 (2-3x) + y_3 (x-1)x))}{\Gamma (\rho + 3)} \right) \quad (39)$$

At $\lambda^1$,

$$\mathcal{A}_i[\mathcal{B}_2] + \left( \frac{1}{s^{1+\rho}} \right) \mathcal{A}_i \left\{ -y_1 \frac{\partial^2 \mathcal{B}_1}{\partial x^2} - y_2 \frac{\partial \mathcal{B}_1}{\partial x} + y_3 \mathcal{B}_1 \right\} = 0. \quad (40)$$

At $\lambda^2$,

$$\mathcal{B}_2 = -2t^{2\rho} \left( - \frac{t^2 (4y_1 (3y_2 - 3y_3 x + y_3) + y_2^2 (6x - 2) + 2y_2 y_3 x (2-3x) + y_3^2 (x-1)x^2)}{\Gamma (2\rho + 3)} - \frac{x (3x - 2) + y_3 (y_3 + 1) (x-1)x^2}{\Gamma (2\rho + 2)} - \frac{2y_1 (3x - 1) - (x (y_2 (2-3x) + y_3 (x-1)x))}{\Gamma (2\rho + 1)} \right)$$

$$- \frac{t (2y_1 (6y_2 - 6y_3 x + 2y_2 x - 3x + 1) + y_2^2 (6x - 2) - y_2 (2y_3 + 1))}{\Gamma (2\rho + 2)} \quad (41)$$

Hence, the continuation of the process gives the fifth-order approximate solution in series form as shown in the following equation:

$$\mathcal{B}(x,t) = \sum_{i=0}^{5} \mathcal{B}_i(x,t). \quad (42)$$

4.1.2. Case 2: Space Fractional. Take $\rho = 1$ in equation (35) and then apply Aboodh transform w.r.t. space ($\mathcal{A}_i$). The acquired homotopy equation is

$$\text{Hom: } (1-\lambda)(\mathcal{A}_i[\mathcal{B}] - Ax) + \lambda \left( \mathcal{A}_i[\mathcal{B}] - \left( \frac{1}{s^{2\rho}} \right) A - \left( \frac{1}{s^{2\rho} y_1} \right) \mathcal{A}_i \left\{ \frac{\partial \mathcal{B}}{\partial t} - y_2 \frac{\partial \mathcal{B}}{\partial x} + y_3 \mathcal{B} - g(x,t) \right\} \right) = 0, \quad (43)$$
where $A$ is the dummy variable which is introduced to convert the boundary condition into the initial condition. A similar procedure as Case 9 leads to the following equation:

$$\mathfrak{U}_x \{ B_1 \} + Ax - \left( \frac{1}{s^{2\eta}} \right) A - \left( \frac{1}{s^{2\eta}/y_1} \right) \mathfrak{U}_x \left\{ \frac{\partial \mathcal{B}_0}{\partial t} - y_2 \frac{\partial^2 \mathcal{B}_0}{\partial x^{2\rho}} + y_3 \mathcal{B}_0 - g(x, t) \right\} = 0. \tag{44}$$

At $\lambda^1$,

$$\mathfrak{U}_x \{ B_2 \} - \left( \frac{1}{s^{2\eta}/y_1} \right) \mathfrak{U}_x \left\{ \frac{\partial \mathcal{B}_1}{\partial t} - y_2 \frac{\partial^2 \mathcal{B}_1}{\partial x^{2\rho}} + y_3 \mathcal{B}_1 \right\} = 0. \tag{45}$$

Taking the inverse Aboudh transform and adding the terms gives the fifth-order series solution. The optimal value of dummy variable $A$ can be found by using the right-side boundary condition of interval.

4.1.3. Case 3: Time-Space Fractional. Aboudh transform with respect to both time ($\mathfrak{U}_t$) and space ($\mathfrak{U}_x$) gives the following equation:

$$\mathfrak{U}_x \mathfrak{U}_t \left\{ \frac{\partial^\rho \mathcal{B}}{\partial t^{\rho}} - y_1 \frac{\partial^2 \mathcal{B}}{\partial x^{2\eta}} - y_2 \frac{\partial^\rho \mathcal{B}}{\partial x^{\rho}} + y_3 \mathcal{B} - g(x, t) \right\} = 0. \tag{46}$$

HPM procedure leads to homotopy.

Example 2 (see [20]).

Applying the double inverse Aboudh transform and then the summation of obtained terms gives a fifth-order approximate series solution.

$$\frac{\partial^\rho \mathcal{B}}{\partial t^{\rho}} - y_1 \frac{\partial^2 \mathcal{B}}{\partial x^{2\eta}} - y_2 \frac{\partial^\rho \mathcal{B}}{\partial x^{\rho}} + y_3 \mathcal{B} - g(x, t) = 0, \quad 0 < \rho \leq 1, 0 < \eta \leq 1, t > 0.$$

$$\mathcal{B}(0, t) = (t + 1)^3, \quad \mathcal{B}(1, t) = 3(t + 1)^3,$$

$$\mathcal{B}(x, 0) = x^3 + x^2 + 1, \tag{50}$$
where \( g(x, t) = ((2t^2 - \rho \Gamma(3 - \rho)) + (2t^1 - \rho \Gamma(2 - \rho))) (x^3 + x^2 + 1) - (t + 1)^2 \ (y_1 (2 + 6\lambda) + y_2 (2x + 3x^2) - y_3 (x^3 + x^2 + 1)) \). At \( \rho = \eta = 1 \), the exact solution of equation (50) is shown in the following equation [56]:

\[
\mathcal{B}(x, t) = (x^3 + x^2 + 1) (t + 1)^2. 
\]

Equation (14) leads to the following equation:

\[
\text{Hom: } (1 - \lambda) \left( \mathcal{B} \mathcal{I}_1[\mathcal{B} - (x^3 + x^2 + 1)] \right) + \lambda \left( \mathcal{B} \mathcal{I}_1[\mathcal{B}] - \left( \frac{x^3 + x^2 + 1}{s^2} \right) \right) + \left( \frac{1}{s^2} \right) \mathcal{I}_1 \left[ -y_1 \frac{\partial^2 \mathcal{B}}{\partial x^2} + y_2 \frac{\partial \mathcal{B}}{\partial x} + y_3 \mathcal{B} - g(x, t) \right] = 0.
\]

At \( \lambda^3 \),

\[
\mathcal{B}(x, t) = \sum_{i=0}^{5} \mathcal{B}_i(x, t).
\]

4.2. Solution

4.2.1. Case 1: Time Fractional. Consider \( \eta = 1 \) and then following the steps given in Section 3, we have a homotopy equation.

\[
\mathcal{B}_1(x, t) + (x^3 + x^2 + 1) - \left( \frac{1}{s^2} \right) \mathcal{I}_1 \left[ -y_1 \frac{\partial^2 \mathcal{B}_1}{\partial x^2} + y_2 \frac{\partial \mathcal{B}_1}{\partial x} + y_3 \mathcal{B}_1 \right] = 0.
\]

At \( \lambda^1 \),

\[
\mathcal{B}_2(x, t) = \sum_{i=0}^{5} \mathcal{B}_i(x, t).
\]

4.2.2. Case 2: Space Fractional. By following the same steps as in Section 3, we get the following equation:

\[
\text{Hom: } (1 - \lambda) \left( \mathcal{B} \mathcal{I}_1[\mathcal{B} - (Ax + (t + 1)^2)] \right) + \lambda \left( \mathcal{B} \mathcal{I}_1[\mathcal{B}] - \left( \frac{A + s2 (t + 1)^2}{s^2} \right) \right) - \left( \frac{1}{s^2} \right) \mathcal{I}_1 \left[ \frac{\partial \mathcal{B}_0}{\partial t} - y_2 \frac{\partial^2 \mathcal{B}_0}{\partial x^2} + y_3 \mathcal{B}_0 - g(x, t) \right] = 0.
\]

Utilizing equation (14) gives the following equation:

At \( \lambda^1 \),

\[
\mathcal{B}_1(x, t) + (Ax + (t + 1)^2) - \left( \frac{A + s2 (t + 1)^2}{s^2} \right) - \left( \frac{1}{s^2} \right) \mathcal{I}_1 \left[ \frac{\partial \mathcal{B}_0}{\partial t} - y_2 \frac{\partial^2 \mathcal{B}_0}{\partial x^2} + y_3 \mathcal{B}_0 - g(x, t) \right] = 0.
\]

At \( \lambda^2 \),

\[
\mathcal{B}_2(x, t) = \sum_{i=0}^{5} \mathcal{B}_i(x, t).
\]
Taking the inverse Aboodh transform and adding the terms gives the fifth-order series solution.

4.2.3. Case 3: Time-Space Fractional. The procedure given in Section 3 leads to the following equation:

\[
\text{Hom: } (1 - \lambda) \left( \mathcal{A}_x \left[ \mathcal{B} \right] - \left( Ax + (t + 1)^2 \right) \right) + \lambda \left( \mathcal{A}_x \left[ \mathcal{B} \right] - \frac{As^1 + (s^1 + 2s^1 + 2)s^2}{s^1s^2} \right)
\]

Substituting equation (14) and comparing similar coefficients of \( \lambda \) gives the following equation:

At \( \lambda^1 \),

\[
\mathcal{A}_x \left[ \mathcal{B}_1 \right] + Ax + (t + 1)^2 - \frac{As^1 + (s^1 + 2s^1 + 2)s^2}{s^1s^2} = \frac{1}{s^1s^2} \mathcal{A}_x \left[ \mathcal{B}_0 \right] \frac{\partial \mathcal{B}_0}{\partial \eta} - y_2 \frac{\partial \mathcal{B}_0}{\partial \eta^2} + y_3 \mathcal{B}_0 - g(x, t)
\]

At \( \lambda^2 \),

\[
\mathcal{A}_x \left[ \mathcal{B}_2 \right] - \frac{1}{s^1s^2} \mathcal{A}_x \left[ \mathcal{B}_1 \right] \left( \frac{\partial \mathcal{B}_1}{\partial \eta} - y_2 \frac{\partial \mathcal{B}_1}{\partial \eta^2} + y_3 \mathcal{B}_1 \right) = 0.
\]

At \( \lambda^3 \),

\[
\mathcal{A}_x \left[ \mathcal{B}_3 \right] - \frac{1}{s^1s^2} \mathcal{A}_x \left[ \mathcal{B}_2 \right] \left( \frac{\partial \mathcal{B}_2}{\partial \eta} - y_2 \frac{\partial \mathcal{B}_2}{\partial \eta^2} + y_3 \mathcal{B}_2 \right) = 0.
\]

Applying the double inverse Aboodh transform and then the summation of obtained terms gives a fifth-order approximate series solution.

5. Results and Discussion

In the current work, Black–Scholes European option pricing model in space- and time-fractional environment is solved through a hybrid technique, the He-Aboodh transform, in which the Aboodh transform and homotopy perturbation method are integrated together. For simulation purposes, Wolfram Mathematica 13.3 is utilized on ThinkPad that has a display size of 14.00-inch, resolution 3840 × 2160 pixels, processor core i9, and 16 GB RAM. Three cases that are time-fractional (Case 6), space-fractional (Case 7), and time-space-fractional (Case 8) are analyzed via 2D and 3D graphs. Absolute errors obtained through the proposed methodology are compared with the multiquadric-radial basis function (MQ-RBF) method errors in Tables 1 and 2. It can be deduced that the He-Aboodh algorithm gives more precise results than the existing technique. For different values of fractional parameter \( \rho \), Tables 3 and 4 exhibit the solutions and errors at varying time \( t \) whereas, in Tables 5 and 6, solutions and errors are calculated throughout the whole domain of stock price \( x \). From these tables, the convergence of the scheme for the whole fractional domain is observed. The consistency of obtained solutions can also be seen from them. It is noted that He-Aboodh transform is a reliable technique for solving space- and time-fractional models as the obtained solutions are nearer to their exact solutions.

In Example 1 for Case 9, Figure 1 displays the solution pattern of the Black–Scholes model at fractional parameter \( \rho = 0.24 \) and 0.86. By considering volatility \( \sigma = 0.37 \), risk-free
rate $r = 0.16$, and dividend rate $D = 0.05$, both plots show that initially increase in time does not have any impact on option price. However, when the time becomes large and the stock price $x$ increases in its domain, the European option pricing also displays an increase in its value. As compared to plot (a), the value of European option pricing $\mathcal{R}$ is greater in plot (b) for larger fractional parameter value. This shows the effects of long-range memory on the dynamics of Black–Scholes model. Figure 2 demonstrates that an increase in stock price and time causes the European option price to rise. For Case 10, Figure 3 with fractional parameter $\eta = 0.24$ and $0.86$ illustrates that option price increases as time and stock price escalates. Moreover, the profile of European option pricing also expands at a greater value of time (see Figure 4). In the case of time-space fractional, plot (a) of Figure 5 at $\rho = 0.24$ and $\eta = 0.5$ exhibits that option price keeps increasing as time and stock price rises. The peak of the European option pricing model is at stock price $x = 0.8$; after that, it started to get lower. Solution at $\rho = 0.84$ and $\eta = 0.7$ in plot (b) also presents the surge in European option pricing model. Additionally, enlarging the values of fractional parameters $\rho$ and $\eta$ in their domains indicates a drop in Black–Scholes

### Table 1: Error comparison of He-Abboodh algorithm and MQ-RBF method (Example 1).

| Solution $\rho$ | $|\mathcal{R}|_{\text{He–Abboodh}}$ | $|\mathcal{R}|_{\text{MQ-RBF}}$ |
|-----------------|----------------------------------|---------------------------------|
| $\rho = 0.2$    | $8.07 \times 10^{-11}$           | $2.30 \times 10^{-5}$          |
| 0.203082        |                                  |                                 |
| 0.221127        | $2.54 \times 10^{-10}$           | $3.24 \times 10^{-4}$          |
| 0.241779        | $5.05 \times 10^{-10}$           | $1.38 \times 10^{-2}$          |
| 0.265120        | $8.30 \times 10^{-10}$           | $2.50 \times 10^{-5}$          |
| 0.291054        | $1.22 \times 10^{-9}$            | $6.79 \times 10^{-4}$          |
| 0.319510        | $1.70 \times 10^{-9}$            | $3.12 \times 10^{-2}$          |
| $\rho = 0.7$    | $8.32 \times 10^{-17}$           | $1.27 \times 10^{-5}$          |
| 0.167321        |                                  |                                 |
| 0.200614        | $1.77 \times 10^{-15}$           | $3.21 \times 10^{-4}$          |
| 0.231784        | $1.59 \times 10^{-14}$           | $1.01 \times 10^{-2}$          |
| 0.262800        | $8.06 \times 10^{-14}$           | $1.67 \times 10^{-5}$          |
| 0.294466        | $2.85 \times 10^{-13}$           | $4.15 \times 10^{-4}$          |
| 0.327210        | $8.04 \times 10^{-13}$           | $2.05 \times 10^{-2}$          |

### Table 2: Error comparison of He-Abboodh algorithm and MQ-RBF method (Example 2).

| Solution $\rho$ | $|\mathcal{R}|_{\text{He–Abboodh}}$ | $|\mathcal{R}|_{\text{MQ-RBF}}$ |
|-----------------|----------------------------------|---------------------------------|
| $\rho = 0.2$    | $6.38 \times 10^{-11}$           | $2.46 \times 10^{-5}$          |
| 5.76684         |                                  |                                 |
| 6.60082         | $2.01 \times 10^{-10}$           | $5.13 \times 10^{-4}$          |
| 7.29695         | $3.99 \times 10^{-10}$           | $1.48 \times 10^{-2}$          |
| 7.93903         | $6.57 \times 10^{-10}$           | $4.36 \times 10^{-5}$          |
| 8.55386         | $9.72 \times 10^{-10}$           | $7.86 \times 10^{-4}$          |
| 9.15321         | $1.34 \times 10^{-9}$            | $4.03 \times 10^{-2}$          |
| $\rho = 0.7$    | $6.10 \times 10^{-16}$           | $3.16 \times 10^{-5}$          |
| 3.31886         |                                  |                                 |
| 4.07935         | $1.33 \times 10^{-15}$           | $5.72 \times 10^{-4}$          |
| 4.80665         | $1.18 \times 10^{-14}$           | $1.85 \times 10^{-2}$          |
| 5.53274         | $6.16 \times 10^{-14}$           | $3.67 \times 10^{-5}$          |
| 6.26966         | $2.27 \times 10^{-13}$           | $8.32 \times 10^{-4}$          |
| 7.02315         | $6.37 \times 10^{-13}$           | $3.26 \times 10^{-2}$          |

### Table 3: Solution and error analysis at $x = 0.83$, $\sigma = 0.5$, $r = 0.3$, and $D = 0.1$ (Example 1).

| $t$   | Solution $\rho = 0.47$ | $|\mathcal{R}|$ | Solution $\rho = 0.81$ | $|\mathcal{R}|$ | Solution $\rho = 1.00$ | $|\mathcal{R}|$ |
|-------|-------------------------|----------------|-------------------------|----------------|-------------------------|----------------|
| 0.0   | 0.117113                | 0.0           | 0.117113                | 5.55 \times 10^{-17} | 0.117113                | 5.55 \times 10^{-17} |
| 0.2   | 0.124053                | 4.68 \times 10^{-7} | 0.175650                | 5.38 \times 10^{-11} | 0.168643                | 4.56 \times 10^{-15} |
| 0.4   | 0.118901                | 6.71 \times 10^{-6} | 0.221004                | 4.98 \times 10^{-9}  | 0.229541                | 2.37 \times 10^{-12} |
| 0.6   | 0.137371                | 3.25 \times 10^{-5} | 0.270332                | 7.14 \times 10^{-8}  | 0.299809                | 9.30 \times 10^{-11} |
| 0.8   | 0.179568                | 1.00 \times 10^{-4} | 0.327652                | 4.77 \times 10^{-7}  | 0.379446                | 1.26 \times 10^{-9}  |
| 1.0   | 0.245842                | 2.45 \times 10^{-4} | 0.395310                | 2.09 \times 10^{-6}  | 0.468452                | 9.58 \times 10^{-9}  |
| 1.2   | 0.336830                | 5.11 \times 10^{-4} | 0.475056                | 7.05 \times 10^{-6}  | 0.566827                | 5.03 \times 10^{-8}  |
| 1.4   | 0.453317                | 9.57 \times 10^{-4} | 0.568346                | 1.97 \times 10^{-5}  | 0.674571                | 2.05 \times 10^{-7}  |
model’s profile which is displayed in Figure 6. It can be seen from these figures that noninteger-order derivatives allow the characterization of financial models more accurately than integer-order derivatives.

By taking the values of volatility \( \sigma = 0.4 \), risk-free rate \( r = 0.21 \), and dividend rate \( D = 0.1 \) in the time-fractional case of Example 2, Figure 7 illustrates that as stock price and fractional parameter expand, the profile of Black–Scholes option pricing model depicts an increasing behaviour. In Case 13 (see Figure 8), it is shown through the arrow that as time increases the option price indicates an elevation. On the other hand, a decline in the option pricing profile is observed for larger values of fractional parameter \( \eta \). 3D Figure 9 in Case 14 demonstrates that in the beginning, the option price was at higher value, but as the stock price and time goes up, it began to decrease. Moreover, the value of fractional parameters \( \rho \) and \( \eta \) is smaller in plot (a) as compared to plot (b). For smaller values of \( \rho \) and \( \eta \), the option price is greater. This indicates the complex and memory-dependent behaviour of option pricing models that are effectively captured through fractional calculus.

### Table 4: Solution and error analysis at \( x = 0.83 \), \( \sigma = 0.5 \), \( r = 0.3 \), and \( D = 0.1 \) (Example 2).

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<tr>
<th>( t )</th>
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<table>
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### Table 6: Solution and error analysis at \( t = 0.4 \), \( \sigma = 0.5 \), \( r = 0.3 \), and \( D = 0.1 \) (Example 2).

<table>
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<tr>
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</table>

\[ \begin{align*}
\rho &= 0.47 \\
\rho &= 0.81 \\
\rho &= 1.00
\end{align*} \]

\[ \begin{align*}
|\mathcal{R}| &= 1.33 \times 10^{-15} \\
|\mathcal{R}| &= 1.33 \times 10^{-11} \\
|\mathcal{R}| &= 2.00 \times 10^{-15}
\end{align*} \]
Figure 1: 3D Black–Scholes European option pricing solution of Example 1 (Case 9) at $\sigma = 0.37$, $r = 0.16$, and $D = 0.05$. (a) Solution at $\rho = 0.24$. (b) Solution at $\rho = 0.86$.

Figure 2: European option pricing model profile at different values of $x$ in Example 1 (Case 9) where $\rho = 0.24$, $\sigma = 2.0$, $r = 1.1$, and $D = 0.8$.

Figure 3: 3D Black–Scholes European option pricing solution of Example 1 (Case 10) at $\sigma = 0.37$, $r = 0.16$, and $D = 0.05$. (a) Solution at $\eta = 0.24$. (b) Solution at $\eta = 0.86$. 
Figure 4: European option pricing model profile at different values of $t$ in Example 1 (Case 10) where $\eta = 0.68$, $\sigma = 2.0$, $r = 1.1$, and $D = 0.8$.

Figure 5: 3D Black–Scholes European option pricing solution of Example 1 (Case 11) at $\sigma = 0.37$, $r = 0.16$, and $D = 0.05$. (a) Solution at $\rho = 0.24$ and $\eta = 0.5$. (b) Solution at $\rho = 0.84$ and $\eta = 0.7$.

Figure 6: European option pricing model profile at different values of $\rho$ and $\eta$ in Example 1 (Case 11) where $t = 1.7$, $\sigma = 2.0$, $r = 1.1$, and $D = 0.8$. 
Figure 7: European option pricing model profile at different values of $x$ and $\rho$ in Example 2 (Case 12) where $\sigma = 0.4$, $r = 0.21$, and $D = 0.1$.

Figure 8: European option pricing model profile at different values of $t$ and $\eta$ in Example 2 (Case 13) where $\sigma = 2.0$, $r = 1.1$, and $D = 0.8$.

Figure 9: 3D Black–Scholes European option pricing solution of Example 2 (Case 14) at $\sigma = 0.37$, $r = 0.16$, and $D = 0.05$. (a) Solution at $\rho = 0.54$ and $\eta = 0.60$. (b) Solution at $\rho = 0.84$ and $\eta = 0.70$. 
6. Conclusion

Fractional analysis of Black–Scholes European option pricing model in both space and time is the primary focus of this research article. The proposed methodology, He-Aboodh algorithm, is utilized to obtain approximate solutions of the given model. Differential property of Aboodh transform on Caputo time-fractional, space-fractional, and time-space-fractional derivatives is applied to efficiently tackle the complexities arising from noninteger-order dynamics. Solution and errors at different values of time, stock price, and fractional parameters are displayed in tabular form which shows that the application of the proposed methodology improves the predictive accuracy of option pricing models especially when dealing with memory-dependent procedures. Additionally, the error comparison of the multiquadric-radial basis function and He-Aboodh algorithm at fractional parameters $\rho = 0.2$ and 0.7 leads to the conclusion that the proposed methodology gives better results in terms of accuracy. By increasing the rates of time, stock price, and fractional parameters, change in the profile of the given Black–Scholes model is demonstrated with the help of two-dimensional figures. The ups and downs in the value of European option pricing are illustrated through three-dimensional figures. Solutions at different values of fractional parameters for all three cases indicate that for Example 1, with an increase in time and stock price, the value of the option price also increases. On the other hand, the option price is initially at a higher value in Example 2, but it started to decline when stock price and time increased. These results provide a beneficial understanding of the interaction between time- and space-fractional dynamics in option pricing models. Thus, it can be concluded that the He-Aboodh algorithm is an efficient technique that can be extended to solve nonlinear complex time- and space-fractional Black–Scholes models arising in the financial market.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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