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Research Article

Existence, Blow-Up, and Blow-Up Rate of Weak Solution to Fourth-Order Non-Newtonian Polytropic Variation-Inequality Arising from Consumption-Investment Models

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This article obtains the conditions for the existence and nonexistence of weak solutions for a variation-inequality problem. This variational inequality is constructed by a fourth-order non-Newtonian polytropic operator which is receiving much attention recently. Under the proper condition of the parameter, the existence of a solution is proved by constructing the initial boundary value problem of an ellipse by time discretization and some elliptic equation theory. Under the opposite parameter condition, we analyze the nonexistence of the solution. The results show that the weak solution will blow up in finite time. Finally, we give the blow-up rate and the upper bound of the blow-up time.

1. Introduction

We consider the existence and blow-up phenomenon for a kind of variation-inequality problem

$$\begin{cases} \mathrm{Lu} \geq 0, & (x,t) \in \Omega_T, \\ u \geq u_0, & (x,t) \in \Omega_T, \\ \mathrm{Lu} \times (u - u_0) = 0, & x \in \Omega; \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1}$$

with the fourth-order non-Newtonian polytropic operator

$$\begin{aligned} \mathrm{Lu} = \partial_t u - u^m \Delta\left(\left|\Delta u^m\right|^{p-2} \Delta u^m\right) - \gamma\left|\Delta u^m\right|^p \\ + h(x,t)u^\alpha, \quad p \geq 2. \end{aligned} \tag{2}$$

Here, $\Omega_T = \Omega \times (0,T)$, $\Omega \in \mathrm{R}_n$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, $m > 0$, and u_0 satisfies $u_0^m \in C_0^\infty(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$.

Variational inequalities similar to model (1) have good applications in option pricing. Han and Yi in [1] studied a kind of irreversible investment problem of the firm on finite horizon T in which they consider the following variation-inequality problem:

$$\begin{cases} \partial_t U \geq -qe^x, & (x,t) \in \Omega_T, \\ L_1 U - \left|e^x - 1\right| e^x \geq 0, & (x,t) \in \Omega_T, \\ \left(U + qe^x\right) \times \left(L_1 U - \left|e^x - 1\right| e^x\right) = 0, & (x,t) \in \Omega_T, \\ U(x,0) = 0, & x \in \Omega, \end{cases} \tag{3}$$

where $L_1 U$ is a linear parabolic operator with constant parameters and q represents the dividend rate of risk assets. The existence of the solution and the regularity of solution were studied. Moreover, the behaviors of free boundaries were considered in [2]. Chen et al. in [3] consider a class of singular control problems whose solution is governed by a time-dependent HJB equation with gradient constraints. The existence, uniqueness, and the behaviors of free boundary face were analyzed.

In recent years, the theoretical research on variation-inequality has also attracted the attention of scholars. In [4], the existence of a weak solution for a class of variational inequality in whole RN is studied, in which the author used the penalization method [5] and the Mountain Pass Theorem shown in [6]. Wang and Zhang in [7] considered the partial differential mixed variational inequality on Hilbert lattices using order-theoretic approaches. The existence of mild solutions to mixed variational inequality is established without Lipschitz continuity. There are some other new studies [8–11], which will not be repeated here.

Unlike classical partial differential equations, parabolic variational inequalities involve inequality constraints. As a result, we introduce a maximal monotone map to characterize these inequality constraints. Furthermore, the existence of weak solutions for parabolic variational inequalities is proven through an auxiliary problem. In this process, we construct a penalty function to approximate the maximal monotone map, which constitutes the first innovation of this paper. To demonstrate the explosive nature of weak solutions in a finite time, we initially analyze the non-negativity of the integral term constructed from the maximal monotone map and weak solution test functions. These test functions also happen to be suitable for constructing differential inequalities based on energy functions, thus completing the proof of the explosive nature of weak solutions, marking another innovative aspect of this paper.

This paper mainly considers the existence and blow-up phenomenon for a kind of variation-inequality problem (1) with the fourth-order non-Newtonian polytropic operator (2). When ≤1, we analyze the existence of weak solutions of variational inequality (1). The time interval $[0, T]$ is discretized, and several estimates of the auxiliary problem are given by using the theory of elliptic equations. Then the existence of weak solutions is given by a limit method. The result is shown in Section 3. When $\gamma > 1$, we show that weak solutions of variational inequality (1) will blow up at finite time and do not exist. Using differential inequality and Sobolev inequality, the existence of blow-up is proved, and the upper bound estimate of blow-up time and blow-up rate is given. The results as well as relative proofs are shown in Section 4.

2. The Main Results and the Financial Background

Variational inequalities are often used to quantify consumption-investment models [12, 13] which contain two factors

$$\begin{aligned} \mathrm{d}D_t &= \mu_1 D\mathrm{d}t + \sigma_1 D\mathrm{d}t, 0 \le t \le T, D_0 = d, \\ \mathrm{d}C &= \mu_2 C\mathrm{d}t + \sigma_2 C\mathrm{d}t, 0 \le t \le T, C_0 = c. \end{aligned} \tag{4}$$

Here, $\{D_t, t \in [0, T]\}$ is the demand of a good in which μ_1 and σ_1 represent the expected rate of return and volatility, respectively. $\{C_t, t \in [0, T]\}$ is the production capacity of the firm, and μ_2 and σ_2 are positive constants called the expected rate of return and volatility of the production process.

The aim of the consumption-investment model is to find an optimal policy to minimize the expected total cost over the finite horizon, such that the value function

$$V(c, d, t) = \inf_{x \longrightarrow \infty} E\left[\int_0^T e^{\mathrm{rt}} g(C_t, D_t) \mathrm{d}t \,\middle|\, C_0 = c, D_0 = d \right], \tag{5}$$

satisfies a variation-inequality

$$\begin{cases} \partial_t V \ge -q, & (c, t) \in \Omega_T, \\ L_1 V - g(c, d) \ge 0, & (c, t) \in \Omega_T, \\ (\partial_t V + q) \times (L_1 U - g(c, d)) = 0, & (c, t) \in \Omega_T, \\ U(x, 0) = 0, & c \in \Omega, \end{cases} \tag{6}$$

with $L_1 V = 1/2\sigma_1^2 d^2 \partial_{\mathrm{dd}} V + 1/2\sigma_2^2 \ c^2 \partial_{\mathrm{cc}} V + \sigma_1 d \partial_d V + \sigma_2 c \partial_c V - rV$. Here, r is the risk-free interest rate in the market.

The investment consumption model involves the specification of goods demand $\{D_t, t \in [0, T]\}$ and production capacity of the firm $\{C_t, t \in [0, T]\}$. Investors must make decisions between current production and future production while considering future income, interest rates, and other economic factors. This balancing act can be seen as an optimization problem, where investors seek to maximize the company's utility or wealth. This optimization process can be formalized as a variational inequality, in which investors strive to find the optimal production strategy to match future demand, maximizing utility or wealth, while satisfying budget constraints. Therefore, the investment consumption model is often described as an optimization problem involving variational inequalities.

This paper investigates a variational inequality that is more extensive than model (6). For this purpose, one gives the maximal monotone mapping G.

$$G = \{g \mid g(x) = 0, x > 0, g(0) \in [0, M_0]\}, \tag{7}$$

where M_0 is a positive constant.

Definition 1. we say that (u, ξ) is a generalized solution of variation-inequality (1), if it satisfies

(a) $(x, t) \ge u_0(x), u(x, 0) = u_0(x)$ for any $(x, t) \in \Omega_T$,

(b) for every test-function $\varphi \in C^1(\overline{\Omega}_T)$ and for almost all $t \in (0, T]$, we get

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t u \cdot \varphi + u^m |\Delta u^m|^{p(x)-2} \Delta u^m \Delta \varphi \mathrm{d}x\, \mathrm{d}t + (1-\gamma) \int_0^t \int_{\Omega_T} |\Delta u^m|^p \varphi \mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_0^t \int_{\Omega_T} |\Delta u^m|^p \varphi + h u^\alpha \mathrm{d}x\, \mathrm{d}t = \int_0^t \int_\Omega \xi \cdot \varphi \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{8}$$

Note that when $u^m = 0$ or $\Delta u^m = 0$, operator Lu degenerates, so the following result of well-posedness for variation-inequality (1) can be obtained by the regular method, and we consider the following regular problem:

$$\begin{cases} L_\varepsilon u_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_0), & (x,t) \in Q_T, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \\ u_\varepsilon(x,t) = \varepsilon, & (x,t) \in \partial Q_T, \end{cases} \tag{9}$$

with $L_\varepsilon u_\varepsilon = \partial_t u_\varepsilon - u_\varepsilon^m \Delta\ ((|\Delta u_\varepsilon^m|^2 + \varepsilon)^{p-2/2} \Delta u_\varepsilon^m) - \gamma(|\Delta u_\varepsilon^m|^2 + \varepsilon)^{p-2/2} |\Delta u_\varepsilon^m|^2$, the penalty map β_ε: $[0, (\infty) \longrightarrow (-\infty, 0]$ satisfies $\beta_\varepsilon(\cdot) \in C^2(\mathrm{R})$,

$$\beta_\varepsilon(x) = 0, x \geq \varepsilon, \beta_\varepsilon(x) \in (-M_0, 0], \quad x \in [0, \varepsilon]. \tag{10}$$

It is noteworthy that $\beta_\varepsilon(\cdot)$ is used to describe inequality constraints, which transforms variation-inequality into parabolic regularized problem with local nonlinear term.

Denote the classical solution of problem (9) by u_ε, whose existence follows, for example, from Theorem 1.1 of [12], such that one can have the weak equation

$$\begin{aligned} &\int_\Omega \left(\partial_t u_\varepsilon \cdot \varphi + u_\varepsilon^m \left(|\Delta u_\varepsilon^m|^2 + \varepsilon\right)^{p-2/2} \Delta u_\varepsilon^m \Delta \varphi + (1-\gamma)\left(|\Delta u_\varepsilon^m|^2 + \varepsilon\right)^{p-2/2} |\Delta u_\varepsilon^m|^2 \varphi\right) \mathrm{d}x \\ &\quad = -\int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\varphi \mathrm{d}x - \int_\Omega h u_\varepsilon^\alpha \mathrm{d}x, \end{aligned} \tag{11}$$

with $\varphi \in C^1(\overline{\Omega}_T)$. Moreover, the solution of (9) from [10, 11] satisfies

$$u_{0\varepsilon} \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon, u_{\varepsilon_1} \leq u_{\varepsilon_2}, \quad \text{for } \varepsilon_1 \leq \varepsilon_2. \tag{12}$$

First, we obtain estimates, independent of, for u_ε and its derivatives. In doing so, we discretize the problem (9) in time domain $[0, T]$. Let $t_k = k\Delta t$ be the uniform partition on the domain $[0, T]$, where $\Delta t = T/N$, $k = 0, 1, \ldots, N$. No matter how small h is, we always choose the suitable h to make sure that $t = n\Delta t$. We denote $u_{\varepsilon,k} = u_\varepsilon(t_k)$ and introduce an approximate solution

$$\begin{aligned} &\omega_\varepsilon^{(N)}(x,t) = \sum_{k=1}^N \chi_k(t) u_{\varepsilon,k}, \\ &\chi_k(t) = \begin{cases} 1, & t \in ((k-1)h, kh], \\ 0, & t \notin ((k-1)h, kh]. \end{cases} \end{aligned} \tag{13}$$

Then, the classical theorem of elliptic equations (see Wu et al. [14]) ensures that problem

$$\frac{1}{\Delta t}\left(u_{\varepsilon,k} - u_{\varepsilon,k-1}\right) - u_{\varepsilon,k}^m \Delta\left(\left(|\Delta u_{\varepsilon,k}^m|^2 + \varepsilon\right)^{p-2/2} \Delta u_{\varepsilon,k}^m\right) - \gamma\left(|\Delta u_{\varepsilon,k}^m|^2 + \varepsilon\right)^{p-2/2} |\Delta u_{\varepsilon,k}^m|^2 = -\beta_\varepsilon\left(u_{\varepsilon,k} - u_0\right) - h u_{\varepsilon,k}^\alpha, \tag{14}$$

has a unique solution $u_{\varepsilon,k}$ for any $u_{\varepsilon,k-1}$, $k = 0, 1, \ldots, N$, and satisfies

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega \left(u_{\varepsilon,k} - u_{\varepsilon,k-1}\right)\varphi \mathrm{d}x + \int_\Omega u_{\varepsilon,k}^m \left(|\Delta u_{\varepsilon,k}^m|^2 + \varepsilon\right)^{p-2/2} \Delta u_{\varepsilon,k}^m \Delta \varphi \mathrm{d}x \\ &\quad + (1-\gamma)\int_\Omega \left(|\Delta u_{\varepsilon,k}^m|^2 + \varepsilon\right)^{p-2/2} |\Delta u_{\varepsilon,k}^m|^2 \varphi \mathrm{d}x - \int_\Omega h u_{\varepsilon,k}^\alpha \mathrm{d}x \\ &\quad = -\int_\Omega \beta_\varepsilon\left(u_{\varepsilon,k} - u_0\right)\varphi \mathrm{d}x. \end{aligned} \tag{15}$$

Next, we give some useful estimates for $u_{\varepsilon,k}$. Choosing $\varphi = u_{\varepsilon,k}$ in (15) gives

$$\frac{1}{\Delta t}\int_{\Omega}\left(u_{\varepsilon,k}-u_{\varepsilon,k-1}\right)u_{\varepsilon,k}\mathrm{d}x+(2-\gamma)\int_{\Omega}u_{\varepsilon,k}^{m}\left(\left|\Delta u_{\varepsilon,k}^{m}\right|^{2}+\varepsilon\right)^{p-2/2}\left|\Delta u_{\varepsilon,k}^{m}\right|^{2}\mathrm{d}x = -\int_{\Omega}\beta_{\varepsilon}\left(u_{\varepsilon,k}-u_{0}\right)u_{\varepsilon,k}\mathrm{d}x-\int_{\Omega}hu_{\varepsilon,k}^{\alpha}\mathrm{d}x. \tag{16}$$

Applying inequality $u_{\varepsilon,k}u_{\varepsilon,k-1}\leq 1/2u_{\varepsilon,k}^{2}+1/2u_{\varepsilon,k-1}^{2}$, one can infer that

$$\frac{1}{2\Delta t}\int_{\Omega}u_{\varepsilon,k}^{2}\mathrm{d}x+(2-\gamma)\int_{\Omega}u_{\varepsilon,k}^{m}\left(\left|\Delta u_{\varepsilon,k}^{m}\right|^{2}+\varepsilon\right)^{p-2/2}\left|\Delta u_{\varepsilon,k}^{m}\right|^{2}\mathrm{d}x \leq -\int_{\Omega}\beta_{\varepsilon}\left(u_{\varepsilon,k}-u_{0}\right)u_{\varepsilon,k}\mathrm{d}x-\int_{\Omega}hu_{\varepsilon,k}^{\alpha}\mathrm{d}x+\frac{1}{2\Delta t}\int_{\Omega}u_{\varepsilon,k-1}^{2}\mathrm{d}x. \tag{17}$$

Recall that $t=n\Delta t$. We sum (17) from $k=1$ to $k=n$,

$$\frac{1}{2}\int_{\Omega}u_{\varepsilon,k}^{2}\mathrm{d}x+(2-\gamma)\int_{\Omega}\Delta t\sum_{k=1}^{n}u_{\varepsilon,k}^{m}\left(\left|\Delta u_{\varepsilon,k}^{m}\right|^{2}+\varepsilon\right)^{p-2/2}\left|\Delta u_{\varepsilon,k}^{m}\right|^{2}\mathrm{d}x \leq -\sum_{k=1}^{n}\Delta t\int_{\Omega}\beta_{\varepsilon}\left(u_{\varepsilon,k}-u_{0}\right)u_{\varepsilon,k}\mathrm{d}x-\sum_{k=1}^{n}\Delta t\int_{\Omega}hu_{\varepsilon,k}^{\alpha}\mathrm{d}x+\frac{1}{2}\int_{\Omega}u_{\varepsilon,0}^{2}\mathrm{d}x. \tag{18}$$

Pass the limit $\Delta t\longrightarrow 0$ and combine it with the definition of integral

$$\frac{1}{2}\int_{\Omega}u_{\varepsilon}^{2}\mathrm{d}x+(2-\gamma)\int_{0}^{t}\int_{\Omega}u_{\varepsilon}^{m}\left(\left|\Delta u_{\varepsilon}^{m}\right|^{2}+\varepsilon\right)^{p-2/2}\left|\Delta u_{\varepsilon}^{m}\right|^{2}\mathrm{d}x\,\mathrm{d}t \leq -\int_{0}^{t}\int_{\Omega}\beta_{\varepsilon}\left(u_{\varepsilon}-u_{0}\right)u_{\varepsilon}\mathrm{d}x-\int_{0}^{t}\int_{\Omega}hu_{\varepsilon,k}^{\alpha}\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_{\Omega}u_{\varepsilon,0}^{2}\mathrm{d}x. \tag{19}$$

Since $h(x,t)\geq 0,\forall (x,t)\in\Omega_{T}$, it follows from (10) and (12) that

$$\int_{0}^{t}\int_{\Omega}\beta_{\varepsilon}\left(u_{\varepsilon}-u_{0}\right)u_{\varepsilon}\mathrm{d}x\leq TM_{0}, \qquad \int_{0}^{t}\int_{\Omega}hu_{\varepsilon,k}^{\alpha}\mathrm{d}x\,\mathrm{d}t\geq 0. \tag{20}$$

Then auxiliary problem, dropping the nonnegative term $\int_{0}^{t}\int_{\Omega}hu_{\varepsilon,k}^{\alpha}\mathrm{d}x\,\mathrm{d}t$, has the following estimates:

$$\int_{\Omega}u_{\varepsilon}^{2}\mathrm{d}x\leq 2TM_{0}+\frac{1}{2}\int_{\Omega}u_{\varepsilon,0}^{2}\mathrm{d}x, \tag{21}$$

$$\int_{0}^{t}\int_{\Omega}u_{\varepsilon,k}^{m}\left(\left|\Delta u_{\varepsilon}^{m}\right|^{2}+\varepsilon\right)^{p-2/2}\left|\Delta u_{\varepsilon}^{m}\right|^{2}\mathrm{d}x\,\mathrm{d}t\leq 2TM_{0}+\frac{1}{2}\int_{\Omega}u_{\varepsilon,0}^{2}\mathrm{d}x, \tag{22}$$

$$\int_{0}^{t}\int_{\Omega}u_{\varepsilon,k}^{m}\left|\Delta u_{\varepsilon}^{m}\right|^{p}\mathrm{d}x\,\mathrm{d}t\leq 2TM_{0}+\frac{1}{2}\int_{\Omega}u_{\varepsilon,0}^{2}\mathrm{d}x. \tag{23}$$

Finally, we consider the time gradient estimation of u_{ε} and can prove it in a rather simple way. Indeed choosing $\varphi=\partial_{t}u_{\varepsilon}$ in (11) or taking $\varphi=u_{\varepsilon,k}-u_{\varepsilon,k-1}$ in (14), it follows from Holder and Young inequalities that

$$\frac{1}{2}\int_0^t\int_\Omega |\partial_s u_\varepsilon|^2 dx\, ds \le \frac{(2-\gamma)}{p}\int_\Omega u_{\varepsilon,0}^m |\Delta u_{\varepsilon,0}^m|^p dx + \frac{1}{2}T|\Omega|M_0. \tag{24}$$

3. Existence of a Generalized Solution

To discuss the existence of solutions, we need to use some weak limit results. Here, we give the details of the proof. (21)–(24) imply that, for any given $\varepsilon \in (0,1)$, the sequence $\{u_\varepsilon, \varepsilon \ge 0\}$ contains a subsequence (for the sake of simplicity, still denoted by itself) and three functions u, I_1, and I_2, such that

$$u_\varepsilon \longrightarrow u \text{ a.e. in } \Omega_T, \quad \text{as } \varepsilon \longrightarrow 0, \tag{25}$$

$$u_\varepsilon^m \overset{\text{weak}}{\longrightarrow} u^m \text{ in } L^\infty\left(0,T;W_0^{2,p}(\Omega)\right), \quad \text{as } \varepsilon \longrightarrow 0, \tag{26}$$

$$\partial_t u_\varepsilon \overset{\text{weak}}{\longrightarrow} \partial_t u \text{ in } L^2(\Omega_T), \quad \text{as } \varepsilon \longrightarrow 0, \tag{27}$$

$$u_\varepsilon^m\left(|\Delta u_\varepsilon^m|^2 + \varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m \overset{\text{weak}}{\longrightarrow} I_1, \quad \text{as } \varepsilon \longrightarrow 0, \tag{28}$$

$$\left(|\Delta u_\varepsilon^m|^2 + \varepsilon\right)^{p-2/2}|\Delta u_\varepsilon^m|^2 \overset{\text{weak}}{\longrightarrow} I_2, \quad \text{as } \varepsilon \longrightarrow 0. \tag{29}$$

By (12), we derive that $u_\varepsilon \le u$, a.e. in Ω_T, so combining with (10) gives

$$\beta_\varepsilon(u_\varepsilon - u_0) \longrightarrow \xi \text{ a.e. in } \Omega_T, \quad \text{as } \varepsilon \longrightarrow 0. \tag{30}$$

Letting $\varepsilon \longrightarrow 0$ in (11) yields

$$\int_\Omega (\partial_t u\cdot\varphi + I_1\Delta\varphi + (1-\gamma)I_2\varphi)dx = \int_\Omega \xi\cdot\varphi dx. \tag{31}$$

Define $\varphi = (u_\varepsilon - u)\Phi$ with $\Phi \in W^{2,p}(\Omega_T)$, and subtract (11) and (31), so

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left(\partial_t(u_\varepsilon - u)^2 dx + \int_\Omega\left[\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m - I_1\right]\Delta(u_\varepsilon - u)dx\right.\\
&\quad + \int_\Omega\left[\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m - I_1\right]\Delta\Phi\cdot(u_\varepsilon - u)dx\\
&\quad + (1-\gamma)\int_\Omega\left[\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}|\Delta u_\varepsilon^m|^2 - I_2\right]\varphi dx\\
&\le \int_\Omega [\beta_\varepsilon(u_\varepsilon - u_0) + \xi]\varphi dx.
\end{aligned} \tag{32}$$

By the boundary condition in (9), it is easy to verify that

$$\begin{aligned}
&\int_0^t\int_\Omega\left(\partial_t(u_\varepsilon - u)^2 dx\right.\\
&= [u_\varepsilon(\cdot,t) - u(\cdot,t)]^2 - [u_\varepsilon(\cdot,0) - u(\cdot,0)]^2 = [u_\varepsilon(\cdot,t) - u(\cdot,t)]^2 - \varepsilon^2.
\end{aligned} \tag{33}$$

It follows from (25) that as $\varepsilon \longrightarrow 0$,

$$\begin{aligned}
&\int_0^t\int_\Omega\left(\partial_t(u_\varepsilon - u)^2 dx \longrightarrow 0,\right.\\
&\int_\Omega\left[\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m - I_1\right]\Delta\Phi\cdot(u_\varepsilon - u)dx \longrightarrow 0,\\
&\int_\Omega\left[\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}|\Delta u_\varepsilon^m|^2 - I_2\right]\varphi dx \longrightarrow 0,\\
&\int_0^t\int_\Omega h(x,t)(u_\varepsilon^\alpha - u^\alpha)dx \longrightarrow 0,\\
&\int_\Omega [\beta_\varepsilon(u_\varepsilon - u_0) + \xi]\varphi dx \longrightarrow 0.
\end{aligned} \tag{34}$$

Combining above, it is clear to see that

$$\lim_{\varepsilon\longrightarrow\infty}\int_\Omega \Phi\cdot\left[\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m - I_1\right]\Delta(u_\varepsilon - u)dx = 0. \tag{35}$$

On the contrary, it follows from [12] that

$$\begin{aligned}
&\left(\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m - |\Delta u^m|^{p-2}\Delta u^m\right)\Delta(u_\varepsilon^m - u^m)\\
&\quad\ge \left(|\Delta u_\varepsilon^m|^{p-2}\Delta u_\varepsilon^m - |\Delta u^m|^{p-2}\Delta u^m\right)\Delta(u_\varepsilon^m - u^m)\\
&\quad\ge C(p)|\Delta u_\varepsilon^m - \Delta u^m|^p \ge 0.
\end{aligned} \tag{36}$$

Note that $\operatorname{sgn}(\Delta(u_\varepsilon^m - u^m)) = \operatorname{sgn}(\Delta(u_\varepsilon - u))$, then

$$\left(\left(|\Delta u_\varepsilon^m|^2+\varepsilon\right)^{p-2/2}\Delta u_\varepsilon^m - |\Delta u^m|^{p-2}\Delta u^m\right)\Delta(u_\varepsilon - u) \ge 0. \tag{37}$$

Subtracting (37) from (35), one can find that

$$\lim_{\varepsilon \longrightarrow \infty} \int_{\Omega} \Phi \cdot \left[I_1 - \left|\Delta u^m\right|^{p-2} \Delta u^m\right] \Delta\left(u_\varepsilon - u\right) \mathrm{d}x \leq 0. \quad (38)$$

Obliviously, if we subtract both hand side of (35) by $((|\Delta u_\varepsilon^m|^2 + \varepsilon)^{p-2/2} \Delta u_\varepsilon^m - |\Delta u^m|^{p-2} \Delta u^m) \Delta(u_\varepsilon - u)$, it is clear to find the invited inequality, such that

$$\lim_{\varepsilon \longrightarrow \infty} \int_{\Omega} \Phi \cdot \left[I_1 - \left|\Delta u^m\right|^{p-2} \Delta u^m\right] \Delta\left(u_\varepsilon - u\right) \mathrm{d}x = 0. \quad (39)$$

By the arbitrariness of Φ, one can establish the following lemma.

Lemma 2. *$I_1 = |\Delta u^m|^{p-2} \Delta u^m$ a.e. in Ω_T.*

Furthermore, taking $\Phi = 1$, employing Holder inequality, and joining with (32), (36), and (37), it can be easily verified that

$$\int_{\Omega} \left|\Delta u_\varepsilon^m - \Delta u^m\right|^p \mathrm{d}x \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0, \quad (40)$$

for any given $t \in (0, T)$. For detailed proof, refer to [14]. Further, we consider the fact $(|\alpha|^2 + \varepsilon)^{p-2/2} |\alpha|^2 \longrightarrow |\alpha|^p$ as $\varepsilon \longrightarrow 0$, then combining with (40), we have the following result.

Lemma 3. *If $\varepsilon \longrightarrow 0$, $(|\Delta u_\varepsilon^m|^2 + \varepsilon)^{p-2/2} |\Delta u_\varepsilon^m|^2$ converges to $|\Delta u^m|^p$ with norm $L^1(\Omega)$, that is to say $I_2 = |\Delta u^m|^p$ a.e. in Ω_T.*

Further, analyze the limit of $\beta_\varepsilon(u_\varepsilon - u_0)$. From (10) and (12), boundedness of $\{\beta_\varepsilon(u_\varepsilon - u_0), \varepsilon \geq 0\}$ gives

$$\beta_\varepsilon\left(u_\varepsilon - u_0\right) \longrightarrow \xi, \quad \text{as } \varepsilon \longrightarrow 0. \quad (41)$$

Here, $\xi \in G$, we will show it later in two cases: $u_\varepsilon \geq u_0 + \varepsilon$ and $0 < u_\varepsilon < u_0 + \varepsilon$. If $u_\varepsilon \geq u_0 + \varepsilon$, we have $\beta_\varepsilon(u_\varepsilon - u_0) = 0$, so together with (7) it gives

$$\xi(x, t) = 0 \Longleftrightarrow u > u_0. \quad (42)$$

If $u_0 \leq u_\varepsilon < u_0 + \varepsilon$, one gets $0 \geq \beta_\varepsilon(0) \geq -M_0$. Passing the limit $\varepsilon \longrightarrow 0$, $\xi(x, t) \in [0, M_0]$. Combining above, it is inferred that

$$-\beta_\varepsilon\left(u_\varepsilon - u_0\right) \longrightarrow \xi \in G, \quad \text{as } \varepsilon \longrightarrow 0, \quad (43)$$

for all $(x, t) \in \Omega_T$. Following a standard limit method, one can show the existence of generalized solution and summarize the following theorem.

Theorem 4. *If $h(x, t) \geq 0, \forall (x, t) \in \Omega_T$, parameter γ given by (1) satisfies $\gamma \leq 1$, then there exists a solution (u, ξ) in the sense of Definition 1 and it satisfies*

$$u \in L^\infty\left(0, T, W^{2,p}(\Omega)\right), \partial_t u \in L^2\left(\Omega_T\right), \xi \in G\left(u - u_0\right). \quad (44)$$

Note that the condition $\gamma \leq 1$ is used in Section 2, so $\gamma \leq 1$ is also required in Theorem 4.

4. Blow-Up of a Generalized Solution

This section is devoted to give the proof of blow-up of solution with the condition $h(x, t) \geq 0, \forall (x, t) \in \Omega_T$ and $\gamma > 1$. In doing so, we introduce the following function:

$$E(t) = \int_{\Omega} u(x, t) \mathrm{d}x. \quad (45)$$

From Definition 1, we know that $(t) \geq 0$. Choosing $\varphi = u^m / u^m + \varepsilon$ in generalized equation gives

$$\begin{aligned}
&\int_{\Omega} \partial_t u \cdot \frac{u^m}{u^m + \varepsilon} + \left|\Delta u^m\right|^p \frac{u^m}{u^m + \varepsilon} - u^{2m} \left|\Delta u^m\right|^p \Delta u^m \frac{1}{\left(u^m + \varepsilon\right)} \mathrm{d}x \\
&\quad + (1 - \gamma) \int_{\Omega} \left|\Delta u^m\right|^p \frac{u^m}{u^m + \varepsilon} \mathrm{d}x = \int_{\Omega} \xi \cdot \varphi \mathrm{d}x - \int_{\Omega} h(x, t) \frac{u^m}{u^m + \varepsilon} \mathrm{d}x, \\
&\text{or} \int_{\Omega} \partial_t u \cdot \frac{u^m}{u^m + \varepsilon} - \frac{u^{2m}}{\left(u^m + \varepsilon\right)^2} \left|\Delta u^m\right|^p \mathrm{d}x + (2 - \gamma) \int_{\Omega} \left|\Delta u^m\right|^p \frac{u^m}{u^m + \varepsilon} \mathrm{d}x = \int_{\Omega} \xi \cdot \frac{u^m}{u^m + \varepsilon} \mathrm{d}x - \int_{\Omega} h(x, t) \frac{u^m}{u^m + \varepsilon} \mathrm{d}x.
\end{aligned} \quad (46)$$

From (25), $u \in L^\infty(0, T, W^{2,p}(\Omega))$, as well as $\partial_t u \in L^2(\Omega_T)$, it is clear to verify that

$$\int_{\Omega}\partial_t u\cdot\frac{u^m}{u^m+\varepsilon}\mathrm{d}x\longrightarrow\int_{\Omega}\partial_t u\mathrm{d}x,\quad\text{as }\varepsilon\longrightarrow 0,\tag{47}$$

$$\int_{\Omega}\frac{u^{2m}}{\left(u^m+\varepsilon\right)^2}\left|\Delta u^m\right|^p\mathrm{d}x\longrightarrow\int_{\Omega}\left|\Delta u^m\right|^p\mathrm{d}x,\quad\text{as }\varepsilon\longrightarrow 0,\tag{48}$$

$$\int_{\Omega}\left|\Delta u^m\right|^p\frac{u^m}{u^m+\varepsilon}\mathrm{d}x\longrightarrow\int_{\Omega}\left|\Delta u^m\right|^p\mathrm{d}x,\quad\text{as }\varepsilon\longrightarrow 0.\tag{49}$$

On the contrary, from (7), Definition 1, and the fact that $h(x,t)\le 0,\forall(x,t)\in\Omega_T$, we can infer $\int_{\Omega}\xi\cdot u/u+\varepsilon\mathrm{d}x\ge 0$ and $\int_{\Omega}h(x,t)u^m/u^m+\varepsilon\mathrm{d}x\ge 0$ which combining with (46)–(49) (note that $d/\mathrm{d}t\,E(t)=\int_{\Omega}\partial_t u\mathrm{d}x$), we see that

$$\frac{d}{\mathrm{d}t}E(t)\ge(\gamma-1)\int_{\Omega}\left|\Delta u^m\right|^p\mathrm{d}x.\tag{50}$$

Next, we reduce $\int_{\Omega}|\Delta u^m|^p\mathrm{d}x$ to make it a function of $E(t)$. Employing Sobolev inequality, one gets

$$\int_{\Omega}\left|\Delta u^m\right|^p\mathrm{d}x\ge C_{\text{sobolev}}^2\int_{\Omega}\left|u^{\mathrm{mp}}\right|dx.\tag{51}$$

Here, C_{sobolev} represents the Sobolev coefficient. According to [15], C_{sobolev} satisfies

$$C_{\text{sobolev}}=\pi^{-1/2}n^{-1/p}\left(\frac{p-1}{n-p}\right)^{1-1/p}\left(\frac{p-1}{\Gamma(n/p)-\Gamma(1+n-n/p)}\right)^{1/n}.\tag{52}$$

Using Holder inequality with parameter $(1/\mathrm{mp},\mathrm{mp}-1/\mathrm{mp})$, we infer for any $t\in(0,T]$ that

$$\int_{\Omega}|u|\mathrm{d}x\le\left(\int_{\Omega}|u|^{\mathrm{mp}}\mathrm{d}x\right)^{1/\mathrm{mp}}|\Omega|^{\mathrm{mp}-1/\mathrm{mp}},\tag{53}$$

that is to say

$$\int_{\Omega}|u|^{\mathrm{mp}}\mathrm{d}x\ge|\Omega|^{1-\mathrm{mp}}\left(\int_{\Omega}|u|\mathrm{d}x\right)^{\mathrm{mp}}=|\Omega|^{1-\mathrm{mp}}E(t)^{\mathrm{mp}}.\tag{54}$$

This, combining with (50) and (51) implies that

$$\frac{d}{\mathrm{d}t}E(t)\ge-C_{\text{sobolev}}^2(\gamma-1)|\Omega|^{1-\mathrm{mp}}E(t)^{\mathrm{mp}}.\tag{55}$$

Now, we analyze the above ordinary differential inequality and separate the variable to obtain

$$\frac{d}{\mathrm{d}t}E(t)^{1-\mathrm{mp}}\le-C_{\text{sobolev}}^2(\gamma-1)|\Omega|^{1-\mathrm{mp}}(\mathrm{mp}-1).\tag{56}$$

Integrating the above inequality over $(0,t]$ gives

$$E(t)^{1-\mathrm{mp}}\le E(0)^{1-\mathrm{mp}}-C_{\text{sobolev}}^2(\gamma-1)|\Omega|^{1-\mathrm{mp}}(\mathrm{mp}-1)t.\tag{57}$$

In this section, we use the condition $\mathrm{mp}-1\ge 0$, such that

$$E(t)\ge\frac{1}{\left(E(0)^{1-\mathrm{mp}}-C_{\text{sobolev}}^2(\gamma-1)|\Omega|^{1-\mathrm{mp}}(\mathrm{mp}-1)t\right)^{1/\mathrm{mp}-1}}.\tag{58}$$

This means that generalized solution (u,ξ) blows up at some finite time T^*, and T^* satisfies

$$T^*\le\frac{E(0)^{1-\mathrm{mp}}}{C_{\text{sobolev}}^2(\gamma-1)|\Omega|^{1-\mathrm{mp}}(\mathrm{mp}-1)}.\tag{59}$$

Since $-1\ge 0$, $E(T^*)^{1-\mathrm{mp}}=0$. Then integrating the value of (56) over $[t,T^*]$, it is inferred that

$$E(t)^{1-\mathrm{mp}}\le C_{\text{sobolev}}^{2/\mathrm{mp}-1}(\gamma-1)^{-1/\mathrm{mp}-1}|\Omega|^{-1}(\mathrm{mp}-1)^{-1/\mathrm{mp}-1}\left(T^*-t\right)^{-1/\mathrm{mp}-1}.\tag{60}$$

Theorem 5. *Assume that $mp-1\ge 0$. If $\gamma>1$, $h(x,t)\le 0,\forall(x,t)\in\Omega_T$, the generalized solution of variation-inequality (1) will blow up at finite time T^*, and the upper bound of blow-up time is given by (42). Moreover, the blow-up rate of the generalized solution can be estimated by (43).*

5. Conclusions

This paper concerns with the existence and nonexistence of weak solutions for a class of variation-inequality problems with the fourth-order non-Newtonian polytropic operator. The existence of weak solutions is analyzed under the condition $\gamma\le 1$ and $h(x,t)\ge 0,\forall(x,t)\in\Omega_T$. First of all, in order to solve the inequality constraints in problem (1), we introduce penalty functions $\beta_\varepsilon(\cdot)$ to construct auxiliary problems. The auxiliary problem approaches the variational inequality (1) as $\varepsilon\longrightarrow 0$. Thirdly, the limit of penalty term $\beta_\varepsilon(u_\varepsilon-u_0)$ is analyzed, and the existence of weak solution is given by combining it with the limit method of other literature [12].

When $\gamma>1$ and $h(x,t)\le 0,\forall(x,t)\in\Omega_T$, we consider the nonexistence of weak solutions. We introduce the function $E(t)=\int_{\Omega}u(x,t)\mathrm{d}x$ and choose $\varphi=u^m/u^m+\varepsilon$ in generalized equation to obtain an ordinary differential inequality. Finally, the existence of the blow-up phenomenon and the upper bound estimate of blow-up rate and blow-up point are obtained by using inequality transformation technique and Sobolev inequality.

The condition $mp - 1 \geq 0$ needs to be supplemented in (56), otherwise the inequality under (57) cannot be obtained, such that the blow-up phenomenon of the generalized solution cannot be verified. We will study those focuses in the future.

Data Availability

No datasets were generated or analyzed during this current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] X. Han and F. Yi, "An irreversible investment problem with demand on a finite horizon: the optimal investment boundary analysis," *Communications in Nonlinear Science and Numerical Simulation*, vol. 109, Article ID 106302, 2022.
[2] X. Han and F. Yi, "An irreversible investment problem with maintenance expenditure on a finite horizon: free boundary analysis," *Journal of Mathematical Analysis and Applications*, vol. 440, no. 2, pp. 597–623, 2016.
[3] X. Chen, Y. Chen, and F. Yi, "Parabolic variational inequality with parameter and gradient constraints," *Journal of Mathematical Analysis and Applications*, vol. 385, no. 2, pp. 928–946, 2012.
[4] C. O. Alves, L. M. Barros, and C. E. Torres Ledesma, "Existence of solution for a class of variational inequality in whole RN with critical growth," *Journal of Mathematical Analysis and Applications*, vol. 494, no. 2, Article ID 124672, 2021.
[5] Z. Li and W. Zhou, "Optimizing UHPC walls into trusses based on the solid isotropic material with penalization method," *Structures*, vol. 47, no. 2023, pp. 573–585, 2023.
[6] R. S. Gvalani and A. Schlichting, "Barriers of the McKean-Vlasov energy via a mountain pass theorem in the space of probability measures," *Journal of Functional Analysis*, vol. 279, no. 11, Article ID 108720, 2020.
[7] Y. Wang and C. Zhang, "Existence results of partial differential mixed variational inequalities without Lipschitz continuity," *Journal of Mathematical Analysis and Applications*, vol. 484, no. 1, Article ID 123710, 2020.
[8] Y. Bai, S. Migorski, and S. Zeng, "A class of generalized mixed variational-hemivariational inequalities I: existence and uniqueness results," *Computers and Mathematics with Applications*, vol. 79, no. 10, pp. 2897–2911, 2020.
[9] M. T. O. Pimenta, R. Servadei, and R. Servadei, "Some existence results for variational inequalities with nonlocal fractional operators," *Nonlinear Analysis*, vol. 189, Article ID 111561, 2019.
[10] J. Li and C. Bi, "Study of weak solutions of variational inequality systems with degenerate parabolic operators and quasilinear terms arising American option pricing problems," *AIMS Mathematics*, vol. 7, no. 11, pp. 19758–19769, 2022.
[11] T. Wu, "Some results for a variation-inequality problem with fourth order p(x)-Kirchhoff operator arising from options on fresh agricultural products," *AIMS Mathematics*, vol. 8, no. 3, pp. 6749–6762, 2023.
[12] W. Chen and T. Zhou, "Existence of solutions for p-Laplacian parabolic Kirchhoff equation," *Applied Mathematics Letters*, vol. 122, Article ID 107527, 2021.
[13] C. Guan, Z. Xu, and F. Yi, "A consumption-investment model with state-dependent lower bound constraint on consumption," *Journal of Mathematical Analysis and Applications*, vol. 516, no. 1, Article ID 126511, 2022.
[14] Z. Wu, J. Zhao, and H. Li, *Nonlinear Diffusion Equations*, World Scientific Publishing, Singapore, 2001.
[15] G. Talenti, "Best constant in Sobolev inequality," *Annali di Matematica Pura ed Applicata*, vol. 110, no. 1, pp. 353–372, 1976.