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Research Article

Singular Value and Matrix Norm Inequalities between Positive Semidefinite Matrices and Their Blocks

Feng Zhang [1], **Rong Ma** [1], **Chunwen Zhang** [2], and **Yuxin Cao** [2]

[1] *Department of Mathematics, Southeast University Chengxian College, Nanjing 210000, China*
[2] *Department of Electrical and Computer Engineering, Southeast University Chengxian College, Nanjing 210000, China*

Correspondence should be addressed to Feng Zhang; fzhang1024@163.com

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In this paper, we obtain some inequalities involving positive semidefinite 2×2 block matrices and their blocks.

1. Introduction

We denote by M_n the vector space of all complex $n \times n$ matrices. For $A \in M_n$, the conjugate transpose of A is denoted by A^*. The notation $A \geq 0$ is used to mean that A is positive semidefinite. If A is a Hermitian element of M_n, then we enumerate its eigenvalues as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The singular values of A are enumerated as $s_1(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^* A)^{1/2}$. Throughout this paper, we assume that H is the positive semidefinite block matrix in the form

$$H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix} \in M_{2n}, \tag{1}$$

where $M, K, N \in M_n$.

The block matrix $\begin{bmatrix} M & K^* \\ K & N \end{bmatrix}$, where $M, K, N \in M_n$, is positive partial transpose (i.e., PPT) if both $\begin{bmatrix} M & K^* \\ K & N \end{bmatrix}$ and $\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ are positive semidefinite.

For $1 \leq k \leq n$, the norm $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$ is called the Fan k-norm. The norm $\|A\|_{(1)}$ is called the spectral norm and the norm $\|A\|_{(n)}$ is called the trace norm. A norm $\|\cdot\|$ on M_n is called unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in M_n$ and any unitary $U, V \in M_n$. Clearly, the spectral norm and the trace norm are unitarily invariant. Recall that a unitarily invariant norm may be considered as defined on M_n for all orders n by the rule $\|A\| = \left\| \begin{matrix} A & 0 \\ 0 & 0 \end{matrix} \right\|$.

Positive semidefinite matrices partitioned into four blocks play important roles in matrix analysis [1–3] and quantum theory [4, 5]. The related inequalities aroused much interest and several applications were given [6–9]. Of these, the one germane to our discussion occurs in the paper of Ulukök [9]. Ulukök in [9] obtained the following results.

Theorem 1

$$H^r \leq 2\left[\lambda_1(M) + \lambda_1(N)\right]^{r-1} (M \oplus N), \tag{2}$$

for $r \geq 1$.

Theorem 2

$$\lambda_j\left(H^r\right) \leq 2^{r-1}\left(\lambda_k^r(M \oplus N) + \lambda_{j-k+1}^r(|K| \oplus |K|)\right), \tag{3}$$

for $r \geq 1$, $j, k = 1, 2, \cdots, n$, *such that* $k \leq j$.

Theorem 3

$$\left\|H^r\right\|^2 \leq 4\left\|M^2 \oplus N^2\right\| \left(\|M \oplus 0\| + \|N \oplus 0\|\right)^{2r-2}, \tag{4}$$

for $r \geq 3/2$.

Theorem 4. *Let f be a nonnegative increasing continuous concave function on $[0,\infty)$. Then,*

$$\|f(H)\| \le \|f(M)\oplus f(N)\| + \|f(|K|)\oplus f(|K|)\|. \tag{5}$$

Theorem 5. *Let f be a nonnegative increasing continuous convex function on $[0,\infty)$. Then,*

$$\|f(H)\| \le \frac{1}{2}\left(\|f(2M\oplus 2N)\| + \|f(2|K|\oplus 2|K|)\|\right). \tag{6}$$

One of the questions that arise from Ulukök's work is the following. Are the conditions in every inequality essential? Furthermore, it is natural to ask whether stronger inequalities of (2)–(6) might be proved. This is the motivation for our study.

In this paper, we present a refinement of inequality (2) and a generalization of inequality (5). Next we derive a result related to inequality (3) and give a new proof of inequality (6). Additionally, we construct some counterexamples to show that the conditions in (2)–(6) are necessary.

2. Main Result

We begin our discussion with inequality (2). We firstly give an example to show inequality (2) is not always true for $0<r<1$.

Example 1. Take $r=1/2$ in inequality (2) and let $H=\begin{bmatrix}4&0&2&1\\0&4&1&2\\2&1&3&1\\1&2&1&3\end{bmatrix}$ with $M=\begin{bmatrix}4&0\\0&4\end{bmatrix}$, $N=\begin{bmatrix}3&1\\1&3\end{bmatrix}$, and $K=\begin{bmatrix}2&1\\1&2\end{bmatrix}$. Then,

$$\lambda_1(M)=\lambda_1(N)=4, M\oplus N=\begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&3&1\\0&0&1&3\end{bmatrix},$$
$$\lambda_4\left(2\left[\lambda_1(M)+\lambda_1(N)\right]^{-1/2}(M\oplus N)-H^{1/2}\right)\approx -0.065. \tag{7}$$

It is known that [6] $A\ge 0$ if and only if $A=A^*$ and $\lambda_j(A)\ge 0$. Using this, we see inequality (2) is not always true for $0<r<1$.

Next, we use block matrix technique to derive some inequalities related to 2×2 positive semidefinite matrices.

Theorem 6. *Let $H=\begin{bmatrix}M&K\\K^*&N\end{bmatrix}$ be PPT. Then,*

$$H^r \le 2\lambda_1^{r-1}(M+N)(M\oplus N), \tag{8}$$

for $r\ge 1$.

Proof. The idea of proof is similar to that in [9], Theorem 3.3. It suffices to show $\lambda_1(XX^*+YY^*)\le\lambda_1(X^*X+Y^*Y)$. In [7], Hiroshima proved that $\|A\|\le\|\sum_{j=1}^m A_{jj}\|$ for PPT matrix $A=[A_{ij}]_{i,j=1}^m$ and any unitarily invariant norm.

Let $H=[X,Y]^*[X,Y]$ be PPT, where X,Y are matrices with $2n$ rows and n columns. Using Hiroshima's result, we obtain $\lambda_1(H)=\lambda_1(XX^*+YY^*)\le\lambda_1(X^*X+Y^*Y)$. □

Example 2. Let $r=1/2$ and $H=\begin{bmatrix}4&0&1&1\\0&4&1&2\\1&1&3&1\\1&2&1&3\end{bmatrix}$ with $M=\begin{bmatrix}4&0\\0&4\end{bmatrix}$, $N=\begin{bmatrix}3&1\\1&3\end{bmatrix}$, $K=\begin{bmatrix}1&1\\1&2\end{bmatrix}$, and $j=k=1$ in inequality (3). A calculation shows that

$$\begin{aligned}\lambda_1\left(H^{1/2}\right)&\approx 2.5667,\\ \lambda_1^{1/2}(|K|\oplus|K|)&\approx 1.6180,\\ \lambda_1^{1/2}(M\oplus N)&=2.\end{aligned} \tag{9}$$

Hence,

$$\lambda_1\left(H^{1/2}\right)-2^{-1/2}\left(\lambda_1^{1/2}(M\oplus N)+\lambda_1^{1/2}(|K|\oplus|K|)\right)\approx 0.008. \tag{10}$$

Inequality (3) is violated in this case.

Lemma 7 (see [6]). *Let $A,B\ge 0$. Then,*

$$\lambda_j(A+B)\le\lambda_k(A)+\lambda_{j-k+1}(B), \tag{11}$$

for $j,k=1,2,\cdots,n$ such that $k\le j$.

Theorem 8. *Let $H=\begin{bmatrix}M&K\\K^*&N\end{bmatrix}$ be positive semidefinite and let $0\le r\le 1$. Then,*

$$\lambda_j(H^r)\le\lambda_k^r(M\oplus N)+\lambda_{j-k+1}^r(|K|+|K|), \tag{12}$$

for $j,k=1,2,\cdots,n$ such that $k\le j$.

Proof. Let $j,k\in\{1,2,\cdots,n\}$ such that $k\le j$. Then, by Lemma 7,

$$\begin{aligned}\lambda_j(H^r)&=\lambda_j^r(H)\\ &\le\lambda_j^r\left(\begin{pmatrix}M&0\\0&N\end{pmatrix}+\begin{pmatrix}|K^*|&0\\0&|K|\end{pmatrix}\right)\\ &\le\left(\lambda_k\begin{pmatrix}M&0\\0&N\end{pmatrix}+\lambda_{j-k+1}\begin{pmatrix}|K^*|&0\\0&|K|\end{pmatrix}\right)^r\\ &\le\lambda_k^r\begin{pmatrix}M&0\\0&N\end{pmatrix}+\lambda_{j-k+1}^r\begin{pmatrix}|K|&0\\0&|K|\end{pmatrix}.\end{aligned} \tag{13}$$

This completes the proof. □

Remark 9. Inequality (3) is a quick consequence of Theorem 8 by using the convexity of $x^r\ (r \geq 1)$.

Example 3. Let $r = 1/5$ and $H = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}$ with $M = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$, $N = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, and $K = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. By using MATLAB software to calculate, we have

$$\begin{aligned} M^2 \oplus N^2 &= \begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 10 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \\ \lambda_1(H) &\approx 5.1401, \\ \lambda_2(H) &\approx 2.5712, \\ \lambda_3(H) &\approx 1.5622, \\ \lambda_4(H) &\approx 0.7265, \\ \left(trH^{1/5}\right)^2 &\approx 21.4062,\ 4tr\left(M^2 \oplus N^2\right) \\ &\cdot \left(tr(M \oplus 0) + tr(N \oplus 0)\right)^{-8/5} \approx 2.8133. \end{aligned} \tag{14}$$

Hence, inequality (4) is violated for these matrices and the trace norm when $0 < r < 3/2$.

We give some unitarily invariant norm inequalities for positive semidefinite 2×2 block matrices. To achieve our goal, we need the following lemmas.

Lemma 10 (see [6]). *Let $A, B \in M_n$. Then, $\|A + B\| \leq \|A\| + \|B\|$.*

Let $x = (x_1, x_2, \cdots, x_n)$ be an element of R^n and $x^{\downarrow}$ be the vectors obtained by rearranging the coordinates of x in decreasing order.

Lemma 11 (see [6]). *Let $f(x)$ be an increasing convex function and $\sum_{i=1}^k x_i^{\downarrow} \leq \sum_{i=1}^k y_i^{\downarrow}\ (k = 1, 2, \cdots, n)$ with $x, y \in R^n$. Then,*

$$\sum_{i=1}^k f\left(x_i^{\downarrow}\right) \leq \sum_{i=1}^k f\left(y_i^{\downarrow}\right), \quad (k = 1, 2, \cdots, n). \tag{15}$$

Lemma 12 (see [10]). *Let $A, B \geq 0$ and let $f(t)$ be a nonnegative concave function on $[0, \infty)$. Then, for all unitarily invariant norms,*

$$\|f(A + B)\| \leq \|f(A) + f(B)\|. \tag{16}$$

Lemma 13 (see [11]). *Let $A, B \in M_n$ be positive semidefinite and let f be an increasing nonnegative continuous convex function on $[0, \infty)$. Then,*

$$\left\| f\left(\frac{A + B}{2}\right) \right\| \leq \frac{\|f(A) + f(B)\|}{2}. \tag{17}$$

Theorem 14. *Let $H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ be positive semidefinite and let f be a nonnegative concave function on $[0, \infty)$. Then,*

$$\|f(H)\| \leq \|f(M) \oplus f(N)\| + \|f(|K|) \oplus f(|K|)\|, \tag{18}$$

for all unitarily invariant norms.

Proof. We need only to prove the theorem when $f(0) = 0$, since the general case follows by a limit argument due to Lee [12].

Notice that for positive definite matrices, singular values and eigenvalues are the same. Since $H \leq \begin{pmatrix} M + |K^*| & 0 \\ 0 & N + |K| \end{pmatrix}$ and using the fact $0 \leq A \leq B$ includes that $\lambda_j(A) \leq \lambda_j(B)$ [1], we obtain

$$s_j(H) \leq s_j\left(\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} |K^*| & 0 \\ 0 & |K| \end{pmatrix}\right). \tag{19}$$

By Fan' s dominance principle [6], we obtain

$$\begin{aligned} \|f(H)\| &\leq \left\| f\left(\left(M + \left|K^*\right|\right) \oplus (N + |K|)\right) \right\| \\ &\leq \left\| f(M) \oplus f(N) + f\left(\left|K^*\right|\right) \oplus f(|K|) \right\| \\ &\leq \|f(M) \oplus f(N)\| + \left\| f\left(\left|K^*\right|\right) \oplus f(|K|) \right\| \\ &= \|f(M) \oplus f(N)\| + \|f(|K|) \oplus f(|K|)\|, \end{aligned} \tag{20}$$

where the first inequality follows from inequality (19) and the fact that $f(t)$ is nondecreasing, the second inequality is due to Lemma 12, and the third inequality follows from Lemma 10.

This completes the proof. □

Corollary 15. *Let f be a nonnegative increasing continuous concave function on $[0, \infty)$. Then,*

$$\|f(H)\| \leq \|f(M) \oplus f(N)\| + \|f(|K|) \oplus f(|K|)\|, \tag{21}$$

for all unitarily invariant norms.

Example 4. Let $H = \begin{bmatrix} 5 & 0 & 1 & -1 \\ 0 & 4 & -1 & 2 \\ 1 & -1 & 2 & -1 \\ -1 & 2 & -1 & 2 \end{bmatrix}$ with $M = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$, $N = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $K = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, and $f(x) = x^2$. Then,

$$\begin{aligned}\lambda_1(f(M)\oplus f(N)) &= 25,\\ \lambda_1(f(|K|)\oplus f(|K|)) &\approx 6.8541,\\ \lambda_1(f(H)) &\approx 40.8662,\\ \lambda_1(f(H))-(\lambda_1(f(M)\oplus f(N))+\lambda_1(f(|K|)\oplus f(|K|))) &\approx 9.0121,\end{aligned} \tag{22}$$

which shows inequality (5) is not always true without the condition that f is concave.

Finally, we give a new proof of inequality (6).

Theorem 16. *Let* $H=\begin{bmatrix} M & K\\ K^* & N\end{bmatrix}$ *be positive semidefinite and let* f *be a nonnegative increasing continuous convex function on* $[0,\infty)$. *Then,*

$$\|f(H)\|\le\frac{1}{2}(\|f(2M\oplus 2N)\|+\|f(2|K|\oplus 2|K|)\|). \tag{23}$$

Proof. An application of the polar decomposition reveals $\begin{bmatrix} 0 & K^*\\ K & 0\end{bmatrix}\le\begin{bmatrix} |K| & 0\\ 0 & |K^*|\end{bmatrix}$, and we see that

$$H\le\frac{1}{2}\left(\begin{bmatrix} 2M & 0\\ 0 & 2N\end{bmatrix}+\begin{bmatrix} 2|K^*| & 0\\ 0 & 2|K|\end{bmatrix}\right). \tag{24}$$

Using the fact $0\le A\le B$ implies that $\lambda_j(A)\le\lambda_j(B)$ [1], and we get

$$\sum_{j=1}^{k}\lambda_j(H)\le\frac{1}{2}\sum_{j=1}^{k}\lambda_j\left(\begin{bmatrix} 2M & 0\\ 0 & 2N\end{bmatrix}+\begin{bmatrix} 2|K^*| & 0\\ 0 & 2|K|\end{bmatrix}\right). \tag{25}$$

By the spectral mapping theorem, we have $f(\lambda_j(A))=\lambda_j(f(A))$ for $A\ge 0$. Since f is an increasing convex function, by Lemma 10, Lemma 11, and Lemma 13 and Fan' s dominance principle [6], we obtain

$$\sum_{j=1}^{k}\lambda_j(f(H))\le\sum_{j=1}^{k}\lambda_j\left(f\left(\frac{1}{2}\left(\begin{bmatrix} 2M & 0\\ 0 & 2N\end{bmatrix}+\begin{bmatrix} 2|K^*| & 0\\ 0 & 2|K|\end{bmatrix}\right)\right)\right) \tag{26}$$

$$\le\frac{1}{2}\sum_{j=1}^{k}\lambda_j\left(f\begin{bmatrix} 2M & 0\\ 0 & 2N\end{bmatrix}\right)+\frac{1}{2}\sum_{j=1}^{k}\lambda_j\left(f\begin{bmatrix} 2|K^*| & 0\\ 0 & 2|K|\end{bmatrix}\right). \tag{27}$$

Inequality (27) is equivalent to

$$\|f(H)\|\le\frac{1}{2}(\|f(2M\oplus 2N)\|+\|f(2|K|\oplus 2|K|)\|). \tag{28}$$

This completes the proof. ☐

Example 5. Let $H=\begin{bmatrix} 2 & 1 & 1 & -1\\ 1 & 4 & -1 & 1\\ 1 & -1 & 2 & 0\\ -1 & 1 & 0 & 4\end{bmatrix}$ with $M=\begin{bmatrix} 2 & 1\\ 1 & 4\end{bmatrix}$, $N=\begin{bmatrix} 2 & 0\\ 0 & 4\end{bmatrix}$, $K=\begin{bmatrix} 1 & -1\\ -1 & 1\end{bmatrix}$, and $f(x)=x^{1/5}$. As we see, f is concave. By computation,

$$\begin{aligned}\sum_{j=1}^{4}\lambda_j(f(H)) &\approx 4.6056,\\ \sum_{j=1}^{4}\lambda_j(f(2M\oplus 2N)) &\approx 5.6408,\\ \sum_{j=1}^{4}\lambda_j(f(2|K|\oplus 2|K|)) &\approx 2.6390.\end{aligned} \tag{29}$$

Hence,

$$\|f(H)\|>\frac{1}{2}(\|f(2M\oplus 2N)\|+\|f(2|K|\oplus 2|K|)\|), \tag{30}$$

for the trace norm, which shows that inequality (6) is not always true if f is not a convex function.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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