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Research Article

Horadam Spinors

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Spinors can be expressed as Lie algebra of infinitesimal rotations. Spinors are also defined as elements of a vector space which carries a linear representation of the Clifford algebra typically. The motivation for this study is to define a new and particular sequence. An essential feature of this sequence is that while a generalization is being made, spinors, which have a lot of use in physics, are used. This new sequence defined using spinor representations is called the Horadam spinor sequence; formulas such as the Binet formula, generating function formula, and Cassini formula are given. The Horadam spinors given in this study are a generalization of the spinor representations of Horadam quaternion sequences.

1. Introduction

Number sequences are quite remarkable in mathematics. The first number sequence that comes to mind is the number sequences obtained by Leonardo Fibonacci (1170–1250). There are many number sequences similar to Fibonacci number sequence written after the second term as the sum of the two preceding terms but defined by different initial values. Some of them, such as Lucas, Pell, modified Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers, were defined with different initial values [1–8]. Fibonacci and Lucas number sequences, which have found themselves in many fields from the past to the present, have been also associated with polynomials and quaternions. The association with polynomials was first given by Belgian mathematician Catalan and German mathematician Jacobsthal in 1883. In [4], the first example of this is the quaternions whose coefficients were formed by the terms of the Fibonacci number sequence. The same author also examined quaternion recurrence relations [5]. Today, many studies have been carried out on the generalizations of number sequences. On the other hand, in the studies [9, 10], besides Fibonacci and Lucas number sequences, there were quaternion sequences formed by taking coefficients from different number sequences. Iyer [11] also studied the relationship between Fibonacci and generalized Fibonacci quaternions. Also, the same author conducted studies on the relationship between Fibonacci and Lucas quaternions in [12]. In [13], Halıcı expressed the Fibonacci and Lucas quaternions. In addition to that, in [14], Polatlı studied on a new generalization of Fibonacci and Lucas quaternions and gave the sum formulas for these new generalized quaternions. Horadam number sequence was defined by Horadam [4]. Then, Haukkanen gave information about linear compositions and generator functions of Horadam sequences in [15]. Moreover, in [16, 17], the new families of Horadam numbers were given. Çimen and İpek introduced a new quaternion sequence, Pell and Pell–Lucas quaternions in [18]. In [19], Flaut and Shpakivskyi studied generalized Fibonacci and Fibonacci–Narayana quaternions. After that, Polatlı and Kesim obtained binomial sum formulas by using Binet formulas in their studies [20]. Halıcı gave the complex Fibonacci quaternions in [21]. Halıcı and Karataş [22] introduced Horadam quaternions, which are a generalization of previously defined quaternion sequences. In the same study, they were interested in the Binet formula, Cassini identity, sum formulas, and norm value of this quaternion sequence. In [23], the Fibonacci generalized quaternions were obtained.

Spinors can be defined, with a simple definition, as vectors of a space whose transformations are related in a particular way to rotations in physical space. Cartan was the first to introduce spinors in geometric meaning [24].

Cartan's study [24] is a remarkable reference to the geometry of spinors because it gave spinor representations of the most basic geometric expressions. Another important study giving spinor representation of motion geometry was given by Vivarelli [25]. In Vivarelli's study [25], a relationship between quaternions and spinors was introduced, and by considering the relationship between quaternions and rotations in 3-dimensional Euclidean space, the spinor representations of these 3-dimensional rotations were obtained. In a study of Del Castillo and Barrales, the spinor representation of curve theory was expressed [26]. Also, in this study, the spinor formulations of Frenet frame and curvatures of a curve in 3-dimensional Euclidean space were given [26]. On the other hand, Kişi and Tosun gave the spinor representation of the Darboux frame in 3-dimensional Euclidean space [27]. In addition, in [28], the spinor equations of the Bishop frame of curves in 3-dimensional Euclidean space were expressed. Then, the hyperbolic spinors were introduced and the hyperbolic spinor representation of a given curve in Minkowski space was expressed [29]. Therefore, many studies for hyperbolic spinors were obtained [30–32]. In addition to these studies, Erişir and Güngör defined Fibonacci spinors, which are spinor representations of Fibonacci and Lucas quaternion sequences [33]. In addition, spinor expressions of Pell and Pell–Lucas quaternions were given in [34]. Apart from these, Jacobsthal spinors were expressed in [35]. Horadam spinors given in this study are a generalization of the spinor representations of all these quaternion sequences.

2. Preliminaries

In this section, the spinors, the relationship between the real quaternions and spinors, and the Horadam quaternions are introduced.

Now, the spinors given by Cartan [24] are given in geometric meaning. Let any isotropic vector be $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{C}^3$, and the three-dimensional complex vector space is $\mathbf{C}^3$, where $v_1{}^2 + v_2{}^2 + v_3{}^2 = 0$. Therefore, the set of isotropic vectors in the complex vector space $\mathbf{C}^3$ forms a two-dimensional surface in the complex space $\mathbf{C}^2$. Let this two-dimensional surface be any surface parameterized by coordinates η_1 and η_2, then $v_1 = \eta_1{}^2 - \eta_2{}^2$, $v_2 = \mathbf{i}(\eta_1{}^2 + \eta_2{}^2)$, and $v_3 = -2\eta_1\eta_2$ are obtained. Moreover, equations $\eta_1 = \pm\sqrt{v_1 - \mathbf{i}v_2/2}$ and $\eta_2 = \pm\sqrt{-v_1 - \mathbf{i}v_2/2}$ are satisfied. Therefore, it is known that every isotropic vector in $\mathbf{C}^3$ corresponds to two vectors, (η_1, η_2) and $(-\eta_1, -\eta_2) \in \mathbf{C}^2$. On the other hand, these vectors correspond to the same isotropic vector $\mathbf{v}$. Two-dimensional complex vectors mentioned above are called as spinor by Cartan such that

$$\eta = (\eta_1, \eta_2) = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \tag{1}$$

in spinor space $\mathbf{S}$ [24].

Let any real quaternion be $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, where $q_0, q_1, q_2, q_3 \in \mathbf{R}$. $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is called the quaternion basis such that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \tag{2}$$

as given in [36]. $q_0 = S_q$ and $\mathbf{V}_q = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ are called scalar and vector parts of the real quaternion q, respectively. This real quaternion can be written as $q = S_q + \mathbf{V}_q$ [36]. In addition to that, suppose that two any real quaternion $p = S_p + \mathbf{V}_p, q = S_q + \mathbf{V}_q$. Hence, the quaternion product of these quaternions is as follows:

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q, \tag{3}$$

where $\langle , \rangle$ is the inner product and $\wedge$ is the vector product in $\mathbf{R}^3$ [36]. It is known that the product of two real quaternions is noncommutative. On the other hand, the quaternion conjugate and the norm of the real quaternion q are defined as $q^* = S_q - \mathbf{V}_q$ and

$$N(q) = \sqrt{q_1{}^2 + q_2{}^2 + q_3{}^2 + q_4{}^2}. \tag{4}$$

Let the norm of q be $N(q) = 1$; therefore, q is defined as unit quaternion [36].

Vivarelli gave a relationship between quaternions and spinors such that

$$\begin{aligned} f\colon \mathbf{H}; &\longrightarrow \mathbf{S} \\ q; &\longrightarrow f(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) \cong \begin{bmatrix} q_3 + \mathbf{i}q_0 \\ q_1 + \mathbf{i}q_2 \end{bmatrix} \equiv \eta, \end{aligned} \tag{5}$$

where $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is any real quaternion [25]. Moreover, Vivarelli expressed a spinor representation of the quaternion product $q \times p$ such that

$$q \times p \longrightarrow -\mathbf{i}\widehat{\eta}\rho, \tag{6}$$

where the spinor ρ corresponds to the quaternion p with the aid of the transformation f in equation (5), and the complex, unitary, square matrix $\widehat{\eta}$ can be written as follows:

$$\hat{\eta} = \begin{bmatrix} q_3 + \mathbf{i}q_0 & q_1 - \mathbf{i}q_2 \\ q_1 + \mathbf{i}q_2 & -q_3 + \mathbf{i}q_0 \end{bmatrix}, \tag{7}$$

as given in [25]. Moreover, the spinor matrix $\eta_L = -\mathbf{i}\widehat{\eta}$, namely,

$$\eta_L = \begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix}, \tag{8}$$

was called the left Hamilton spinor matrix or fundamental spinor matrix of the real quaternion q [37].

Now, the some equalities about the Horadam quaternions given in [22] can be obtained.

For $n \geq 2$, the nth Horadam quaternion is defined as

$$Q_{w,n} = W_n + \mathbf{i}W_{n+1} + \mathbf{j}W_{n+2} + \mathbf{k}W_{n+3}, \tag{9}$$

where the nth Horadam number $W_n = W_n(a,b;p,q) = pW_{n-1} + qW_{n-2}$ and $W_0 = a$ and $W_1 = b$. Therefore, the recurrence relation of the Horadam quaternions is as follows:

$$Q_{w,n} = pQ_{w,n-1} + qQ_{w,n-2}, \tag{10}$$

with initial conditions

$$\begin{aligned} Q_{w,0} &= a + \mathbf{i}b + \mathbf{j}(pb + qa) + \mathbf{k}\left(p^2b + pqa + qb\right), \\ Q_{w,1}; &= b + \mathbf{i}(pb + qa) + \mathbf{j}\left(p^2b + pqa + qb\right) + \mathbf{k}\left(p^3b + p^2qa + 2pqb + q^2a\right), \end{aligned} \tag{11}$$

as given in [22]. Moreover, Binet's formula for the Horadam quaternions is given as follows:

$$Q_{w,n} = \frac{1}{\alpha - \beta}\left(A\,\underset{\sim}{\alpha}\,\alpha^n - B\,\underset{\sim}{\beta}\,\beta^n\right), \tag{12}$$

where the quaternions $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are $\underset{\sim}{\alpha} = 1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$ and $\underset{\sim}{\beta} = 1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$, and $\alpha = (p + \sqrt{p^2 + 4q}/2)$ and $\beta = (p - \sqrt{p^2 + 4q}/2)$ are roots of the characteristic equation $\{x^2\} - px - q = 0$, and $A = b - a\beta$ and $B = b - a\alpha$ [22]. On the other hand, the generating function of the Horadam quaternions is found such that

$$g(t) = \frac{Q_{w,0} + \left(Q_{w,1} - pQ_{w,0}\right)t}{1 - pt - qt^2}, \tag{13}$$

and the Cassini formula for the Horadam quaternions is obtained as follows:

$$Q_{w,n-1}Q_{w,n+1} - Q_{w,n}^2 = \frac{AB\alpha^{n-1}\beta^{n-1}}{\alpha - \beta}\left(\beta\,\underset{\sim}{\alpha}\,\underset{\sim}{\beta} - \alpha\,\underset{\sim}{\beta}\,\underset{\sim}{\alpha}\right), \tag{14}$$

as given in [22]. Consequently, the sum formula of the Horadam quaternions is given by the following equation:

$$\sum_{k=0}^{n} Q_{w,k} = \frac{1}{\alpha - \beta}\left(\frac{B\,\underset{\sim}{\beta}\,\beta^{n+1}}{1 - \beta} - \frac{A\,\underset{\sim}{\alpha}\,\alpha^{n+1}}{1 - \alpha}\right) + K, \tag{15}$$

where

$$K = \frac{(a + b - pa) + \mathbf{i}(b + qa) + \mathbf{j}(pb + qa + qb) + \mathbf{k}\left(\left(p^2 + q\right)b + pq(a + b) + q^2a\right)}{(1 - p - q)}, \tag{16}$$

as given in [22].

3. Main Theorems and Proofs

It is known that there is a spinor for every real quaternion by means of the transformation f in equation (5). With the help of this information in this study, a new transformation between Horadam quaternions and spinors is defined and the spinors corresponding to Horadam quaternions are given. Therefore, these spinors associated with Horadam quaternions are called as Horadam spinors. Then, some formulas, such as Binet, Cassini, sum formulas, and generating functions for Horadam spinors and theorems, are given.

Definition 1. Let $Q_{w,n} = W_n + \mathbf{i}W_{n+1} + \mathbf{j}W_{n+2} + \mathbf{k}W_{n+3}$ be nth Horadam quaternion, where W_n is nth Horadam number and the set of Horadam quaternions be $\mathbf{Q_w}$. Therefore, considering the linear transformation between the spinors and quaternions in equation (5) the following linear transformation:

$$\begin{aligned} &f_w\colon \mathbf{Q_w} \longrightarrow \mathbf{S}, \\ &Q_{w,n} \longrightarrow f_w\left(W_n + \mathbf{i}W_{n+1} + \mathbf{j}W_{n+2} + \mathbf{k}W_{n+3}\right) \cong S_{w,n} = \begin{bmatrix} W_{n+3} + \mathbf{i}W_n \\ W_{n+1} + \mathbf{i}W_{n+2} \end{bmatrix}, \end{aligned} \tag{17}$$

can be defined, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ coincide with basis vectors given for Horadam quaternions and $\mathbf{i}^2 = -1$. Therefore, a new sequence for the spinors related with Horadam quaternions is introduced, and this sequence is called as "Horadam spinor sequence." The set of this Horadam spinor sequence is defined as follows:

$$\left\{S_{w,n}\right\}_{n\in\mathbf{N}}^{\infty} = \left\{\begin{bmatrix} \left(p^2b+pqa+qb\right)+\mathbf{i}a \\ b+\mathbf{i}(pb+qa) \end{bmatrix}, \begin{bmatrix} \left(p^3b+p^2qa+2pqb+q^2a\right)+\mathbf{i}b \\ (pb+qa)+\mathbf{i}\left(p^2b+pqa+qb\right) \end{bmatrix}, \ldots\ldots, \begin{bmatrix} W_{n+3}+\mathbf{i}W_n \\ W_{n+1}+\mathbf{i}W_{n+2} \end{bmatrix}, \ldots\ldots\right\}, \quad (18)$$

where $S_{w,n} = \begin{bmatrix} W_{n+3}+\mathbf{i}W_n \\ W_{n+1}+\mathbf{i}W_{n+2} \end{bmatrix}$ is nth Horadam spinor and W_n is nth Horadam number.

Now, the recurrence relation of Horadam spinor sequence with the following equations can be obtained.

Let the (n)th and $(n+1)$th Horadam spinors be $S_{w,n}$ and $S_{w,n+1}$, respectively. In this case, for the Horadam spinor $pS_{w,n+1}+qS_{w,n}$, the following equation can be written:

$$\begin{aligned} pS_{w,n+1}+qS_{w,n} &= p\begin{bmatrix} W_{n+4}+\mathbf{i}W_{n+1} \\ W_{n+2}+\mathbf{i}W_{n+3} \end{bmatrix} + q\begin{bmatrix} W_{n+3}+\mathbf{i}W_n \\ W_{n+1}+\mathbf{i}W_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} pW_{n+4}+qW_{n+3}+\mathbf{i}\left(pW_{n+1}+qW_n\right) \\ pW_{n+2}+qW_{n+1}+\mathbf{i}\left(pW_{n+3}+qW_{n+2}\right) \end{bmatrix}. \end{aligned} \quad (19)$$

It is known that the recurrence relation of the Horadam number sequence is $W_n = pW_{n-1}+qW_{n-2}$, where $n\geq 2$ and with initial conditions $W_0 = a, W_1 = b$. Therefore, the equation

$$pS_{w,n+1}+qS_{w,n} = \begin{bmatrix} W_{n+5}+\mathbf{i}W_{n+2} \\ W_{n+3}+\mathbf{i}W_{n+4} \end{bmatrix} = S_{w,n+2}, \quad (20)$$

is obtained, and there is the recurrence relation $S_{w,n+2} = pS_{w,n+1}+qS_{w,n}$ for the Horadam spinors, where considering the transformation f_w in equation (17), the initial condition for $n=0$ is as follows:

$$S_{w,0} = \begin{bmatrix} \left(p^2b+pqa+qb\right)+\mathbf{i}a \\ b+\mathbf{i}(pb+qa) \end{bmatrix}, \quad (21)$$

and the initial condition for $n=1$ is as follows:

$$S_{w,1} = \begin{bmatrix} \left(p^3b+p^2qa+2pqb+q^2a\right)+\mathbf{i}b \\ (pb+qa)+\mathbf{i}\left(p^2b+pqa+qb\right) \end{bmatrix}. \quad (22)$$

Consequently, we can give the following definition.

Definition 2. Assume that $n\geq 0$ and $n\in Z$ and the $(n+1)th$, $(n+2)th$ Horadam spinors are $S_{w,n+1}$ and $S_{w,n+2}$, respectively. Then, the recurrence relation of this Horadam spinor is as follows:

$$S_{w,n+2} = pS_{w,n+1}+qS_{w,n}, \quad (23)$$

with initial conditions

$$\begin{aligned} S_{w,0} &= \begin{bmatrix} \left(p^2b+pqa+qb\right)+\mathbf{i}a \\ b+\mathbf{i}(pb+qa) \end{bmatrix}, \\ S_{w,1} &= \begin{bmatrix} \left(p^3b+p^2qa+2pqb+q^2a\right)+\mathbf{i}b \\ (pb+qa)+\mathbf{i}\left(p^2b+pqa+qb\right) \end{bmatrix}. \end{aligned} \quad (24)$$

Considering the Horadam spinors, Table 1 can be given.

Proposition 3. *Let the nth Horadam spinor $S_{w,n}$ be the spinor corresponding to the nth Horadam quaternion $Q_{w,n}$. In this case, the Horadam spinor representation of the quaternion norm is as follows:*

$$N^2\left(Q_{w,n}\right) = \overline{S_{w,n}}^{t} S_{w,n}. \quad (25)$$

Proof. Assume that nth Horadam spinor $S_{w,n}$ corresponds to the nth Horadam quaternion $Q_{w,n}$. Therefore, considering the transformation f_w in equation (17), the following equation can be obtained as:

$$\overline{S_{w,n}}^{t} S_{w,n} = \begin{bmatrix} W_{n+3}-\mathbf{i}W_n & W_{n+1}-\mathbf{i}W_{n+2} \end{bmatrix} \begin{bmatrix} W_{n+3}+\mathbf{i}W_n \\ W_{n+1}+\mathbf{i}W_{n+2} \end{bmatrix} = {W_n}^2+{W_{n+1}}^2+{W_{n+2}}^2+{W_{n+3}}^2. \quad (26)$$

Consequently, it can be said that the Horadam spinor representation $\overline{S_{w,n}}^{t} S_{w,n}$ gives the norm of the Horadam quaternions. □

It is known that there is a spinor representation of the quaternion product given by equation (6). In this case, with the aid of equations (6) and (8), the following definition can be given.

Definition 4. Assume that $S_{w,n}$ is the nth Horadam spinor corresponding to the nth Horadam quaternion $Q_{w,n}$. Then, the fundamental Horadam spinor matrix is as follows:

$$\left(S_{w,n}\right)_L = -\mathbf{i}\widehat{S_{w,n}} = \begin{bmatrix} W_n-\mathbf{i}W_{n+3} & -W_{n+2}-\mathbf{i}W_{n+1} \\ W_{n+2}-\mathbf{i}W_{n+1} & W_n+\mathbf{i}W_{n+3} \end{bmatrix}, \quad (27)$$

Table 1: Horadam spinors according to initial values.

Horadam spinors	Initial conditions $(a, b; p, q)$
Fibonacci spinors	(0, 1; 1, 1) [33]
Lucas spinors	(2, 1; 1, 1) [33]
Pell spinors	(0, 1; 2, 1) [34]
Pell–Lucas spinors	(2, 2; 2, 1) [34]
Jacobsthal spinors	(0, 1; 1, 2) [35]
Jacobsthal–Lucas spinors	(2, 1; 1, 2) [35]

where $\widehat{S_{w,n}} = \begin{bmatrix} W_{n+3} + \mathbf{i}W_n & W_{n+1} - \mathbf{i}W_{n+2} \\ W_{n+1} + \mathbf{i}W_{n+2} & -W_{n+3} + \mathbf{i}W_n \end{bmatrix}$. At the same time, the fundamental Horadam spinor matrix $(S_{w,n})_L$ corresponds to the left Hamilton spinor matrix of the nth Horadam quaternion.

Now, Binet's formula for the Horadam spinors can be given with the following theorem.

Theorem 5. *Let the nth Horadam spinor be $S_{w,n}$. In this case, Binet's formula for the Horadam spinors is given by the following expression:*

$$S_{w,n} = \frac{1}{\alpha - \beta}\left(A\alpha^n S_\alpha - B\beta^n S_\beta\right), \tag{28}$$

where $\alpha = (p + \sqrt{p^2 + 4q}/2)$, $\beta = (p - \sqrt{p^2 + 4q}/2)$, $A = b - a\beta$, $B = b - a\alpha$, $S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix}$, and $S_\beta = \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}$.

Proof. Suppose the $n$thHoradam spinor be $S_{w,n}$. It is known that for the nth Horadam number, Binet's formula is $W_n = (A\alpha^n - B\beta^n/\alpha - \beta)$ where $\alpha = (p + \sqrt{p^2 + 4q}/2)$, $\beta = (p - \sqrt{p^2 + 4q}/2)$, $A = b - a\beta$, and $B = b - a\alpha$. Therefore, for the nth Horadam spinor, the following expression can be given:

$$S_{w,n} = \frac{1}{\alpha - \beta}\begin{bmatrix} A\alpha^{n+3} - B\beta^{n+3} + \mathbf{i}\left(A\alpha^n - B\beta^n\right) \\ A\alpha^{n+1} - B\beta^{n+1} + \mathbf{i}\left(A\alpha^{n+2} - B\beta^{n+2}\right) \end{bmatrix}. \tag{29}$$

Consequently, if the necessary arrangements can be made, it can obtain the following equation:

$$S_{w,n} = \begin{bmatrix} W_{n+3} + \mathbf{i}W_n \\ W_{n+1} + \mathbf{i}W_{n+2} \end{bmatrix} = \frac{1}{\alpha - \beta}\left(A\alpha^n \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} - B\beta^n \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}\right), \tag{30}$$

or

$$S_{w,n} = \frac{1}{\alpha - \beta}\left(A\alpha^n S_\alpha - B\beta^n S_\beta\right), \tag{31}$$

where $S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix}$ and $S_\beta = \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}$ are the spinors corresponding to the quaternions $\alpha = 1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$ and $\beta = 1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$, respectively. □

Especially, for Fibonacci spinors as $a = 0, b = 1, p = 1, q = 1$, we can calculate $A = b - a\beta = 1$ and $B = b - a\alpha = 1$, where $\alpha = (1 + \sqrt{5}/2)$ and $\beta = (1 - \sqrt{5}/2)$. Therefore, the Binet formula in equation (28) for Horadam spinors corresponds to the Binet formula for Fibonacci spinors in [33]. Moreover, for Lucas spinors as $a = 2, b = 1, p = 1, q = 1$, we can calculate $A = \sqrt{5}$ and $B = -\sqrt{5}$ and this Binet formula pairs with in [33]. Similarly, for Pell spinors as $a = 0, b = 1, p = 2, q = 1$, the equalities $A = 1$ and $B = 1$ and the Binet formula for Pell spinors are obtained in [34]; for Pell–Lucas spinors as $a = 2, b = 2, p = 2, q = 1$, the equalities $A = 2\sqrt{2}$ and $B = -2\sqrt{2}$ and the Binet formula for Pell–Lucas spinors are obtained in [34]; for Jacobsthal spinors as $a = 0, b = 1, p = 1, q = 2$, the equalities $A = 1$ and $B = 1$ and the Binet formula for Jacobsthal spinors are obtained in [35]; and finally, for Jacobsthal–Lucas spinors as $a = 2, b = 1, p = 1, q = 2$, the equalities $A = 3$ and $B = -3$ and the Binet formula for Jacobsthal–Lucas spinors are obtained in [35].

Theorem 6. *The generating function for the nth Horadam spinor is obtained by the following expression:*

$$G_s(t) = \frac{1}{1 - pt - qt^2}\left(S_{w,0} + t\left(S_{w,1} - pS_{w,0}\right)\right), \tag{32}$$

where

$$\begin{aligned} S_{w,0} &= \begin{bmatrix} \left(p^2 b + pqa + qb\right) + \mathbf{i}a \\ b + \mathbf{i}(pb + qa) \end{bmatrix}, \\ S_{w,1} &= \begin{bmatrix} \left(p^3 b + p^2 qa + 2pqb + q^2 a\right) + \mathbf{i}b \\ (pb + qa) + \mathbf{i}\left(p^2 b + pqa + qb\right) \end{bmatrix}. \end{aligned} \tag{33}$$

Proof. Assume that $S_{w,n}$ is the nth Horadam spinor and the generating function of the Horadam spinors is $G_s(t)$. Therefore, if the generating function formula $G_s(t) = \sum_{n=0}^{\infty} t^n S_{w,n}$ is used, it can obtain the generating formula for the Horadam spinors as follows. First, the functions $-pt\,G_s(t)$ and $-qt^2 G_s(t)$ are calculated. Therefore, the following equation is found:

$$G_s(t) - ptG_s(t) - qt^2 G_s(t) = S_{w,0} + tS_{w,1} - ptS_{w,0}, \tag{34}$$

with the aid of the recurrence relation of Horadam spinor sequence in equation (23). Consequently, for the Horadam spinors, the generating function is obtained as follows:

$$G_s(t) = \frac{1}{1 - pt - qt^2}\left(S_{w,0} + t\left(S_{w,1} - pS_{w,0}\right)\right). \tag{35}$$

In addition to Theorem 6, if the Horadam spinors $S_{w,0} + tS_{w,1} - ptS_{w,0}$ are calculated, the following corollary can be given as a result of Theorem 6. □

Corollary 7. *The generating function for nth Horadam spinor can be written as*

$$G_s(t) = \frac{1}{1 - pt - qt^2}\begin{bmatrix} p^2b + pqa + qb + t\left(pqb - q^2a\right) + \mathbf{i}(a + t(b - pa)) \\ b + tqa + \mathbf{i}(pb + qa + tqb) \end{bmatrix}. \tag{36}$$

Theorem 8. *Assume that $m, n \in Z$ and the $(n+m)th$ Horadam spinor is $S_{w,n+m}$. In this case, the generating function for $(n+m)th$ Horadam spinor is given by the following expression:*

$$G_s(t) = \frac{1}{1 - pt - qt^2}\left(S_{w,m} + qtS_{w,m-1}\right). \tag{37}$$

Proof. Let $S_{w,n+m}$ be the $(n+m)th$ Horadam spinor. It is known that with the aid of equation (28), Binet's formula for the $(n+m)th$ Horadam spinor $S_{w,n+m}$ is as follows:

$$S_{w,n+m} = \frac{1}{\alpha - \beta}\left(A\alpha^{n+m}\begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} - B\beta^{n+m}\begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}\right). \tag{38}$$

Therefore, if the generating function formula $G_s(t) = \sum_{n=0}^{\infty} t^n S_{w,n+m}$ is used for the $(n+m)th$ Horadam spinor $S_{w,n+m}$, the following equation:

$$G_s(t) = \frac{1}{\alpha - \beta}\left(A\alpha^m \sum_{n=0}^{\infty} \alpha^n t^n \begin{bmatrix} \alpha^3 + \mathbf{i}, \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} - B\beta^m \sum_{n=0}^{\infty} \beta^n t^n \begin{bmatrix} \beta^3 + \mathbf{i}, \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}\right), \tag{39}$$

is calculated. Now, assume that $f(t) = \sum_{n=0}^{\infty} \alpha^n t^n$ and $g(t) = \sum_{n=0}^{\infty} \beta^n t^n$. In this case, one can see that $f(t) = (1/1 - \alpha t)$ and $g(t) = (1/1 - \beta t)$. Then, the following equation can be obtained:

$$G_s(t) = \frac{1}{(\alpha - \beta)(1 - \alpha t)(1 - \beta t)}\left(A\alpha^m(1 - \beta t)\begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} - B\beta^m(1 - \alpha t)\begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}\right). \tag{40}$$

If the necessary arrangements in the last equation are made consequently, the following equation is obtained:

$$G_s(t) = \frac{1}{1 - pt - qt^2}\left(S_{w,m} + qtS_{w,m-1}\right), \tag{41}$$

or

$$G_s(t) = \frac{1}{1 - pt - qt^2}\begin{bmatrix} W_{m+3} + qtW_{m+2} + \mathbf{i}\left(W_m + qtW_{m-1}\right) \\ W_{m+1} + qtW_m + \mathbf{i}\left(W_{m+2} + qtW_{m+1}\right) \end{bmatrix}. \tag{42}$$

□

Now, the Cassini formula for the Horadam spinors is given. For this, it is useful to remember one subject first. That is to say that the product of two real quaternions $q \times p$ is represented by the spinor product $\eta_L \rho$, where η_L is the fundamental spinor matrix of the real quaternion q and ρ is the spinor corresponding to the real quaternion p with the aid of the transformation f according to equations (7) and (8). Therefore, for the product of the Horadam quaternions $Q_{w,n-1}Q_{w,n+1} - Q_{w,n}^2$, the Horadam spinor product $(S_{w,n-1})_L S_{w,n+1} - (S_{w,n})_L S_{w,n}$ should be written (where $Q_{w,n+1}Q_{w,n-1} - Q_{w,n}^2$ can be also chosen). In this case, $(S_{w,n+1})_L S_{w,n-1} - (S_{w,n})_L S_{w,n}$ should be calculated. Therefore, the Cassini formula for the Horadam spinors can be given.

Theorem 9. *Let the nth Horadam spinor be $S_{w,n}$. In this case, the spinor representation of the Cassini formula for Horadam spinors is as follows:*

$$\left(S_{w,n-1}\right)_L S_{w,n+1} - \left(S_{w,n}\right)_L S_{w,n} = -AB(-q)^{n-1}\begin{bmatrix} p\left(p^2+2q\right) + \mathbf{i}\left(q^3 - q^2 + q + 1\right) \\ p\left(1+q^2\right) + \mathbf{i}\left(\left(p^2+q\right)(1+q) - q^2 + q\right) \end{bmatrix}, \tag{43}$$

where $\alpha = (p + \sqrt{p^2+4q}/2), \beta = (p - \sqrt{p^2+4q}/2), A = b - a\beta$, *and* $B = b - a\alpha$.

Proof. Let the $(n-1)th$, $(n)th$, and $(n+1)th$ Horadam spinors be $S_{w,n-1}$, $S_{w,n}$, and $S_{w,n+1}$, respectively. In this case, it can be written as the product of these Horadam spinors $(S_{w,n-1})_L S_{w,n+1} - (S_{w,n})_L S_{w,n}$ corresponding to the product of Horadam quaternions $Q_{w,n-1}Q_{w,n+1} - Q_{w,n}^2$. Hence, with the aid of Binet's formula in equation (28), the fundamental Horadam spinor matrices

$$\begin{aligned} \left(S_{w,n}\right)_L &= \frac{1}{\alpha-\beta}\left(A\alpha^n \left(S_\alpha\right)_L - B\beta^n\left(S_\beta\right)_L\right), \\ \left(S_{w,n-1}\right)_L &= \frac{1}{\alpha-\beta}\left(A\alpha^{n-1} \left(S_\alpha\right)_L - B\beta^{n-1}\left(S_\beta\right)_L\right), \end{aligned} \tag{44}$$

are obtained with the aid of equation (27), where

$$\begin{aligned} &S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix}, S_\beta = \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix}, \left(S_\alpha\right)_L = \begin{bmatrix} 1 - \mathbf{i}\alpha^3 & -\alpha^2 - \mathbf{i}\alpha \\ \alpha^2 - \mathbf{i}\alpha & 1 + \mathbf{i}\alpha^3 \end{bmatrix}, \\ &\left(S_\beta\right)_L = \begin{bmatrix} 1 - \mathbf{i}\beta^3 & -\beta^2 - \mathbf{i}\beta \\ \beta^2 - \mathbf{i}\beta & 1 + \mathbf{i}\beta^3 \end{bmatrix}. \end{aligned} \tag{45}$$

If the long algebraic calculations are made in the products of the Horadam spinors, $(S_{w,n-1})_L S_{w,n+1} - (S_{w,n})_L S_{w,n}$ can be obtained as follows:

$$\begin{aligned} &\left(S_{w,n-1}\right)_L S_{w,n+1} - \left(S_{w,n}\right)_L S_{w,n} \\ &= \frac{AB}{(\alpha-\beta)^2}(\alpha\beta)^{n-1}\left(\left(\alpha\beta-\beta^2\right)\left(S_\alpha\right)_L S_\beta + \left(\alpha\beta-\alpha^2\right)\left(S_\beta\right)_L S_\alpha\right) \\ &= \frac{AB}{\alpha-\beta}(-q)^{n-1}\left(\beta\left(S_\alpha\right)_L S_\beta - \alpha\left(S_\beta\right)_L S_\alpha\right), \end{aligned} \tag{46}$$

and consequently,

$$\left(S_{w,n-1}\right)_L S_{w,n+1} - \left(S_{w,n}\right)_L S_{w,n} = -AB(-q)^{n-1}\begin{bmatrix} p\left(p^2+2q\right) + \mathbf{i}\left(q^3 - q^2 + q + 1\right) \\ p\left(1+q^2\right) + \mathbf{i}\left(\left(p^2+q\right)(1+q) - q^2 + q\right) \end{bmatrix}, \tag{47}$$

where $\alpha = (p + \sqrt{p^2 + 4q}/2), \beta = (p - \sqrt{p^2 + 4q}/2), A = b - a\beta$, and $B = b - a\alpha$. □

Theorem 10. *Assume that the nth Horadam spinor is $S_{w,n}$. Therefore, the sum formula of the Horadam spinors is as follows:*

$$\sum_{k=0}^{n} S_{w,k} = \frac{1}{p+q-1}\left(S_{w,n+1} - S_{w,0} + q\left(S_{w,n} - S_{w,-1}\right)\right). \quad (48)$$

Proof. Let $S_{w,n}$ be the nth Horadam spinor. Hence, with the aid of Binet's formula, the sum formula of the Horadam spinors can be written as follows:

$$\sum_{k=0}^{n} S_{w,k} = \sum_{k=0}^{n}\left(\frac{1}{\alpha-\beta}\right)\left(A\alpha^k\begin{bmatrix}\alpha^3 + \mathbf{i}\\ \alpha + \mathbf{i}\alpha^2\end{bmatrix} - B\beta^k\begin{bmatrix}\beta^3 + \mathbf{i}\\ \beta + \mathbf{i}\beta^2\end{bmatrix}\right). \quad (49)$$

It is known as the geometric sequences $\sum_{k=0}^{n}\alpha^k = (1 - \alpha^{n+1})/(1-\alpha)$ and $\sum_{k=0}^{n}\beta^k = (1-\beta^{n+1})/(1-\beta)$. In this case, the following equation is obtained:

$$\sum_{k=0}^{n} S_{w,k} = \frac{1}{\alpha-\beta}\left(\frac{A\left(1-\alpha^{n+1}\right)}{1-\alpha}\begin{bmatrix}\alpha^3 + \mathbf{i}\\ \alpha + \mathbf{i}\alpha^2\end{bmatrix} - \frac{B\left(1-\beta^{n+1}\right)}{1-\beta}\begin{bmatrix}\beta^3 + \mathbf{i}\\ \beta + \mathbf{i}\beta^2\end{bmatrix}\right). \quad (50)$$

Consequently, if some necessary arrangements in the last equation are made, it is obtained as follows:

$$\begin{aligned}\sum_{k=0}^{n} S_{w,k} &= \frac{1}{(1-\alpha)(1-\beta)}\left((1-p)S_{w,0} + S_{w,1} + (p-1)S_{w,n+1} - S_{w,n+2}\right),\\ \sum_{k=0}^{n} S_{w,k} &= \frac{1}{p+q-1}\left(S_{w,n+1} - S_{w,0} + q\left(S_{w,n} - S_{w,-1}\right)\right).\end{aligned} \quad (51)$$

The proof is completed. □

4. Conclusion

The concept of spinors, which is frequently encountered in physics, is defined as the elements of a minimal left ideal of Clifford algebra. This left ideal is called spinor space. Inner products can be defined on minimal left ideals by also making use of minimal right ideal representations. Clifford algebras can be constructed over any number field. Among these, the complex number field is important. The aim of this study is to connect the spinor structure, which is frequently used in physics and algebra, with quaternions used in geometry and algebra. In this context, the counterparts of quaternions in spinor space have an important place. Therefore, in this study, the counterparts of Horadam quaternions, which are a generalization of quaternion sequences, in spinor space are investigated. With this matching, the importance of this study is that Horadam spinors given in this study are a generalization of the spinor representations of all these quaternion sequences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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