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Research Article

Geometric Classifications of Perfect Fluid Space-Time Admit Conformal Ricci-Bourguignon Solitons

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This paper is dedicated to the study of the geometric composition of a perfect fluid space-time with a conformal Ricci-Bourguignon soliton, which is the extended version of the soliton to the Ricci-Bourguignon flow. Here, we have delineated the conditions for conformal Ricci-Bourguignon soliton to be expanding, steady, or shrinking. We have studied certain curvature conditions on the spacetime that admit conformal Ricci-Bourguignon soliton. We have also discussed conformal Ricci-Bourguignon soliton on some special types of perfect fluid spacetime such as dust fluid, dark fluid, and radiation era.

1. Motivations and Introduction

The Ricci-Bourguignon flow (RB flow for short) was first discovered by Jean-Pierre Bourguignon [1] in 1981. This notion was developed with the help of Lichnerowicz and Aubin [2]. Now, the RB flow can be defined as follows [3].

Definition 1. A family of metrics $g(t)$ on an n-dimensional Riemannian manifold (M^n, g) is defined as the Ricci-Bourguignon flow if $g(t)$ admits the following:

$$\frac{\partial g}{\partial t} = -2(S - \omega Rg), \tag{1}$$

where S is a Ricci tensor and R is the scalar curvature, respectively, and $\omega \in \mathbb{R}$ is a constant.

Using the abovementioned definition, we can easily identify the above equation as a Ricci flow when ω equals zero. Now, from [3], we can get the Einstein tensor, traceless Ricci tensor, Schouten tensor, and Ricci tensor for $\omega = 1/2$, $\omega = 1/n$, $\omega = 1/2(n-1)$, and $\omega = 0$. Recently, Shubham Dwivedi [3] introduced the following definition for the Ricci-Bourguignon soliton (RBS).

Definition 2. A Riemannian manifold (M^n, g) is a Ricci-Bourguignon soliton (RBS) if it satisfies the following equations:

$$\pounds_V g + 2S = 2\Lambda_1 g + 2\omega \mathrm{Rg}, \tag{2}$$

where $\Lambda_1 \in \mathbb{R}$ is a constant.

Based on Ricci's soliton equation, Basu and Ariesh Bhattacharyya developed the Ricci soliton equation and proposed the concept of conformal Ricci soliton [4]. Based on the conformal Ricci soliton, we propose the following.

Definition 3. It is said that a Riemannian manifold (M^n, g) is a conformal Ricci-Bourguignon soliton (CRBS) when it admits

$$\pounds_V g + 2S = \left\{2\Lambda_1 - \left(p + \frac{2}{n}\right)\right\} g + 2\omega \mathrm{R}g, \tag{3}$$

for the constant $\Lambda_1 \in \mathbb{R}$. Here, p is a scalar field with no dynamical characteristic (a time-dependent scalar field).

CRBS becomes shrinking, steady, or expanding based on $\Lambda_1 > 0$, $\Lambda_1 = 0$, and $\Lambda_1 < 0$, respectively. The gradient CRBS of the manifold is obtained by considering the potential vector field V as the gradient of a smooth function f. Also, if Λ_1 is a smooth function, then (M, g) is called an almost CRBS. In [5], the authors found a spacetime-admitting Ricci soliton. Later, Blaga [6] developed a complete fluid spacetime characterization using η-Ricci solitons and η-Einstein solitons. Venkatesha et al. [7] also studied Ricci solitons in a completely fluid spacetime. Some Ricci solitons associated with spacetime are improbable by several authors (see [8–10]). Recently, Ricci solitons and their extensions have been studied by several authors. For example, conformal Ricci solitons and quasi-Yamabe solitons on trans-Sasakian 3-manifolds as generalized Sasakian space forms [11], η-"Einstein's solitons" [12], conformal Einstein soliton in the framework of quasi-Kähler manifold [13], Yamabe soliton on $(\mathrm{LCS})_n$-manifolds, conformal Yamabe soliton with torsion formation, and $*$ Yamabe soliton potential vector field [14] and many others (see [15, 16]).

In 1915, Albert Einstein discovered the theory of general relativity, also known as general relativity (GTR), a geometric theory of gravity. In this theory, the gravitational field and its sources are the spacetime curvature and the energy-momentum tensor. The geometry of Lorentzian manifolds begins with the study of the causal properties of the vectors of the manifold. This causal relationship makes Lorentzian manifolds a convenient choice for studying general relativity. EMT plays an important substantive role in space and time. Matter can also be considered a fluid with dynamic and kinematic quantities such as density, pressure, velocity, acceleration, vortex forces, shear forces, and extension [17, 18]. For standard cosmological models, we can assume that the material content of the universe behaves like a perfect fluid. The most appropriate example of a perfect liquid is a dusty liquid. In the present paper, we will consider some notations throughout the paper as follows:

Energy-momentum tensor as EMT,
Perfect fluid spacetime as PFS,
Dust fluid spacetime as DFS,
Dark fluid spacetime as $\mathrm{D_1FS}$, and
Radiation fluid spacetime as RFS.

2. Emergence of PFS with Torse-Forming Vector Field

A perfect fluid is one whose rest frame mass density and isotropic pressure may be used to entirely define it. The EMT $\mathcal{T}_1$ of the form [19] on a PFS is as follows:

$$\mathcal{T}_1(\mathbb{V}_1, \mathbb{V}_2) = \rho_1 g(\mathbb{V}_1, \mathbb{V}_2) + (\sigma_1 + \rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2), \tag{4}$$

where the isotropic pressure and energy density are denoted by ρ_1 and σ_1, respectively, and the unit vector ξ satisfies $g(\xi, \xi) = -1$. Einstein's gravitational equation (19) with cosmological constant and is denoted by

$$S(\mathbb{V}_1, \mathbb{V}_2) + \left\{\lambda_1 - \frac{R}{2}\right\} g(\mathbb{V}_1, \mathbb{V}_2) = \kappa_1 T_1(\mathbb{V}_1, \mathbb{V}_2), \tag{5}$$

where λ_1 and κ are the cosmological constant and gravitational constant, respectively. Similarly, S and R are the Ricci tensor and scalar curvature of g. We can consider κ to be equal to $8\pi G$, where G stands for the gravitational constant of the universe. Using (4), the previous identity becomes

$$\begin{aligned} S(\mathbb{V}_1, \mathbb{V}_2) = {} & \left\{-\lambda_1 + \frac{R}{2} + \kappa_1\rho_1\right\} g(\mathbb{V}_1, \mathbb{V}_2) \\ & + \kappa_1(\sigma_1 + \rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \end{aligned} \tag{6}$$

We contract the identity (6) on relativistic PFS which admits (5) to yield

$$R = 4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1). \tag{7}$$

Using the value of R from the above identity into (6) reads

$$\begin{aligned} S(\mathbb{V}_1, \mathbb{V}_2) = {} & \left\{\lambda_1 + \frac{\kappa_1(\sigma_1 - \rho_1)}{2}\right\} g(\mathbb{V}_1, \mathbb{V}_2) \\ & + \kappa_1(\sigma_1 + \rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \end{aligned} \tag{8}$$

So, we can write the Ricci operator Q as follows:

$$Q\mathbb{V}_1 = \left\{\lambda_1 + \frac{\kappa_1(\sigma_1 - \rho_1)}{2}\right\} X + \kappa_1(\sigma_1 + \rho_1)\eta(\mathbb{V}_1)\xi, \tag{9}$$

where $g(Q\mathbb{V}_1, \mathbb{V}_2) = S(\mathbb{V}_1, \mathbb{V}_2)$.

Example 1. If σ_1 is equal to $3\rho_1$, then radiation fluid becomes a perfect fluid and EMT T is

$$\mathcal{T}_1(\mathbb{V}_1, \mathbb{V}_2) = \rho_1[g(\mathbb{V}_1, \mathbb{V}_2) + 4\eta(\mathbb{V}_1)\eta(\mathbb{V}_2)]. \tag{10}$$

From (7), we get that the radiation fluid consists of constant scalar curvature R, which is equal to $4\lambda_1$.

Now, when ξ is a torse-forming vector field [6, 20], then we have the following form:

$$\nabla_{\mathbb{V}_1}\xi = \mathbb{V}_1 + \eta(\mathbb{V}_1)\xi. \tag{11}$$

Also, because PFS admits a torse-forming vector field, then the following relations are satisfied [6]:

$$\nabla_\xi \xi = 0, \tag{12}$$

$$\left(\nabla_{\mathbb{V}_1}\eta\right)(\mathbb{V}_2) = g(\mathbb{V}_1, \mathbb{V}_2) + \eta(\mathbb{V}_1)\eta(\mathbb{V}_2), \tag{13}$$

$$R(\mathbb{V}_1, \mathbb{V}_2)\xi = \eta(\mathbb{V}_2)\mathbb{V}_1 - \eta(\mathbb{V}_1)\mathbb{V}_2, \tag{14}$$

$$\eta(R(\mathbb{V}_1,\mathbb{V}_2)\mathbb{V}_3) = \eta(\mathbb{V}_1)g(\mathbb{V}_2,\mathbb{V}_3) - \eta(\mathbb{V}_2)g(\mathbb{V}_1,\mathbb{V}_3), \tag{15}$$

for all vector fields $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$. Now, using identity (11), we get

$$\begin{aligned}(\pounds_\xi g)(\mathbb{V}_1,\mathbb{V}_2) &= g(\nabla_{\mathbb{V}_1}\xi,\mathbb{V}_2) + g(\mathbb{V}_1,\nabla_{\mathbb{V}_2}\xi)\\ &= 2\{g(\mathbb{V}_1,\mathbb{V}_2) + \eta(\mathbb{V}_1)\eta(\mathbb{V}_2)\},\end{aligned} \tag{16}$$

for all vector fields $\mathbb{V}_1, \mathbb{V}_2$.

Perfect fluids are used in general relativity to simulate the idealized distribution of matter, such as an isotropic cosmos. The source of the gravitational field is frequently inserted in general relativity and spacetime symmetries using a PFEMT (4), and PF also has two thermodynamic degrees of freedom. A perfect fluid solution serves as a precise solution of the Einstein field equation in general relativity. It may be beneficial to consider a perfect gas as a unique instance of a perfect fluid. Fluid solutions are frequently employed in cosmology as cosmological models. There are some special cases of fluid solutions.

(i) A dust is a pressureless perfect fluid with the energy-momentum tensor $\mathcal{T}_1(\mathbb{V}_1,\mathbb{V}_2) = \sigma_1\eta(\mathbb{V}_1)\eta(\mathbb{V}_2)$.

(ii) A radiation fluid is considered a perfect fluid with (10).

The two situations mentioned above can be used as cosmological models. Finding such solutions is significantly simpler than finding a generic fluid solution. A perfect fluid's Einstein tensor's characteristic polynomial also has the following shape:

$$\chi(\tau) = (\tau - 8\pi\sigma_1)(\tau - 8\pi\rho_1)^3, \tag{17}$$

where σ_1 and ρ_1 are the density and pressure of fluid, respectively. Perfect fluid solutions attribute positive pressure, including

(i) FRW radiation fluids pointed out to us the radiation-dominated FRW models with various radiation fluid models from cosmology.

(ii) We can derive the evolution of the universe from Friedmann–Lemaitre–Robertson Walker equations by using the equation of state of the perfect fluid.

3. Main Results

Let the metric of a relativistic PFS (M^4, g) satisfies a CRBS (3). Then, using (8), equation (3) can be written as

$$(\pounds_V g)(\mathbb{V}_1,\mathbb{V}_2) + 2\left\{\lambda_1 + \frac{\kappa_1(\sigma_1-\rho_1)}{2}\right\}g(\mathbb{V}_1,\mathbb{V}_2) \tag{18}$$

$$+ 2\kappa_1(\sigma_1+\rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2),$$

$$= \left\{2\Lambda_1 - \left(p + \frac{1}{2}\right)\right\}g + 2\omega \mathrm{R}g, \tag{19}$$

for all vector fields $\mathbb{V}_1, \mathbb{V}_2$ on M. Now, contracting the above identity, we get

$$\operatorname{div}V = 4\left\{\Lambda_1 - \lambda_1 - \frac{1}{2}\left(p+\frac{1}{2}\right) - \frac{\kappa_1(\sigma_1-3\rho_1)}{4} + \omega R\right\}, \tag{20}$$

where $\operatorname{div}V$ is the divergence of the vector field V. In light of (7), we obtain

$$\Lambda = \frac{\operatorname{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1-3\rho_1)}{4}\right\}(1-4\omega) + \frac{1}{2}\left(p+\frac{1}{2}\right). \tag{21}$$

So, we can state the following.

Theorem 4. *If CRBS $(g, V, \Lambda_1, \omega)$ admits PFS, then the soliton becomes shrinking, steady, expanding according to*

$$\frac{\operatorname{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1-3\rho_1)}{4}\right\}(1-4\omega) + \frac{1}{2}\left(p+\frac{1}{2}\right) \underset{<}{\overset{\geq}{=}} 0. \tag{22}$$

In (21), $\operatorname{div}V = 0$ if and only if

$$\Lambda_1 = \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1-3\rho_1)}{4}\right\}(1-4\omega) + \frac{1}{2}\left(p+\frac{1}{2}\right). \tag{23}$$

Hence, the following is established.

Corollary 5. *Let a PFS satisfies a CRBS $(g, V, \Lambda_1, \omega)$, then V is solenoidal if and only if*

$$\Lambda_1 = \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1-3\rho_1)}{4}\right\}(1-4\omega) + \frac{1}{2}\left(p+\frac{1}{2}\right). \tag{24}$$

Now, a vector field V is said to be a conformal Killing vector field if and only if we have

$$(\pounds_V g)(\mathbb{V}_1,\mathbb{V}_2) = 2\Phi g(\mathbb{V}_1,\mathbb{V}_2), \tag{25}$$

where Φ is some function of the co-ordinates(conformal scalar). Moreover, the conformal Killing vector field V becomes a proper and homothetic vector field when Φ is not constant and Φ is constant. Also, V is said to be a proper homothetic vector field when the constant Φ becomes nonzero. In the above equation, V is called a Killing vector field if Φ is zero. We consider conformal Killing vector field into the identity (3) by taking the identities (3) and (25) to achieve

$$S(\mathbb{V}_1,\mathbb{V}_2) = \left\{\Lambda_1 - \frac{1}{2}\left(p+\frac{1}{2}\right) + \omega R - \Phi\right\}g(\mathbb{V}_1,\mathbb{V}_2). \tag{26}$$

This shows that the spacetime becomes Einstein. Conversely, we assume that the spacetime is Einstein. Let $S(\mathbb{V}_1,\mathbb{V}_2) = \theta g(\mathbb{V}_1,\mathbb{V}_2)$, where θ is smooth function. Then, (3) becomes

$$(\pounds_V g)(\mathbb{V}_1,\mathbb{V}_2) = 2\left\{\Lambda_1 - \frac{1}{2}\left(p+\frac{1}{2}\right) + \omega R - \theta\right\}g(\mathbb{V}_1,\mathbb{V}_2). \tag{27}$$

Hence, V is a conformal Killing vector field. So, we have the following theorem.

Theorem 6. *Let a PFS admits CRBS $(g, V, \Lambda_1, \omega)$ with conformal Killing potential vector field V if the spacetime is Einstein.*

Using (8) and (26), we can write

$$\left\{\Lambda_1 + \omega R - \Phi - \lambda_1 - \frac{\kappa_1(\sigma_1 - \rho_1)}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right\} g(\mathbb{V}_1, \mathbb{V}_2) = \kappa_1(\sigma_1 + \rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \quad (28)$$

Now, we plug $\mathbb{V}_2 = \xi$ into (24) and considering $\eta(\xi) = -1$ to acquire

$$\left\{\Lambda_1 + \omega R - \Phi - \lambda_1 - \frac{\kappa_1(\sigma_1 - \rho_1)}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right) + \kappa_1(\sigma_1 + \rho_1)\right\}\eta(\mathbb{V}_1) = 0. \quad (29)$$

Since $\eta(\mathbb{V}_1) \neq 0$, one obtains

$$\Phi = \Lambda_1 + \omega R - \lambda_1 + \frac{\kappa_1(\sigma_1 + 3\rho_1)}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right). \quad (30)$$

We get the following identity by replacing the value of R from (7):

$$\Phi = \frac{\kappa_1}{2}\left[(1 + 2\omega)\sigma_1 + 3(1 - 2\omega_1)\rho_1\right] - (1 - 4\omega)\lambda_1 + \Lambda_1 - \frac{1}{2}\left(p + \frac{1}{2}\right). \quad (31)$$

Hence, we give the following theorem.

Theorem 7. *If CRBS endows a PFS (g, V, Λ, ω) admitting conformal Killing, then vector field becomes*

(a) Proper conformal Killing vector field if p is not constant.

(b) Homothetic vector field if p is constant.

If we consider $\Phi = 0$ in (25), then V becomes Killing vector field, and consequently (31) gives

$$\Lambda_1 = (1 - 4\omega)\lambda_1 + \frac{1}{2}\left(p + \frac{1}{2}\right) - \frac{\kappa_1}{2}\left[(1 + 2\omega)\sigma_1 + 3(1 - 2\omega)\rho_1\right]. \quad (32)$$

This gives the following statement.

Corollary 8. *If PFS satisfies a CRBS $(g, V, \Lambda_1, \omega)$, where the potential vector field V is Killing, then the soliton is shrinking, steady, and expanding according to*

$$(1 - 4\omega)\lambda_1 + \frac{1}{2}\left(p + \frac{1}{2}\right) - \frac{\kappa_1}{2}\left[(1 + 2\omega)\sigma_1 + 3(1 - 2\omega)\rho_1\right] \overset{\geq}{\underset{<}{=}} 0. \quad (33)$$

Definition 9. If $\mathscr{W}_2$-curvature tensor [21],

$$\mathscr{W}_2(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4) = \grave{R}(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4) + \frac{1}{n - 1} \cdot \left[g(\mathbb{V}_1, \mathbb{V}_3)S(\mathbb{V}_2, \mathbb{V}_4) - g(\mathbb{V}_2, \mathbb{V}_3)S(\mathbb{V}_1, \mathbb{V}_4)\right], \quad (34)$$

for all vector fields $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4$ and is defined on an n-dimensional manifold, zero identically, then spacetime is said to be $\mathscr{W}_2$-flat.

We obtain the following identity by taking $\mathscr{W}_2$-flat relativistic PFS invoking (34) to find

$$\grave{R}(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4) = -\frac{1}{3}\left[g(\mathbb{V}_1, \mathbb{V}_3)S(\mathbb{V}_2, \mathbb{V}_4) - g(\mathbb{V}_2, \mathbb{V}_3)S(\mathbb{V}_1, \mathbb{V}_4)\right]. \quad (35)$$

In the previous identity, taking $\mathbb{V}_1 = \mathbb{V}_4 = e_i$, we get

$$S(\mathbb{V}_2, \mathbb{V}_3) = \frac{R}{4} g(\mathbb{V}_2, \mathbb{V}_3). \quad (36)$$

In view of (3), the above equation can be written as

$$(\pounds_V)(\mathbb{V}_2, \mathbb{V}_3) = \left\{2\Lambda_1 + \frac{R}{2}(4\omega - 1) - \left(p + \frac{1}{2}\right)\right\} g(\mathbb{V}_2, \mathbb{V}_3). \quad (37)$$

Now, we take $\mathbb{V}_2 = \mathbb{V}_3 = e_i$ into identity (37), taking summation over i, $1 \leq i \leq 4$ and using (7) to yield

$$\Lambda = \frac{\mathrm{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1)}{4}\right\}(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{38}$$

This leads to the following.

Theorem 10. *Let a PFS, which is $\mathscr{W}_2$-flat, admit a CRBS $(g, V, \Lambda_1, \omega)$. Then, the soliton is shrinking, steady, and expanding according to*

$$\frac{\mathrm{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1)}{4}\right\}(1 - 4\omega) + \left(\frac{p}{2} + \frac{1}{4}\right) \underset{<}{\overset{\geq}{=}} 0. \tag{39}$$

Definition 11. If the pseudoprojective curvature tensor [22],

$$\begin{aligned}\overline{\mathscr{P}}(\mathbb{V}_1, \mathbb{V}_2)\mathbb{V}_3 = {} & aR(\mathbb{V}_1, \mathbb{V}_2)\mathbb{V}_3 + b[S(\mathbb{V}_2, \mathbb{V}_3)\mathbb{V}_1 - S(\mathbb{V}_1, \mathbb{V}_3)\mathbb{V}_2] \\ & - \frac{R}{n}\left(\frac{a}{n-1} + b\right)[g(\mathbb{V}_2, \mathbb{V}_3)\mathbb{V}_1 - g(\mathbb{V}_1, \mathbb{V}_3)\mathbb{V}_2],\end{aligned} \tag{40}$$

for all fields $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ defined on $n-$ dimensional manifold is identically zero, where a, $b \neq 0$ are constants, then a spacetime is said to be pseudoprojectively flat.

In particular, if $a = 1, b = -1/n - 1$, the above equation reduces to

$$\begin{aligned}\overline{\mathscr{P}}(\mathbb{V}_1, \mathbb{V}_2)\mathbb{V}_3 &= R(\mathbb{V}_1, \mathbb{V}_2)\mathbb{V}_3 - \frac{1}{(n-1)}[S(\mathbb{V}_2, \mathbb{V}_3)\mathbb{V}_1 - S(\mathbb{V}_1, \mathbb{V}_3)\mathbb{V}_2] \\ &= \mathscr{P}(\mathbb{V}_1, \mathbb{V}_2)\mathbb{V}_3,\end{aligned} \tag{41}$$

where $\mathscr{P}$ is the projective curvature tensor [23]. We acquire the following identity by considering pseudoprojectively flat relativistic PFS turning into an inner product with W in (40) to read

$$\begin{aligned}a\grave{R}(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4) = {} & \frac{R}{4}\left[\frac{a}{3} + b\right][g(\mathbb{V}_2, \mathbb{V}_3)g(\mathbb{V}_1, \mathbb{V}_4) - g(\mathbb{V}_1, \mathbb{V}_3)g(\mathbb{V}_2, \mathbb{V}_4)] \\ & - b[S(\mathbb{V}_2, \mathbb{V}_3)g(\mathbb{V}_1, \mathbb{V}_4) - S(\mathbb{V}_1, \mathbb{V}_3)g(\mathbb{V}_2, \mathbb{V}_4)].\end{aligned} \tag{42}$$

Taking trace of the previous equation by setting $\mathbb{V}_1 = \mathbb{V}_4 = e_i$, we find

$$S(\mathbb{V}_2, \mathbb{V}_3) = \frac{R}{4}g(\mathbb{V}_2, \mathbb{V}_3). \tag{43}$$

Since $(a + 3b) \neq 0$. In view of (3), the above equation can be written as

$$(\pounds_V)(\mathbb{V}_2, \mathbb{V}_3) = \left\{2\Lambda + \frac{R}{2}(4\omega - 1) - \left(p + \frac{1}{2}\right)\right\}g(\mathbb{V}_2, \mathbb{V}_3). \tag{44}$$

We fetch $\mathbb{V}_2 = \mathbb{V}_3 = e_i$ in the previous identity, summing over $1 \leq i \leq 4$ and using (7) to find

$$\Lambda = \frac{\mathrm{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1)}{4}\right\}(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{45}$$

So, we achieve the following.

Theorem 12. *If pseudoprojectively flat PFS endows a CRBS $(g, V, \Lambda_1, \omega)$, then the soliton is shrinking, steady, and expanding according to*

$$\frac{\mathrm{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1)}{4}\right\}(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right) \underset{<}{\overset{\geq}{=}} 0. \tag{46}$$

Definition 13. If conharmonic curvature tensor $\mathscr{H}$ [24]

$$\mathscr{H}(\mathbb{V}_1,\mathbb{V}_2)\mathbb{V}_3 = R(\mathbb{V}_1,\mathbb{V}_2)\mathbb{V}_3 - \frac{1}{(n-2)} \cdot [g(\mathbb{V}_{2,3})Q\mathbb{V}_1 - g(\mathbb{V}_1,\mathbb{V}_3)Q\mathbb{V}_2 + S(\mathbb{V}_2,\mathbb{V}_3)\mathbb{V}_1 - S(\mathbb{V}_1,\mathbb{V}_3)\mathbb{V}_2], \tag{47}$$

for all fields $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ defined on n-dimensional manifold is identically zero, then spacetime is denoted conharmonically flat.

We obtain the following identity using conharmonically flat relativistic PFS and taking an inner product with W in (47) to yield

$$\grave{R}(\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3,\mathbb{V}_4) = \frac{1}{2}[g(\mathbb{V}_2,\mathbb{V}_3)S(\mathbb{V}_1,\mathbb{V}_4) - g(\mathbb{V}_1,\mathbb{V}_3)s(\mathbb{V}_2,\mathbb{V}_4) + S(\mathbb{V}_2,\mathbb{V}_3)g(\mathbb{V}_1,\mathbb{V}_4) - S(\mathbb{V}_1,\mathbb{V}_3)g(\mathbb{V}_2,\mathbb{V}_4)]. \tag{48}$$

Putting $\mathbb{V}_1 = \mathbb{V}_4 = e_i$ in the above equation, taking summation over $1 \le i \le 4$, we get

$$R = 0. \tag{49}$$

Hence, (6) takes the form

$$S(\mathbb{V}_1,\mathbb{V}_2) = \{-\lambda_1 + \kappa_1\rho_1\}g(\mathbb{V}_1,\mathbb{V}_2) + \kappa_1(\sigma_1+\rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \tag{50}$$

In view of (49) and (50), (3) reads

$$(\pounds_V g)(\mathbb{V}_1,\mathbb{V}_2) - 2\left\{\Lambda_1 + \lambda_1 - \frac{1}{2}\left(p+\frac{1}{2}\right) - \kappa_1\rho_1\right\}g(\mathbb{V}_1,\mathbb{V}_2) + 2\kappa_1(\sigma_1+\rho_1)\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \tag{51}$$

Taking $\mathbb{V}_1 = \mathbb{V}_2 = e_i$ in the above equation, summing over $1 \le i \le 4$, we obtain

$$\Lambda = \frac{\mathrm{div}V}{4} - \lambda_1 - \frac{\kappa_1(\sigma_1 - 3\rho_1)}{4} + \frac{1}{2}\left(p+\frac{1}{2}\right). \tag{52}$$

This leads to the following.

Theorem 14. *Let a PFS, which is conharmonically flat, admits a CRBS $(g, V, \Lambda_1, \omega)$. Then, the soliton is shrinking, steady, and expanding according to*

$$\frac{\mathrm{div}V}{4} - \lambda_1 - \frac{\kappa_1(\sigma_1 - 3\rho_1)}{4} + \frac{1}{2}\left(p+\frac{1}{2}\right) \underset{<}{\overset{\geq}{=}} 0. \tag{53}$$

Very recently, new curvature tensor $\mathscr{Q}_1$ in an n-dimensional Riemannian manifold has been developed by Mantica et al. [25] and is as follows:

$$\mathscr{Q}_1(\mathbb{V}_1,\mathbb{V}_2)\mathbb{V}_3 = R(\mathbb{V}_1,\mathbb{V}_2)\mathbb{V}_3 - \frac{\psi}{(n-1)}[g(\mathbb{V}_2,\mathbb{V}_3)\mathbb{V}_1 - g(\mathbb{V}_1,\mathbb{V}_3)\mathbb{V}_2], \tag{54}$$

where ψ_1 denotes an arbitrary scalar function.

Definition 15. If $\mathscr{Q}_1$-curvature tensor is zero identically, then a spacetime is said to be $\mathscr{Q}_1$-flat.

We derive the following identity by taking $\mathscr{Q}_1$-flat relativistic PFS invoking an inner product with U_1 in (54) to find

$$\grave{R}(\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3,\mathbb{V}_4) = \frac{\psi_1}{3}[g(\mathbb{V}_2,\mathbb{V}_3)g(\mathbb{V}_1,\mathbb{V}_4) - g(\mathbb{V}_1,\mathbb{V}_3)g(\mathbb{V}_2,\mathbb{V}_4)]. \tag{55}$$

Now, we take trace the previous identity by considering $\mathbb{V}_1 = \mathbb{V}_4 = e_i$ to achieve

$$S(\mathbb{V}_2,\mathbb{V}_3) = \psi_1 g(\mathbb{V}_2,\mathbb{V}_3). \tag{56}$$

From the above result, the spacetime becomes Einstein. In light of fd3(3) and (56) takes the form

$$(\pounds_V g)(\mathbb{V}_2,\mathbb{V}_3) + \left\{2\psi_1 - 2\Lambda_1 - 2\omega R + \left(p+\frac{1}{2}\right)\right\}g(\mathbb{V}_2,\mathbb{V}_3) = 0. \tag{57}$$

Tracing the above equation to take $\mathbb{V}_2 = \mathbb{V}_3 = e_i$ and using (7) to find

$$\Lambda = \frac{\mathrm{div}V}{4} - [4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1)]\omega + \psi_1 + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{58}$$

So, we get the following theorem.

Theorem 16. *Let a PFS, which is $\mathcal{Q}_1$-flat, admits a CRBS $(g, V, \Lambda_1, \omega)$. Then, the soliton is shrinking, steady, and expanding according to*

$$\frac{\mathrm{div}V}{4} - [4\lambda_1 + \kappa_1(\sigma_1 - 3\rho_1)]\omega + \psi_1 + \frac{1}{2}\left(p + \frac{1}{2}\right) \mathop{\gtreqless} 0. \tag{59}$$

4. CRBS Associated on Dust Fluid Spacetime (DFS)

In a dust or pressureless fluid spacetime (DFS), the EMT looks like [26]

$$\mathcal{T}_1(\mathbb{V}_1, \mathbb{V}_2) = \sigma_1\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \tag{60}$$

Now, we insert the identities (5) and (51) to obtain

$$S(\mathbb{V}_1, \mathbb{V}_2) = \left\{-\lambda_1 + \frac{R}{2}\right\}g(\mathbb{V}_1, \mathbb{V}_2) + \kappa_1\sigma_1\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \tag{61}$$

(M^4, g) is a DFS if (61) holds. Then, contracting (61), we get

$$R = 4\lambda_1 + \kappa_1\sigma_1. \tag{62}$$

Again, we contract (3) and using (62), we get

$$\Lambda = \frac{\mathrm{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1\sigma_1}{4}\right\}(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{63}$$

Then, we have

Theorem 17. *If a CRBS $(g, V, \Lambda_1, \omega)$ is endowed with PFS $(g, V, \Lambda_1, \omega)$, then the soliton is shrinking, steady, and expanding according to*

$$\frac{\mathrm{div}V}{4} + \left\{\frac{4\lambda_1 + \kappa_1\sigma_1}{4}\right\}(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right) \mathop{\gtreqless} 0. \tag{64}$$

5. CRBS on Dark Fluid Spacetime ($\mathbf{D_1FS}$)

In D_1FS, ρ_1 is characterized by $-\sigma_1$. Then, the EMT (4) satisfies

$$\mathcal{T}_1(\mathbb{V}_1, \mathbb{V}_2) = \rho_1 g(\mathbb{V}_1, \mathbb{V}_2). \tag{65}$$

Using the identities (5) and (65) to arrive

$$S(\mathbb{V}_1, \mathbb{V}_2) = \left\{\kappa_1\rho_1 - \lambda_1 + \frac{R}{2}\right\}g(\mathbb{V}_1, \mathbb{V}_2). \tag{66}$$

Let (M^4, g) is a D_1FS admits (66). Then, by contracting (66) to consider $g(\xi, \xi) = -1$, we achieve

$$R = 4(\lambda_1 - \kappa_1\rho_1). \tag{67}$$

We contract in (3) and invoking (67) to acquire

$$\Lambda = \frac{\mathrm{div}V}{4} + (\lambda_1 - \kappa_1\rho_1)(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{68}$$

So, we get the following.

Theorem 18. *If a D_1FS is associated with a CRBS $(g, V, \Lambda_1, \omega)$, then the soliton is shrinking, steady, and expanding according to*

$$\frac{\mathrm{div}V}{4} + (\lambda_1 - \kappa_1\rho_1)(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right) \mathop{\gtreqless} 0. \tag{69}$$

6. CRBS Admitting Radiation Era in PFS

The characterization of radiation era, denoted by σ is equal to 3ρ. So, EMT (4) becomes

$$\mathcal{T}_1(\mathbb{V}_1, \mathbb{V}_2) = \rho[g(\mathbb{V}_1, \mathbb{V}_2) + 4\eta(\mathbb{V}_1)\eta(\mathbb{V}_2)]. \tag{70}$$

Using (5) and (70), we get

$$S(\mathbb{V}_1, \mathbb{V}_2) = \left\{\kappa_1\rho_1 - \lambda_1 + \frac{R}{2}\right\}g(\mathbb{V}_1, \mathbb{V}_2) + 4\kappa_1\rho_1\eta(\mathbb{V}_1)\eta(\mathbb{V}_2). \tag{71}$$

Let (M^4, g) be a radiation fluid spacetime(RFS) that admits (71). Then, we contract the identity (71) and using the fact that $g(\xi, \xi) = -1$ to yield

$$R = 4\lambda_1. \tag{72}$$

Now, we contract the identity (3) and using (72) to deduce

$$\Lambda = \frac{\mathrm{div}V}{4} + \lambda_1(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{73}$$

Then we have

Theorem 19. *If an RFS contains a CRBS $(g, V, \Lambda_1, \omega)$, then the soliton is shrinking, steady, expanding according as*

$$\frac{\mathrm{div}V}{4} + \lambda_1(1 - 4\omega) + \frac{1}{2}\left(p + \frac{1}{2}\right) \mathop{\gtreqless} 0. \tag{74}$$

7. Conclusion Remarks on Perfect Fluid and Robertson-Walker Spacetime

As the setting for large-scale cosmology, generalized Robertson-Walker (GRW) spacetimes are a logical and extensive extension of RW spacetime. They are a metric-defined Lorentzian manifold of size n [27],

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + q(t)^2 g^*_{\gamma\delta}(x_2, x_3, \ldots, x_n)\mathrm{dx}^\gamma \mathrm{dx}^\delta, \quad \gamma, \delta = 2, 3, \ldots .n, \tag{75}$$

where $g^*_{\gamma\delta}(x_2, x_3, \ldots, x_n)$ is the metric tensor of the wrapped product $(-I)\times_{q^2} M^*$ ([28–30]). A Robertson-Walker(RW) spacetime is one in which $M*$ has dimension 3 and constant curvature. A Lorentzian manifold is referred to as a time-oriented Lorentzian manifold, or space-time if it permits a globally time-like vector field. Since time is oriented in four dimensions, space-time is a Lorentzian manifold. Lorentzian manifolds are widely used in applied physics, particularly in relativity and cosmology. It is a wise decision for scholars to investigate relativity and cosmology since Lorentzian manifolds are crucial to the study of causation. A Ricci tensor of the form endowed with Lorentzian manifolds is defined by

$$\mathrm{Ag}_{\mathrm{ij}} + Bu_i u_j = R_{\mathrm{ij}}, \tag{76}$$

such that $u_i u^i = -1$ commonly referred to as ideal fluid spacetimes and A and B are referred to as scalar fields. If the energy-matter composition of space-time is a perfect fluid with velocity vector field u, then Einstein's equation implies the form of the above Ricci tensor equation. The pressure and energy density observed in the locally moving inertial frame are linearly connected to the scalars A and B. According to the Bianchi identity, $\nabla^m R_{\mathrm{im}} = 1/2\nabla_i R$, they are not independent. Shepley and Taub examined a perfect fluid spacetime in dimension $n = 4$ using the equation of state and the extra requirement that the Weyl tensor has null divergence [31, 32].

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no competing interests.

Authors' Contributions

All authors have made equal contributions and finalized them.

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