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Research Article

An Unconditionally Stable Numerical Method for Space Tempered Fractional Convection-Diffusion Models

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A second-order numerical method for two-sided tempered fractional convection-diffusion equations is studied in this paper, both convection term and diffusion term are approximated by the tempered weighted and shifted Grünwald difference operators, the first time partial derivative is discretized by the Crank–Nicolson method, and then a class of second-order numerical schemes is derived. By means of matrix method, numerical schemes are proved to be unconditionally stable and convergent with order $O(\tau^2 + h^2)$. The validity of the proposed numerical scheme is verified by numerical experiments.

1. Introduction

In this paper, the following two-sided space tempered fractional convection-diffusion equations is considered

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\left(l_\alpha \dfrac{\partial_-^{\alpha,\lambda_1} u(x,t)}{\partial x^\alpha} + r_\alpha \dfrac{\partial_+^{\alpha,\lambda_1} u(x,t)}{\partial x^\alpha} \right), & \\ + l_\beta \dfrac{\partial_-^{\beta,\lambda_2} u(x,t)}{\partial x^\beta} + r_\beta \dfrac{\partial_+^{\beta,\lambda_2} u(x,t)}{\partial x^\beta} + f(x,t), & (x,t) \in (a,b) \times (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_l(t), u(b,t) = \psi_r(t), & t \in [0,T], \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, and $\lambda_1, \lambda_2 \geq 0$, and parameters l_α, r_α, l_β, and r_β are nonnegative constants, which satisfy that $l_\xi + r_\xi \neq 0 (\xi \in \{\alpha, \beta\})$ and $\psi_l(t) \equiv 0$ if $l_\xi \neq 0$, and $\psi_r(t) \equiv 0$ if $r_\xi \neq 0$. $(\partial_-^{\xi,\lambda} u(x,t)/\partial x^\xi)$ and $(\partial_+^{\xi,\lambda} u(x,t)/\partial x^\xi)$ represent the normalized left and right Riemann–Liouville tempered fractional derivatives, respectively, which are defined as [1–3].

$$\begin{cases} \dfrac{\partial_-^{\xi,\lambda} u(x,t)}{\partial x^\xi} = {}_aD_x^{\xi,\lambda} u(x,t) - \lambda^\xi u(x,t), & 0 < \xi < 1, \\ \dfrac{\partial_-^{\xi,\lambda} u(x,t)}{\partial x^\xi} = {}_aD_x^{\xi,\lambda} u(x,t) - \lambda^\xi u(x,t) - \xi \lambda^{\xi-1} \dfrac{\partial u(x,t)}{\partial x}, & 1 < \xi < 2, \end{cases}$$
$$\begin{cases} \dfrac{\partial_+^{\xi,\lambda} u(x,t)}{\partial x^\xi} = {}_xD_b^{\xi,\lambda} u(x,t) - \lambda^\xi u(x,t), & 0 < \xi < 1, \\ \dfrac{\partial_+^{\xi,\lambda} u(x,t)}{\partial x^\xi} = {}_xD_b^{\xi,\lambda} u(x,t) - \lambda^\xi u(x,t) + \xi \lambda^{\xi-1} \dfrac{\partial u(x,t)}{\partial x}, & 1 < \xi < 2, \end{cases} \tag{2}$$

let $n = \lceil \xi \rceil$, where ${}_aD_x^{\xi,\lambda} u(x,t)$ and ${}_xD_b^{\xi,\lambda} u(x,t)$ represent the left and right Riemann–Liouville tempered fractional derivatives, respectively, are defined by the following equation:

$$\begin{cases} {}_aD_x^{\xi,\lambda} u(x,t) = \dfrac{e^{-\lambda x}}{\Gamma(n-\xi)} \dfrac{\partial^n}{\partial x^n} \left(\displaystyle\int_a^x \dfrac{e^{\lambda \tau} u(\tau,t)}{(x-\tau)^{\xi-n+1}} d\tau \right), \\ {}_xD_b^{\xi,\lambda} u(x,t) = \dfrac{(-1)^n e^{\lambda x}}{\Gamma(n-\xi)} \dfrac{\partial^n}{\partial x^n} \left(\displaystyle\int_x^b \dfrac{e^{-\lambda \tau} u(\tau,t)}{(\tau-x)^{\xi-n+1}} d\tau \right), \end{cases} \tag{3}$$

the $\Gamma(\cdot)$ in here is the gamma function. In fact, when $\lambda_1 = \lambda_2 = 0$, the equation is a well-known space fractional advection-dispersion equation [4], and in particular, if $l_\alpha = r_\alpha = -(1/(2\cos(\pi\alpha/2)))$ and $l_\beta = r_\beta = -(1/(2\cos(\pi\alpha/2)))$, the equation is the Riesz space fractional advection-dispersion equation [5]. In addition, when $l_\alpha = r_\alpha = 0$, the equation is the space tempered fractional diffusion equation [2, 3]; especially, if $l_\beta = r_\beta = -(1/(2\cos(\pi\alpha/2)))$, the equation is the Riesz space tempered fractional diffusion equation [6, 7].

Fractional derivatives have a long history, but due to the lack of practical background, until the recent decades, many scholars find that the application of fractional derivatives in plasma physics, groundwater hydrology, fluid mechanics, and many other fields can better describe the actual phenomenon and thus obtain many fractional models [1, 3, 4, 8–12]. Due to the nonlocality of fractional derivatives, analytical solutions for fractional models are often difficult to obtain, so it is urgent to develop numerical methods with high accuracy. Fortunately, many scholars have made a lot of important achievements [2, 6, 12–26] on the corresponding fractional model.

Among them, Meerschaert and Tadjeran [16] find that for the advection-dispersion equation, if the standard Grünwald difference operator is used to approximate the space fractional derivative, then either the explicit or implicit Euler method is unconditionally unstable, so in order to overcome this drawback, the shifted Grünwald difference operator is proposed, which combine with the implicit Euler method to derive an unconditionally stable and convergent numerical scheme with order $O(\tau + h)$. Later, Meerschaert and Tadjeran [17] develop a numerical scheme for the two-sided space fractional diffusion equation by utilizing the shifted Grünwald difference operators to approximate the left and right Riemann–Liouville fractional derivatives, but the convergence order is only $O(\tau + h)$. In order to improve the convergence order, Tian et al. [20] propose the weighted and shifted Grünwald difference operator and combine Crank–Nicolson time discretization to construct a class of second-order numerical schemes with $O(\tau^2 + h^2)$ for solving the two-sided space fractional diffusion equation. Thereafter, the unconditionally stable quasicompact scheme [23] and compact scheme [15] for the two-sided space fractional diffusion equation are quickly proposed, and the convergence order reaches $O(\tau^2 + h^3)$ and $O(\tau^2 + h^4)$, respectively.

Recently, a new variant of fractional calculus, in which power laws are tempered by an exponential factor, has attracted the attention of researchers because of its mathematical and practical advantages [3]. In fact, the space tempered fractional derivative and the time tempered fractional derivative obtain the space tempered fractional diffusion equation [8] and the time tempered fractional diffusion equation [27], respectively. It is important to develop the effective high-precision algorithms for the tempered fractional diffusion equation. Baeumer and Meerschaert [1] first propose the tempered shifted Grünwald difference operator to approximate the Riemann–Liouville tempered fractional derivative. Based on this, a class of the tempered weighted and shifted Grünwald difference operators with second-order precision is constructed by Li and Deng [2] to approximate the left and right

Riemann–Liouville tempered fractional derivatives, and numerical schemes with convergent order $O(\tau^2 + h^2)$ for solving the two-sided tempered fractional diffusion equation is obtained by combining the Crank–Nicolson method. Due to the influence of the exponential factor in the tempered fractional derivative, Yu et al. [12] only consider the compact technique in the numerical solution of one-sided tempered fractional diffusion equation, and combine with the implicit Euler method to obtain a numerical scheme with unconditional stability and convergence order $O(\tau + h^3)$, higher-order numerical schemes need to be further developed. The nonlinear tempered fractional diffusion equation is also studied by Zhao et al. [28], where a robust preconditioner is developed and analyzed to speed up the solution of the Jacobian matrix for the nonlinear all-at-once system obtained from the equation. For the time tempered fractional models, Feng et al. obtain some important results as detailed in [29, 30]. Since finite difference schemes for the (tempered) fractional diffusion equations usually produce Toeplitz matrices, which are expensive to compute directly, fast computation methods have been developed [31]. More work on tempered fractional models is shown in references [6, 7, 10, 13, 14, 32–34]. According to the existing literature, there are many researches on the effective numerical methods for space tempered fractional diffusion equation, but there are few research studies on the effective numerical methods for space tempered fractional convection-diffusion equation. In this paper, an unconditionally stable finite difference method for solving the equation is presented.

The rest of this article is arranged as follows. In Section 2, second-order difference approximations for the tempered fractional derivatives are introduced. Numerical schemes for solving the problem (1) are derived in Section 3. Section 4 discusses the stability and convergence theory of numerical schemes. In Section 5, the effectiveness of the proposed numerical scheme is demonstrated by numerical experiments. Section 6 concludes the work of this paper in brief.

2. Second-Order Difference Approximations for the Tempered Fractional Derivatives

In order to derive numerical schemes, we need some auxiliary knowledge.

$$S_\lambda^{n+\xi}(\mathbb{R}) = \left\{ \nu \mid \nu \in L_1(\mathbb{R}), \text{and} \int_{\mathbb{R}} (|\lambda| + |w|)^{n+\xi} |\widehat{\nu}(w)| \mathrm{d}w < \infty \right\}, \tag{4}$$

is a fractional Sobolev space $S_\lambda^{n+\xi}(\mathbb{R})$, where $\widehat{\nu}(w) = \int_{\mathbb{R}} \nu(x) e^{-iwx} \mathrm{d}x$ is the Fourier transform of $\nu(x)$.

Lemma 1 (see [1]). *Let $n-1<\xi<n(n=1,2)$ and $\lambda \geq 0$, the shift number p is an integer, h is the step size, and $\nu(x)$ is defined on the bounded interval $[a,b]$ and belongs to $S_\lambda^{2+\xi}(\mathbb{R})$ after zero extension on the interval $x \in (-\infty, a) \cup (b, +\infty)$. The tempered and shifted Grünwald type difference operators are defined as*

$$\begin{cases} A_{h,p}^{\xi,\lambda} \nu(x) = \dfrac{1}{h^\xi} \displaystyle\sum_{k=0}^{[(x-a)/h]+p} g_k^{(\xi)} e^{-(k-p)\lambda h} \nu(x-(k-p)h) - \dfrac{1}{h^\xi} G_p^{(\xi)}(1) \nu(x), \\ \widehat{A}_{h,p}^{\xi,\lambda} \nu(x) = \dfrac{1}{h^\xi} \displaystyle\sum_{k=0}^{[(b-x)/h]+p} g_k^{(\xi)} e^{-(k-p)\lambda h} \nu(x+(k-p)h) - \dfrac{1}{h^\xi} G_p^{(\xi)}(1) \nu(x), \end{cases} \tag{5}$$

then

$$\begin{cases} A_{h,p}^{\xi,\lambda} \nu(x) = {}_a D_x^{\xi,\lambda} \nu(x) - \lambda^\xi \nu(x) + O(h), \\ \widehat{A}_{h,p}^{\xi,\lambda} \nu(x) = {}_x D_b^{\xi,\lambda} \nu(x) - \lambda^\xi \nu(x) + O(h), \end{cases} \tag{6}$$

where $g_k^{(\xi)} = (-1)^k \binom{\xi}{k} (k \geq 0)$ denotes the normalized Grünwald weights

$$G_p^{(\xi)}(s) = e^{p\lambda h} \left(1 - e^{-\lambda h} s\right)^\xi = \sum_{k=0}^{+\infty} g_k^{(\xi)} e^{-(k-p)\lambda h} s^k. \tag{7}$$

Lemma 2 (see [2]). *Let $\nu(x) \in S_\lambda^{2+\xi}(\mathbb{R})$, $n-1<\xi<n(n=1,2)$, and $\lambda \geq 0$, if two difference operators are defined as*

$$
\begin{aligned}
B_h^{\xi,\lambda}\nu(x) &= \gamma_1^{(\xi)}A_{h,1}^{\xi,\lambda}\nu(x) + \gamma_2^{(\xi)}A_{h,0}^{\xi,\lambda}\nu(x) + \gamma_3^{(\xi)}A_{h,-1}^{\xi,\lambda}\nu(x)\\
&= \frac{1}{h^\xi}\sum_{k=0}^{[(x-a)/h]+1} w_k^{(\xi)}\nu(x-(k-1)h) - \frac{1}{h^\xi}\widehat{G}^{(\xi)}(1)\nu(x),\\
\widehat{B}_h^{\xi,\lambda}\nu(x) &= \gamma_1^{(\xi)}\widehat{A}_{h,1}^{\xi,\lambda}\nu(x) + \gamma_2^{(\xi)}\widehat{A}_{h,0}^{\xi,\lambda}\nu(x) + \gamma_3^{(\xi)}\widehat{A}_{h,-1}^{\xi,\lambda}\nu(x)\\
&= \frac{1}{h^\xi}\sum_{k=0}^{[(b-x)/h]+1} w_k^{(\xi)}\nu(x+(k-1)h) - \frac{1}{h^\xi}\widehat{G}^{(\xi)}(1)\nu(x),
\end{aligned}
\tag{8}
$$

then

$$
\begin{cases}
B_h^{\xi,\lambda}\nu(x) = {}_aD_x^{\xi,\lambda}\nu(x) - \lambda^\xi\nu(x) + O(h^2),\\
\widehat{B}_h^{\xi,\lambda}\nu(x) = {}_xD_b^{\xi,\lambda}\nu(x) - \lambda^\xi\nu(x) + O(h^2),
\end{cases}
\tag{9}
$$

where

$$
\begin{aligned}
w_k^{(\xi)} &= \left(\gamma_1^{(\xi)}g_k^{(\xi)} + \gamma_2^{(\xi)}g_{k-1}^{(\xi)} + \gamma_3^{(\xi)}g_{k-2}^{(\xi)}\right)e^{-(k-1)\lambda h} \quad \left(k\ge 0, g_{-2}^{(\xi)} = g_{-1}^{(\xi)} = 0\right),\\
\gamma_2^{(\xi)} &= \frac{2+\xi}{2} - 2\gamma_1^{(\xi)},\\
\gamma_3^{(\xi)} &= \gamma_1^{(\xi)} - \frac{\xi}{2},\\
\widehat{G}^{(\xi)}(1) &= \gamma_1^{(\xi)}G_1^{(\xi)}(1) + \gamma_2^{(\xi)}G_0^{(\xi)}(1) + \gamma_3^{(\xi)}G_{-1}^{(\xi)}(1).
\end{aligned}
\tag{10}
$$

In this part, because $\gamma_1^{(\xi)}$ is a freely variable quantity, therefore, a class of second-order operators for the Riemann–Liouville tempered fractional derivatives is given. In the later part, we will find that, in fact, numerical schemes are valid only when $\gamma_1^{(\xi)}$ takes a certain range of values.

3. Numerical Discretization

The spatial interval $[a,b]$ and temporal interval $[0,T]$ are uniformly meshed, respectively, the spatial step size is denoted as $h=(b-a)/M$ and the temporal step size is denoted as $\tau = T/N$, therefore $x_i = a+ih, 0\le i\le M, t_n = n\tau$, and $0\le n\le N$. Let $u_i^n = u(x_i,t_n)$ and U_i^n represent the exact solution and the numerical solution at point (x_i,t_n), respectively. Denoting $t_{n+1/2} = (t_n+t_{n+1})/2$, $u_i^{n+1/2} = (u_i^n + u_i^{n+1})/2$, $f_i^{n+1/2} = f(x_i,t_{n+1/2})$.

In this article, we always assume that the function $u(x,\cdot)$ in problem (1) belongs to $S_\lambda^{2+\beta}(\mathbb{R})$ after zero extension.

The Crank–Nicolson method is used to discrete problem (1) at point $(x_i,t_{n+1/2})$, and then, a semidiscretization form is obtained

$$
\begin{cases}
\dfrac{u_i^{n+1}-u_i^n}{\tau} = -\left(l_\alpha\left(\dfrac{\partial_-^{\alpha,\lambda_1}u}{\partial x^\alpha}\right)_i^{n+1/2} + r_\alpha\left(\dfrac{\partial_+^{\alpha,\lambda_1}u}{\partial x^\alpha}\right)_i^{n+1/2}\right) + l_\beta\left(\dfrac{\partial_-^{\beta,\lambda_2}u}{\partial x^\beta}\right)_i^{n+1/2}\\
+ r_\beta\left(\dfrac{\partial_+^{\beta,\lambda_2}u}{\partial x^\beta}\right)_i^{n+1/2} + f_i^{n+1/2} + O(\tau^2), \quad 0\le n\le N-1.
\end{cases}
\tag{11}
$$

According to Lemma 2, we further obtain

$$\begin{cases} \dfrac{u_i^{n+1}-u_i^n}{\tau} = -\left(l_\alpha B_h^{\alpha,\lambda_1} u_i^{n+1/2} + r_\alpha \widehat{B}_h^{\alpha,\lambda_1} u_i^{n+1/2}\right) + l_\beta\left(B_h^{\beta,\lambda_2} u_i^{n+1/2} - \beta\lambda_2^{\beta-1}\delta_x u_i^{n+1/2}\right) \\ + r_\beta\left(\widehat{B}_h^{\beta,\lambda_2} u_i^{n+1/2} + \beta\lambda_2^{\beta-1}\delta_x u_i^{n+1/2}\right) + f_i^{n+1/2} + O\left(\tau^2+h^2\right), \quad 1 \le i \le M-1, \end{cases} \tag{12}$$

where $\delta_x u_i^n = (u_{i+1}^n - u_{i-1}^n)/(2h)$.

Denoting $U_i^{n+1/2} = ((U_i^n + U_i^{n+1})/2)$, numerical schemes can be obtained by eliminating the local truncation error in (12)

$$\left\{ \frac{U_i^{n+1}-U_i^n}{\tau} = -\left(l_\alpha B_h^{\alpha,\lambda_1} U_i^{n+1/2} + r_\alpha \widehat{B}_h^{\alpha,\lambda_1} U_i^{n+1/2}\right) + l_\beta\left(B_h^{\beta,\lambda_2} U_i^{n+1/2} - \beta\lambda_2^{\beta-1}\delta_x U_i^{n+1/2}\right) + r_\beta\left(\widehat{B}_h^{\beta,\lambda_2} U_i^{n+1/2} + \beta\lambda_2^{\beta-1}\delta_x U_i^{n+1/2}\right) + f_i^{n+1/2}, \right. \tag{13}$$

that is

$$\begin{cases} U_i^{n+1} = U_i^n - \tau\left(l_\alpha \dfrac{1}{h^\alpha}\displaystyle\sum_{k=0}^{i+1} w_k^{(\alpha)} \dfrac{U_{i-k+1}^n + U_{i-k+1}^{n+1}}{2} - \left(l_\alpha + r_\alpha\right)\dfrac{1}{h^\alpha}\widehat{G}^{(\alpha)}(1)\dfrac{U_i^n + U_i^{n+1}}{2} + r_\alpha \dfrac{1}{h^\alpha}\displaystyle\sum_{k=0}^{M-i+1} w_k^{(\alpha)} \dfrac{U_{i+k-1}^n + U_{i+k-1}^{n+1}}{2}\right) \\ + \tau\left(l_\beta \dfrac{1}{h^\beta}\displaystyle\sum_{k=0}^{i+1} w_k^{(\beta)} \dfrac{U_{i-k+1}^n + U_{i-k+1}^{n+1}}{2} - \left(l_\beta + r_\beta\right)\dfrac{1}{h^\beta}\widehat{G}^{(\beta)}(1)\dfrac{U_i^n + U_i^{n+1}}{2} + r_\beta \dfrac{1}{h^\beta}\displaystyle\sum_{k=0}^{M-i+1} w_k^{(\beta)} \dfrac{U_{i+k-1}^n + U_{i+k-1}^{n+1}}{2} - \beta\lambda_2^{\beta-1}\left(l_\beta - r_\beta\right)\dfrac{U_{i+1}^{n+1} + U_{i+1}^n - U_{i-1}^{n+1} - U_{i-1}^n}{4h}\right) \\ + \tau f_i^{n+1/2}. \end{cases} \tag{14}$$

Further, the matrix form of numerical schemes (14) is

$$\begin{cases} \left(I + \dfrac{\tau\left(l_\alpha B^{(\alpha)} + r_\alpha B^{(\alpha)^T}\right)}{2h^\alpha} - \dfrac{\tau\left(l_\beta B^{(\beta)} + r_\beta B^{(\beta)^T}\right)}{2h^\beta} + \dfrac{\tau\beta\lambda_2^{\beta-1}\left(l_\beta - r_\beta\right)}{4h} C\right) U^{n+1} \\ = \left(I - \dfrac{\tau\left(l_\alpha B^{(\alpha)} + r_\alpha B^{(\alpha)^T}\right)}{2h^\alpha} + \dfrac{\tau\left(l_\beta B^{(\beta)} + r_\beta B^{(\beta)^T}\right)}{2h^\beta} - \dfrac{\tau\beta\lambda_2^{\beta-1}\left(l_\beta - r_\beta\right)}{4h} C\right) U^n + \tau F^{n+1/2}, \end{cases} \tag{15}$$

where $U^n = (U_1^n, U_2^n, \ldots, U_{M-2}^n, U_{M-1}^n)^T$ and $C = \text{tridiag}\{-1, 0, 1\}$ is $(M-1)$ order tridiagonal matrix

$$B^{(\xi)} = \begin{pmatrix} w_1^{(\xi)} - \widehat{G}^{(\xi)}(1) & w_0^{(\xi)} & & & & \\ w_2^{(\xi)} & w_1^{(\xi)} - \widehat{G}^{(\xi)}(1) & w_0^{(\xi)} & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ w_{M-2}^{(\xi)} & w_{M-3}^{(\xi)} & w_{M-4}^{(\xi)} & \cdots & w_1^{(\xi)} - \widehat{G}^{(\xi)}(1) & w_0^{(\xi)} \\ w_{M-1}^{(\xi)} & w_{M-2}^{(\xi)} & w_{M-3}^{(\xi)} & \cdots & w_2^{(\xi)} & w_1^{(\xi)} - \widehat{G}^{(\xi)}(1) \end{pmatrix}, \quad \xi \in (0,1) \text{ or } \xi \in (1,2), \tag{16}$$

the term $F^{n+1/2}$ is given by

$$\begin{cases} F^{n+1/2} = \left(f_1^{n+1/2}, f_2^{n+1/2}, \ldots, f_{M-1}^{n+1/2}\right)^T - \dfrac{U_0^{n+1/2}}{h^\alpha}\left(l_\alpha w_2^{(\alpha)} + r_\alpha w_0^{(\alpha)}, l_\alpha w_3^{(\alpha)}, \ldots, l_\alpha w_M^{(\alpha)}\right)^T \\ - \dfrac{U_M^{n+1/2}}{h^\alpha}\left(r_\alpha w_M^{(\alpha)}, \ldots, r_\alpha w_3^{(\alpha)}, r_\alpha w_2^{(\alpha)} + l_\alpha w_0^{(\alpha)}\right)^T + \dfrac{U_0^{n+1/2}}{h^\beta}\left(l_\beta w_2^{(\beta)} + r_\beta w_0^{(\beta)}, l_\beta w_3^{(\beta)}, \ldots, l_\beta w_M^{(\beta)}\right)^T \\ + \dfrac{U_M^{n+1/2}}{h^\beta}\left(r_\beta w_M^{(\beta)}, \ldots, r_\beta w_3^{(\beta)}, r_\beta w_2^{(\beta)} + l_\beta w_0^{(\beta)}\right)^T - \dfrac{\left(l_\beta - r_\beta\right)\beta\lambda_2^{\beta-1}}{2h}\left(-U_0^{n+1/2}, 0, \ldots, 0, U_M^{n+1/2}\right)^T. \end{cases} \tag{17}$$

4. Theoretical Analysis of Numerical Schemes

Before giving a detailed theoretical analysis, let's first give some lemmas that will be used.

Lemma 3 (see [35]). *Real matrix A of order M is positive definite if and only if $D = (A + A^T)/2$ is positive definite.*

Lemma 4 (see [2]). *For $1 < \beta < 2$, $\lambda_2 \geq 0$, and $h > 0$, if max $\left\{(2(\beta^2 + 3\beta - 4))/(\beta^2 + 3\beta + 2), (\beta^2 + 3\beta)/(\beta^2 + 3\beta + 4)\right\} \leq \gamma_1^{(\beta)} \leq (3(\beta^2 + 3\beta - 2))/(2(\beta^2 + 3\beta + 2))$, then satisfy*

$$w_0^{(\beta)} + w_2^{(\beta)} \geq 0, w_1^{(\beta)} \leq 0, w_k^{(\beta)} \geq 0 \ (k \geq 3), \tag{18}$$

the matrix $B^{(\beta)}$ in (15) is negative definite.

Lemma 5. *For $0 < \alpha < 1$, $\lambda_1 \geq 0$, $h > 0$, if the following condition is satisfied*

$$\frac{2\left(\alpha^2 + 3\alpha - 4\right)}{\alpha^2 + 3\alpha + 2} \leq \gamma_1^{(\alpha)} \leq 0, \tag{19}$$

then the matrix $B^{(\alpha)}$ in (15) is positive definite.

Proof. Note that

$$\begin{aligned}
g_k^{(\alpha)} &= (-1)^k \binom{\alpha}{k}, \quad k \geq 0,\\
g_0^{(\alpha)} &= 1,\\
g_1^{(\alpha)} &= -\alpha,\\
g_k^{(\alpha)} &= \frac{k-1-\alpha}{k} g_{k-1}^{(\alpha)} \leq 0, \quad (k \geq 2),\\
w_k^{(\alpha)} &= \left(\gamma_1^{(\alpha)} g_k^{(\alpha)} + \gamma_2^{(\alpha)} g_{k-1}^{(\alpha)} + \gamma_3^{(\alpha)} g_{k-2}^{(\alpha)}\right) e^{-(k-1)\lambda_1 h} \quad \left(k \geq 0, g_{-2}^{(\alpha)} = g_{-1}^{(\alpha)} = 0\right)
\end{aligned} \tag{20}$$

If $\gamma_1^{(\alpha)} \leq 0$,

$$\begin{aligned}
w_1^{(\alpha)} &= \gamma_1^{(\alpha)} g_1^{(\alpha)} + \gamma_2^{(\alpha)} g_0^{(\alpha)} = -(\alpha+2)\gamma_1^{(\alpha)} + \frac{\alpha+2}{2} \geq 0,\\
w_0^{(\alpha)} + w_2^{(\alpha)} &= \gamma_1^{(\alpha)} g_0^{(\alpha)} e^{\lambda_1 h} + \left(\gamma_1^{(\alpha)} g_2^{(\alpha)} + \gamma_2^{(\alpha)} g_1^{(\alpha)} + \gamma_3^{(\alpha)} g_0^{(\alpha)}\right) e^{-\lambda_1 h}\\
&= \gamma_1^{(\alpha)} e^{\lambda_1 h} + \left(\frac{\alpha^2+3\alpha+2}{2}\gamma_1^{(\alpha)} - \frac{\alpha^2+3\alpha}{2}\right) e^{-\lambda_1 h}\\
&\leq \left(\frac{\alpha^2+3\alpha+4}{2}\gamma_1^{(\alpha)} - \frac{\alpha^2+3\alpha}{2}\right) e^{-\lambda_1 h} \leq 0.
\end{aligned} \tag{21}$$

More generally, if $\gamma_1^{(\alpha)} \geq (k(\alpha^2+3\alpha+4-2k))/(2(\alpha^2+3\alpha+2))$ $(k \geq 3)$, we obtain

$$\begin{aligned}
w_k^{(\alpha)} &= \left(\gamma_1^{(\alpha)} \frac{(k-1-\alpha)(k-2-\alpha)}{k(k-1)} + \gamma_2^{(\alpha)} \frac{k-2-\alpha}{k-1} + \gamma_3^{(\alpha)}\right) g_{k-2}^{(\alpha)} e^{-(k-1)\lambda_1 h}\\
&= \left(\frac{\alpha^2+3\alpha+2}{k(k-1)}\gamma_1^{(\alpha)} - \frac{\alpha^2+3\alpha+4-2k}{2(k-1)}\right) g_{k-2}^{(\alpha)} e^{-(k-1)\lambda_1 h} \leq 0,
\end{aligned} \tag{22}$$

because $(k(\alpha^2+3\alpha+4-2k))/(2(\alpha^2+3\alpha+2))$ is decreasing monotonically, so make $\gamma_1^{(\alpha)} \geq (3(\alpha^2+3\alpha-2))/(2(\alpha^2+3\alpha+2))$.

Further, positive root of equation $(3(\alpha^2+3\alpha-2))/(2(\alpha^2+3\alpha+2)) = 0$ is $\alpha = ((\sqrt{17}-3)/2)$, when $0 < \alpha \leq ((\sqrt{17}-3)/2)$ and $\gamma_1^{(\alpha)}$ has a range of values $(3(\alpha^2+3\alpha-2))/(2(\alpha^2+3\alpha+2)) \leq \gamma_1^{(\alpha)} \leq 0$, therefore,

$$w_0^{(\alpha)} + w_2^{(\alpha)} \leq 0, \quad w_1^{(\alpha)} \geq 0, w_k^{(\alpha)} \leq 0\,(k \geq 3). \tag{23}$$

Let $D = (B^{(\alpha)} + B^{(\alpha)T})/2 = (d_{i,j})_{M-1,M-1}$, we have

$$D = \begin{pmatrix} w_1^{(\alpha)} - \widehat{G}^{(\alpha)}(1) & \frac{1}{2}\left(w_0^{(\alpha)} + w_2^{(\alpha)}\right) & \frac{1}{2}w_3^{(\alpha)} & \cdots & \frac{1}{2}w_{M-2}^{(\alpha)} & \frac{1}{2}w_{M-1}^{(\alpha)} \\ \frac{1}{2}\left(w_0^{(\alpha)} + w_2^{(\alpha)}\right) & w_1^{(\alpha)} - \widehat{G}^{(\alpha)}(1) & \frac{1}{2}\left(w_0^{(\alpha)} + w_2^{(\alpha)}\right) & \cdots & \frac{1}{2}w_{M-3}^{(\alpha)} & \frac{1}{2}w_{M-2}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{2}w_{M-2}^{(\alpha)} & \frac{1}{2}w_{M-3}^{(\alpha)} & \frac{1}{2}w_{M-4}^{(\alpha)} & \cdots & w_1^{(\alpha)} - \widehat{G}^{(\alpha)}(1) & \frac{1}{2}\left(w_0^{(\alpha)} + w_2^{(\alpha)}\right) \\ \frac{1}{2}w_{M-1}^{(\alpha)} & \frac{1}{2}w_{M-2}^{(\alpha)} & \frac{1}{2}w_{M-3}^{(\alpha)} & \cdots & \frac{1}{2}\left(w_0^{(\alpha)} + w_2^{(\alpha)}\right) & w_1^{(\alpha)} - \widehat{G}^{(\alpha)}(1) \end{pmatrix}. \quad (24)$$

Because the following relationship is established

$$\begin{aligned} \sum_{k=0}^{+\infty} w_k^{(\alpha)} &= \sum_{k=0}^{+\infty}\left(\gamma_1^{(\alpha)} g_k^{(\alpha)} + \gamma_2^{(\alpha)} g_{k-1}^{(\alpha)} + \gamma_3^{(\alpha)} g_{k-2}^{(\alpha)}\right)e^{-(k-1)\lambda_1 h} \\ &= \gamma_1^{(\alpha)} G_1^{(\alpha)}(1) + \gamma_2^{(\alpha)} G_0^{(\alpha)}(1) + \gamma_3^{(\alpha)} G_{-1}^{(\alpha)}(1) \\ &= \widehat{G}^{(\alpha)}(1), \end{aligned} \quad (25)$$

it is easy to check from (23)

$$\sum_{k=0}^{M-1} w_k^{(\alpha)} - \widehat{G}^{(\alpha)}(1) > 0, \quad (26)$$

that is

$$d_{i,i} > -\sum_{j\neq i}^{M-1} d_{i,j}, \quad i = 1, 2, \ldots, M-1. \quad (27)$$

Utilizing the Gershgorin theorem [36], we know that the eigenvalues of matrix D are all positive. That is, matrix D is positive definite, from Lemma 3, matrix $B^{(\alpha)}$ is positive definite.

When $((\sqrt{17} - 3)/2) < \alpha < 1$, we find that $\gamma_1^{(\alpha)}$ has no range, so we take $\gamma_1^{(\alpha)} \geq (2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2)$, that is $0 \geq \gamma_1^{(\alpha)} \geq (2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2)$, but in this case,

$$w_0^{(\alpha)} + w_2^{(\alpha)} \leq 0, \quad w_1^{(\alpha)} \geq 0, w_3^{(\alpha)} \geq 0, w_k^{(\alpha)} \leq 0 \, (k \geq 4), \quad (28)$$

let us construct a new symmetric Toeplitz matrix $P^- \in \mathbb{R}^{(M-1)\times(M-1)}$,

$$P^- = \begin{pmatrix} p_1 & p_2 & p_3 & & & \\ p_2 & p_1 & p_2 & p_3 & & \\ p_3 & p_2 & p_1 & p_2 & p_3 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & p_3 & p_2 & p_1 & p_2 \\ & & & p_3 & p_2 & p_1 \end{pmatrix}, \quad (29)$$

where $p_3 = -w_3^{(\alpha)}/2$ and $p_1 + 2p_2 + 2p_3 = 0$, which should be negative definite and can make $D + P^-$ is positive definite. In order to get the negative definite result, a similar calculation idea [12] is used to get $p_1 = 6p_3$ and $p_2 = -4p_3$. It is easy to check matrix $D + P^-$ is positive definite, from Weyl's theorem [37], matrix D is positive definite, that is, $B^{(\alpha)}$ is positive definite. When $0 < \alpha \le (\sqrt{17} - 3)/2$, let $(2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2) \le \gamma_1^{(\alpha)} \le (3(\alpha^2 + 3\alpha - 2))/(2(\alpha^2 + 3\alpha + 2))$, similar provability, $B^{(\alpha)}$ is positive definite.

In summary, for all $\alpha \in (0,1)$, when $(2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2) \le \gamma_1^{(\alpha)} \le 0$, the conclusion of the theorem is obtained.

Define $\mathbf{U}_h = \{\mathbf{u}|\mathbf{u} = \{\mathbf{u}_i\}$ is a grid function defined on $\{x_i = a + ih\}_{i=1}^{M-1}\}$. For $\mathbf{u} \in \mathbf{U}_h$, the corresponding discrete L_2-norm is defined as $\|\mathbf{u}\|_{L_2} = (h\sum_{i=1}^{M-1}\mathbf{u}_i^2)^{1/2}$.

Theorem 6. *For $\alpha \in (0,1)$ and $\beta \in (1,2)$, let $\gamma_1^{(\alpha)} \in ((2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2), 0)$ and $\gamma_1^{(\beta)} \in (\max\{(2(\beta^2 + 3\beta - 4))/(\beta^2 + 3\beta + 2), (\beta^2 + 3\beta)/(\beta^2 + 3\beta + 4)\}, (3(\beta^2 + 3\beta - 2))/(2(\beta^2 + 3\beta + 2)))$, then numerical schemes (14) are unconditionally stable.*

Proof. The proof is similar to [2]. Denoting $Q = (\tau(l_\alpha B^{(\alpha)} + r_\alpha B^{(\alpha)^T}))/(2h^\alpha) - (\tau(l_\beta B^{(\beta)} + r_\beta B^{(\beta)^T}))/(2h^\beta) + (\tau\beta\lambda_2^{\beta-1}(l_\beta - r_\beta)/4h)C$, then the matrix form of numerical schemes (15) is simplified as

$$(I + Q)U^{n+1} = (I - Q)U^n + \tau F^{n+1/2}. \quad (30)$$

Let the eigenvalue of matrix Q be $\lambda(Q)$, then $(1 - \lambda(Q))/(1 + \lambda(Q))$ is the eigenvalue of matrix $(I + Q)^{-1}(I - Q)$. Since $(Q + Q^T)/2 = (\tau(l_\alpha + r_\alpha)(B^{(\alpha)} + B^{(\alpha)^T}))/(4h^\alpha) - (\tau(l_\beta + r_\beta)(B^{(\beta)} + B^{(\beta)^T}))/(4h^\beta)$, it is easy to know that $\lambda(Q) > 0$ by Lemma 3–5, so $|(1 - \lambda(Q))/(1 + \lambda(Q))| < 1$. Furthermore, we can obtain that the spectral radius of matrix $(I + Q)^{-1}(I - Q)$ is less than one, thus numerical schemes are unconditionally stable.

Lemma 7. *For the matrix Q in (30), there exists*

$$\left\|(I + Q)^{-1}\right\|_2 \le 1, \left\|(I + Q)^{-1}(I - Q)\right\|_2 \le 1, \quad (31)$$

where $\|\cdot\|_2$ denotes 2-norm(spectral norm).

Proof. Since the matrix Q in (30) is positive definite, according to Lemma 7 in [2], the two inequalities presented by the lemma are true.

Theorem 8. *For $\alpha \in (0,1)$, $\beta \in (1,2)$, let $\gamma_1^{(\alpha)} \in ((2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2), 0)$ and $\gamma_1^{(\beta)} \in (\max\{(2(\beta^2 + 3\beta - 4))/(\beta^2 + 3\beta + 2), (\beta^2 + 3\beta)/(\beta^2 + 3\beta + 4)\}, (3(\beta^2 + 3\beta - 2))/(2(\beta^2 + 3\beta + 2)))$, then numerical schemes (14) are convergent, that is*

$$\left\|e^n\right\|_{L_2} \le C_1\left(\tau^2 + h^2\right), \quad n = 1, 2, \ldots, N, \quad (32)$$

where $e^n = (e_1^n, e_2^n, \ldots, e_{M-1}^n)^T$, $e_i^n = u_i^n - U_i^n$, and C_1 is a existed constant independent of time and space steps.

Proof. The proof is similar to [2]. Subtracting (13) from (12), we know that

$$(I + Q)e^{n+1} = (I - Q)e^n + R^n, \quad (33)$$

where $R^n = (R_1^n, R_2^n, \ldots, R_{M-1}^n)^T$, and the local truncation error $R_i^n = O(\tau^3 + \tau h^2)$.

Furthermore, (33) can obtain

$$e^{n+1} = (I + Q)^{-1}(I - Q)e^n + (I + Q)^{-1}R^n. \quad (34)$$

By taking Euclidean norm $\|\cdot\|_2$ on both sides of (34) at the same time, we can see

$$\begin{aligned}\left\|e^n\right\|_2 &\le \left\|(I + Q)^{-1}(I - Q)e^{n-1}\right\|_2 + \left\|(I + Q)^{-1}R^{n-1}\right\|_2 \\ &\le \left\|(I + Q)^{-1}(I - Q)\right\|_2\left\|e^{n-1}\right\|_2 + \left\|(I + Q)^{-1}\right\|_2\left\|R^{n-1}\right\|_2.\end{aligned} \quad (35)$$

Noting that $\|\cdot\|_{L_2} = h^{1/2}\|\cdot\|_2$, by Lemma 7, we have

$$\begin{aligned}\left\|e^n\right\|_{L_2} &\le \left\|(I + Q)^{-1}(I - Q)\right\|_2\left\|e^{n-1}\right\|_{L_2} + \left\|(I + Q)^{-1}\right\|_2\left\|R^{n-1}\right\|_{L_2} \\ &\le \left\|e^{n-1}\right\|_{L_2} + \left\|R^{n-1}\right\|_{L_2} \\ &\le \left\|e^{n-2}\right\|_{L_2} + \left\|R^{n-2}\right\|_{L_2} + \left\|R^{n-1}\right\|_{L_2} \\ &\le \left\|e^0\right\|_{L_2} + \sum_{k=0}^{n-1}\left\|R^k\right\|_{L_2} \le C_1\left(\tau^2 + h^2\right).\end{aligned} \quad (36)$$

Thus, the theorem is proved.

5. Numerical Experiments

Some numerical experiments are given in this section to verify the precision and validity of the proposed numerical scheme.

$$\text{Order} = \log_m\left(\frac{\|e\|_{L_2,h}}{\|e\|_{L_2,h/m}}\right), \tag{37}$$

is the order of measurement.

Example 1. Consider the following tempered fractional convection-diffusion equations

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\left(l_\alpha \dfrac{\partial_-^{\alpha,\lambda_1} u(x,t)}{\partial x^\alpha} + r_\alpha \dfrac{\partial_+^{\alpha,\lambda_1} u(x,t)}{\partial x^\alpha}\right) + l_\beta \dfrac{\partial_-^{\beta,\lambda_2} u(x,t)}{\partial x^\beta} + r_\beta \dfrac{\partial_+^{\beta,\lambda_2} u(x,t)}{\partial x^\beta} + f(x,t), & (x,t) \in (0,1)\times(0,1], \\ u(0,t) = 0, u(1,t) = 0, & t \in [0,1], \\ u(x,0) = e^{-\lambda_1 x} x^n (1-x)^n, & x \in (0,1), \end{cases} \tag{38}$$

where $0<\alpha<1$, $1<\beta<2$, and $n \in \mathbb{N}(n \geq 2)$, and the linear source term is

$$\begin{aligned} f(x,t) = {} & \alpha e^{\alpha t-\lambda_1 x} x^n (1-x)^n + e^{\alpha t}\Bigg[l_\alpha e^{-\lambda_1 x} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(n+1+k)}{\Gamma(n+1+k-\alpha)} x^{n+k-\alpha} \\ & + r_\alpha e^{\lambda_1 x} \sum_{i=0}^{+\infty} \frac{(-2\lambda_1)^i}{i!} \sum_{k=0}^{n+i} (-1)^k \binom{n+i}{k} \frac{\Gamma(n+1+k)}{\Gamma(n+1+k-\alpha)} (1-x)^{n+k-\alpha} - (l_\alpha + r_\alpha)\lambda_1^\alpha e^{-\lambda_1 x} x^n (1-x)^n \Bigg] \\ & - e^{\alpha t}\Bigg[l_\beta e^{-\lambda_2 x} \sum_{i=0}^{+\infty} \frac{(\lambda_2-\lambda_1)^i}{i!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(n+1+k+i)}{\Gamma(n+1+k+i-\beta)} x^{n+k+i-\beta} \\ & + r_\beta e^{\lambda_2 x} \sum_{i=0}^{+\infty} \frac{(-\lambda_1-\lambda_2)^i}{i!} \sum_{k=0}^{n+i} (-1)^k \binom{n+i}{k} \frac{\Gamma(n+1+k)}{\Gamma(n+1+k-\beta)} (1-x)^{n+k-\beta} \\ & - (l_\beta + r_\beta)\lambda_2^\beta e^{-\lambda_1 x} x^n (1-x)^n + (r_\beta - l_\beta)\beta\lambda_2^{\beta-1} e^{-\lambda_1 x}\left(-\lambda_1 x^n (1-x)^n + n\left(x-x^2\right)^{n-1}(1-2x)\right)\Bigg] \end{aligned} \tag{39}$$

the exact solution is $u(x,t) = e^{\alpha t-\lambda_1 x} x^n (1-x)^n$.

Let $l_\alpha = 1/5$, $r_\alpha = 1$, $l_\beta = 2$, $r_\beta = 3$, and $n = 2$. Choose different α, β, and λ_1, λ_2, the proposed numerical scheme ($\gamma_1^{(\alpha)} = (2(\alpha^2+3\alpha-4))/(\alpha^2+3\alpha+2)$, and $\gamma_1^{(\beta)} = \beta/2$) is adopted to solve Example 1, the errors and measurement order results are displayed in Table 1. It can be seen from Table 1 that the proposed numerical scheme has second-order precision, and the numerical experimental results are consistent with the convergence analysis.

Let $l_\alpha = 1$, $r_\alpha = 3$, $l_\beta = 5$, $r_\beta = 7$, and $n = 2$. Choose different α, β, and λ_1, λ_2, the proposed numerical scheme is

Table 1: Errors and corresponding measurement order results at $t = 1$.

(α, β)	$h = \tau$	$\lambda_1 = 1/10$ $\lambda_2 = 1$ $\|e\|_{L_2}$	Order	$\lambda_1 = 1$ $\lambda_2 = 2$ $\|e\|_{L_2}$	Order	$\lambda_1 = 2$ $\lambda_2 = 5$ $\|e\|_{L_2}$	Order
(0.2, 1.2)	1/20	1.6077e − 03		2.3196e − 03		4.4553e − 03	
	1/40	4.1307e − 04	1.9605	6.1026e − 04	1.9263	1.3047e − 03	1.7718
	1/80	1.0465e − 04	1.9808	1.5496e − 04	1.9775	3.4072e − 04	1.9370
	1/160	2.6431e − 05	1.9853	3.9055e − 05	1.9883	8.6292e − 05	1.9813
(0.5, 1.5)	1/16	3.6870e − 04		1.0821e − 03		3.5873e − 03	
	1/32	1.0548e − 04	1.8055	2.8301e − 04	1.9349	1.0204e − 03	1.8138
	1/64	2.9817e − 05	1.8228	7.3480e − 05	1.9454	2.6661e − 04	1.9363
	1/128	8.3152e − 06	1.8423	1.9065e − 05	1.9464	6.8031e − 05	1.9705
(0.8, 1.8)	1/10	5.0678e − 04		5.1402e − 04		1.7371e − 04	
	1/20	1.3156e − 04	1.9456	1.3326e − 04	1.9476	4.5103e − 05	1.9454
	1/40	3.3388e − 05	1.9783	3.3800e − 05	1.9791	1.1761e − 05	1.9392
	1/80	8.4106e − 06	1.9890	8.5114e − 06	1.9896	3.0136e − 06	1.9645

adopted to solve Example 1, the errors and measurement order results are displayed in Table 2. As can be seen from Table 2, the numerical scheme has second-order precision in both spatial and temporal directions, and the calculated results are in complete agreement with the theoretical results.

Example 2. Consider the following tempered fractional convection-diffusion equations

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\left(l_\alpha \dfrac{\partial_-^{\alpha,\lambda_1} u(x,t)}{\partial x^\alpha} + r_\alpha \dfrac{\partial_+^{\alpha,\lambda_1} u(x,t)}{\partial x^\alpha} \right) + l_\beta \dfrac{\partial_-^{\beta,\lambda_2} u(x,t)}{\partial x^\beta} + r_\beta \dfrac{\partial_+^{\beta,\lambda_2} u(x,t)}{\partial x^\beta} + f(x,t), & (x,t) \in (0,1) \times (0,1], \\ u(0,t) = 0, u(1,t) = 0, & t \in [0,1], \\ u(x,0) = e^{\lambda_2 x} x^m (1-x)^m, & x \in (0,1), \end{cases} \tag{40}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, and $m \in \mathbb{N} (m \geq 2)$, and the linear source term is

$$\begin{aligned} f(x,t) = & -\beta \sin \beta t \cdot e^{\lambda_2 x} x^m (1-x)^m + \cos \beta t \Bigg[l_\alpha e^{-\lambda_1 x} \sum_{i=0}^{+\infty} \frac{(\lambda_1 + \lambda_2)^i}{i!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \cdot \frac{\Gamma(m+1+k+i)}{\Gamma(m+1+k+i-\alpha)} x^{m+k+i-\alpha} \\ & + r_\alpha e^{\lambda_1 x} \sum_{i=0}^{+\infty} \frac{(-\lambda_1 + \lambda_2)^i}{i!} \sum_{k=0}^{m+i} (-1)^k \binom{m+i}{k} \cdot \frac{\Gamma(m+1+k)}{\Gamma(m+1+k-\alpha)} (1-x)^{m+k-\alpha} - (l_\alpha + r_\alpha) \lambda_1^\alpha e^{\lambda_2 x} x^m (1-x)^m \Bigg] \\ & - \cos \beta t \Bigg[l_\beta e^{-\lambda_2 x} \sum_{i=0}^{+\infty} \frac{(2\lambda_2)^i}{i!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma(m+1+k+i)}{\Gamma(m+1+k+i-\beta)} x^{m+k+i-\beta} \\ & + r_\beta e^{\lambda_2 x} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma(m+1+k)}{\Gamma(m+1+k-\beta)} (1-x)^{m+k-\beta} \\ & - (l_\beta + r_\beta) \lambda_2^\beta e^{\lambda_2 x} x^m (1-x)^m + (r_\beta - l_\beta) \beta \lambda_2^{\beta-1} e^{\lambda_2 x} \left(\lambda_2 x^m (1-x)^m + m(x - x^2)^{m-1} (1-2x) \right) \Bigg], \end{aligned} \tag{41}$$

Table 2: Numerical results for different conditions at $t=1$.

(α,β)	$h=\tau$	$\lambda_1=1/10$ $\lambda_2=1/5$ $\|e\|_{L_2}$	Order	$\lambda_1=1$ $\lambda_2=2$ $\|e\|_{L_2}$	Order	$\lambda_1=2.5$ $\lambda_2=5$ $\|e\|_{L_2}$	Order
(0.2, 1.2)	1/20	$9.6201e-04$		$2.8343e-03$		$4.3376e-03$	
	1/40	$2.4581e-04$	1.9685	$7.6226e-04$	1.8946	$1.3554e-03$	1.6782
	1/80	$6.2406e-05$	1.9777	$1.9469e-04$	1.9691	$3.6113e-04$	1.9081
	1/160	$1.5817e-05$	1.9802	$4.9125e-05$	1.9866	$9.1987e-05$	1.9730
(0.5, 1.5)	1/20	$2.9138e-04$		$8.2818e-04$		$2.1142e-03$	
	1/40	$7.4756e-05$	1.9626	$2.1545e-04$	1.9425	$5.8274e-04$	1.8592
	1/80	$1.8969e-05$	1.9785	$5.5742e-05$	1.9505	$1.5104e-04$	1.9479
	1/160	$4.8432e-06$	1.9696	$1.4400e-05$	1.9527	$3.8495e-05$	1.9722
(0.8, 1.8)	1/20	$7.8062e-04$		$1.9177e-04$		$1.0822e-03$	
	1/40	$1.9158e-04$	2.0266	$4.9995e-05$	1.9395	$2.8578e-04$	1.9210
	1/80	$4.6421e-05$	2.0450	$1.2584e-05$	1.9901	$7.3308e-05$	1.9629
	1/160	$1.1186e-05$	2.0531	$3.1796e-06$	1.9847	$1.8689e-05$	1.9718

Table 3: Numerical experimental results at the end time under different parameters.

(α,β)	$h=\tau$	$\lambda_1=1/10$ $\lambda_2=1/5$ $\|e\|_{L_2}$	Order	$\lambda_1=1$ $\lambda_2=2$ $\|e\|_{L_2}$	Order	$\lambda_1=2$ $\lambda_2=3$ $\|e\|_{L_2}$	Order
(0.2, 1.2)	1/10	$1.4810e-03$		$1.6693e-02$		$4.5001e-02$	
	1/20	$4.0031e-04$	1.8873	$5.9209e-03$	1.4953	$1.9241e-02$	1.2257
	1/40	$1.0299e-04$	1.9586	$1.6551e-03$	1.8388	$5.8019e-03$	1.7295
	1/80	$2.6261e-05$	1.9715	$4.2900e-04$	1.9478	$1.5364e-03$	1.9169
(0.5,1.5)	1/10	$8.9199e-05$		$9.7279e-04$		$3.8466e-03$	
	1/20	$2.1687e-05$	2.0401	$2.8074e-04$	1.7928	$1.2082e-03$	1.6707
	1/40	$5.1075e-06$	2.0861	$7.6602e-05$	1.8737	$3.3638e-04$	1.8446
	1/80	$1.2247e-06$	2.0601	$2.0466e-05$	1.9041	$8.9203e-05$	1.9149
(0.8,1.8)	1/10	$4.0676e-04$		$1.2477e-03$		$4.0271e-03$	
	1/20	$9.8779e-05$	2.0419	$1.5072e-04$	3.0493	$4.8425e-04$	3.0559
	1/40	$2.3833e-05$	2.0512	$3.4398e-05$	2.1314	$9.2126e-05$	2.3940
	1/80	$5.7239e-06$	2.0578	$7.6320e-06$	2.1721	$2.4161e-05$	1.9309

Table 4: Results of error and observation order under different conditions at $t=1$.

(α,β)	$h=\tau$	$\lambda_1=1/10$ $\lambda_2=1/5$ $\|e\|_{L_2}$	Order	$\lambda_1=1$ $\lambda_2=2$ $\|e\|_{L_2}$	Order	$\lambda_1=2$ $\lambda_2=3$ $\|e\|_{L_2}$	Order
(0.2, 1.2)	1/10	$1.5096e-02$		$6.3176e-02$		$9.2840e-02$	
	1/20	$5.7957e-03$	1.3811	$2.5168e-02$	1.3278	$3.8253e-02$	1.2792
	1/40	$1.9178e-03$	1.5955	$8.0822e-03$	1.6388	$1.2195e-02$	1.6493
	1/80	$5.9597e-04$	1.6861	$2.3900e-03$	1.7577	$3.5337e-03$	1.7870
(0.5, 1.5)	1/10	$9.6364e-04$		$2.2824e-03$		$4.7234e-03$	
	1/20	$7.8141e-05$	3.6243	$7.6761e-04$	1.5721	$1.9661e-03$	1.2645
	1/40	$1.5580e-05$	2.3264	$2.4260e-04$	1.6618	$6.3040e-04$	1.6410
	1/80	$4.7875e-06$	1.7024	$7.1658e-05$	1.7594	$1.8159e-04$	1.7956
(0.8,1.8)	1/10	$1.5970e-02$		$2.1523e-02$		$1.2134e-02$	
	1/20	$6.8271e-03$	1.2260	$8.5664e-03$	1.1698	$5.6779e-03$	1.0956
	1/40	$2.3147e-03$	1.5604	$3.0524e-03$	1.4887	$2.2788e-03$	1.3171
	1/80	$6.8285e-04$	1.7612	$9.5351e-04$	1.6786	$7.3702e-04$	1.6285

the exact solution is $u(x,t)=\cos\beta t\cdot e^{\lambda_2 x}x^m(1-x)^m$.

Let $l_\alpha=1$, $r_\alpha=3$, $l_\beta=5$, $r_\beta=7$, and $m=2$. Choose different α, β, and λ_1, λ_2, the proposed numerical scheme ($\gamma_1^{(\alpha)}=(2(\alpha^2+3\alpha-4))/(\alpha^2+3\alpha+2)$, $\gamma_1^{(\beta)}=(\beta/2)$) is used to solve Example 2, and the errors and measurement order results are shown in Table 3. As can be seen from Table 3, the numerical solutions both in space and time direction has second-order accuracy, and the calculation results fit well with the theoretical results.where $0 < \alpha < 1$ and $1 < \beta < 2$.

Example 3. Consider the initial-boundary value problem of tempered fractional convection-diffusion equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\left(\dfrac{\partial_{-}^{\alpha,\lambda_1} u(x,t)}{\partial x^{\alpha}} + \dfrac{\partial_{+}^{\alpha,\lambda_1} u(x,t)}{\partial x^{\alpha}}\right) + \dfrac{\partial_{-}^{\beta,\lambda_2} u(x,t)}{\partial x^{\beta}} + \dfrac{\partial_{+}^{\beta,\lambda_2} u(x,t)}{\partial x^{\beta}}, & (x,t) \in (0,1) \times (0,1], \\ u(0,t) = 0, u(1,t) = 0, & t \in [0,1], \\ u(x,0) = \dfrac{1}{\sqrt{2\pi}} e^{-(x^2/2)}, & x \in (0,1), \end{cases} \tag{42}$$

Use the proposed method to take the step size $h = \tau = 1/400$, the result obtained in solving Example 3 is taken as the reference solution. Choose different α, β, and λ_1, λ_2, the proposed numerical scheme $(\gamma_1^{(\alpha)} = (2(\alpha^2 + 3\alpha - 4))/(\alpha^2 + 3\alpha + 2), \gamma_1^{(\beta)} = \beta/2)$ is used to solve Example 3, the errors and measurement order results are displayed in Table 4. It can be seen from Table 4 that both the time and space convergence order are two.

6. Conclusions

This paper aims to numerically study the tempered fractional convection-diffusion equations. First of all, a class of second-order numerical schemes for solving the equations is proposed. Then, the theoretical results of unconditional stability and convergence of numerical schemes are obtained. Finally, the validity of the proposed numerical scheme is verified by numerical experiments.

Data Availability

The author declares that all data and material in the paper are available and veritable.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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