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Research Article

Characterization of a Cournot–Nash Equilibrium for a Fishery Model with Fuzzy Utilities

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The article deals with the extensions of discrete-time games with infinite time horizon and their application in a fuzzy context to fishery models. The criteria for these games are the total discounted utility and the average utility in a fishing problem. However, in the fuzzy case, game theory is not the best way to represent a real fishing problem because players do not always have enough information to accurately estimate their utility in the context of fishing. For this reason, in this paper, trapezoidal-type fuzzy utility values are considered for a fishing model, and the terms of the Nash equilibrium are given in the fuzzy context, i.e., this equilibrium is represented using the partial order of the α-cuts of the fuzzy numbers; to the best of the authors' knowledge, there is no work with this type of treatment. To obtain each equilibrium, a suitable fully determined fuzzy game is used in combination with the dynamic programming technique applied to this game in the context of fishing. The main results are (i) the Nash equilibria of the fuzzy games coincide with the Nash equilibria of the nonfuzzy games and are explicitly determined in a fishery model and (ii) the values of the fuzzy games are of trapezoidal type and are also explicitly given in the fishery model.

1. Introduction

This paper deals with the fuzzy game applied to fisheries under both the discounted and the average criteria. The manuscript presents a two-player dynamic game with reproduction dynamics. However, it is not always admissible for players to have knowledge of the exact values of their utility function, under this approach it is necessary to propose a game in a fuzzy environment. In this case, the main difference with the crisp game lies in that the utility function considered for each player is a trapezoidal fuzzy function. These fuzzy utility functions represent that imprecision or vagueness in the data. The objective is to guarantee the existence of a Nash equilibrium in a fuzzy appropriate sense for this noncompetitive game; to the best of the authors' knowledge, there are no works with this kind of treatment.

Specifically, this work concerns a class of games where, in contrast with the classical framework, the utility function $\widetilde{U}_i$ of player i $(i = 1, 2)$ is a fuzzy function of a trapezoidal type, which is determined from a classical utility function U_i by applying an affine transformation with the fuzzy coefficients. Under the conditions ensuring that the classical model with the utility function U_i has an average optimal stationary policy $\pi_{i,o}$ with the optimal average utility j_i^*, it is shown that such a policy is also optimal for the fuzzy model with function $\widetilde{U}_i$, and that the optimal fuzzy average value $\widetilde{j}_i^*$ is obtained from j_i^* via the same affine transformation used to go from U_i to $\widetilde{U}_i$. And with $(\widetilde{j}_1^*, \widetilde{j}_2^*)$ and $(\pi_{1,o}, \pi_{2,o})$, the corresponding Nash equilibrium for the game is obtained (see Lemmas 16 and 17). Similar results are also obtained for the fuzzy discounted case (see Theorem 13 and Corollary 14).

Finally, the authors propose specific values for a game model for fisheries in the diffuse context to illustrate the theory presented. In which, a comparison is made between the average and discounted cases to show that in the average case, the biomass level decreases more slowly. This type of conclusion makes it possible to describe a strategy that preserves the level of biomass in an ecosystem.

2. Literature Review

Now some comments on the literature are related to the theme of this paper. In general situations, marine fish resources are typically overexploited. One reason for this situation is associated with the overcapacity of the fishing gear. In this sense, it is important to propose adequate extraction strategies that consider each country's benefit and the preservation of the biomass. Extensive literature on the subject in this direction can be found about the fisheries and the game theory (see, for instance, [1, 2]). On the other hand, for the fuzzy games, these references can be consulted: [3–6]. However, for the fuzzy dynamic games, the literature is scarce: this is one of the reasons that motivate the current work. Papers [7–9] deserve special mention as they deal with a certain class of games with applications in the logistic management during the COVID-19 pandemic outbreak. However, they are not developed in a fuzzy context and in [9] some parameters are considered as uncertainty; specifically, they are assumed as fuzzy trapezoidal numbers, and these fuzzy parameters are substituted by the mean values of their components in the treated mathematical models.

The problem studied in this work has its origins in the great fish war game by Levhari and Mirman [10] and its subsequent references in [11–13]. All of them are under the approach of nonfuzzy (or crisp) game theory.

3. Motivation

Now, in this paper, a fuzzy version of this fisheries game is proposed. Specifically, suppose that two countries extract fish from the same region, and each of them obtains profits from the extracted biomass via a trapezoidal fuzzy utility. It will be assumed that the system presents a dynamic induced by a difference equation via a stock-recruitment function (see (7)).

Hence, in contrast with these previous papers, this manuscript presents an infinite-horizon sequential dynamic game with the following characteristics:

(a) It models a version of a fishery between two countries with the fuzzy utilities of a trapezoidal type.

(b) As the objective functions, the fuzzy versions of the total discounted utility and the average utility are taken into account. And with respect to these fuzzy functions, the corresponding fuzzy versions of the Nash equilibria are established with respect to the partial order of the α-cuts of the fuzzy numbers.

(c) In order to obtain the Nash equilibria, the dynamic programming approach on a suitable nonfuzzy problem is used.

(d) The Nash equilibria for both the discounted and the average cases are explicitly obtained and they are given in terms of the parameters of the model. Also, the fuzzy values of the games are determined.

In addition to points (a–d), it is important to mention that the results obtained in this paper extend to the fuzzy context ones obtained in paper [10] (see Remark 15). This extension allows ambiguity, vagueness, or approximate features of the problem being modelled to be considered in the objective function, unlike in the case of the nonfuzzy one.

It is relevant to remark that the approach developed and analyzed here considers that the uncertainty is measured through a trapezoidal fuzzy utility function, and this permits modelling the fact that the utility U_i is approximately in a certain interval instead of receiving U_i directly (see Remark 3.1). Moreover, in the literature of fuzzy set theory, the class of trapezoidal numbers is considered a sufficiently robust family, in the sense that any fuzzy number, under certain conditions, can be approximated by fuzzy trapezoidal numbers (see, for instance, [14, 15]). On the other hand, in the literature referring to optimization theory, several applications can be found in which fuzzy payment/cost functions are considered, for instance, [16–18]. Also, in Markov decision processes, fuzzy payment/cost functions have been applied as part of the objective function to be optimized ([19–21]).

The paper is organized as follows. In Section 4, some definitions and basic results of the fuzzy numbers are presented. In Section 5, the fuzzy fisheries problem formulation is introduced. Later, in Sections 6 and 7, the existence of Nash equilibrium is demonstrated for the discounted and the average criterion, respectively. Section 8 provides some numerical results, and in Section 9, the conclusions are given.

4. Preliminaries

In this section, definitions and results of the fuzzy theory are presented. For a detailed exposition of the topics, you can consult, for instance, [22–24]. To this end, firstly some notation about the fuzzy numbers is introduced, which is used in rest of the paper. The following standard mathematical symbols will be distinguished in the fuzzy context with the asterisk symbol "$*$." That is, in the fuzzy context, "$<$," "$+$," and "$\sum$" will be denoted by "$<*$," "$+*$," and "$\sum *$," respectively. Similarly, in the fuzzy context, the limit "lim" and the supremum "sup" will be denoted by "$\lim *$" and "$\sup *$," respectively. And the product of a real number λ and a fuzzy number Υ will be denoted by $\lambda\Upsilon$. Moreover, $\mathbb{R}$ is the set of all real numbers. The set of the fuzzy numbers will be denoted by $\mathfrak{F}(\mathbb{R})$.

In the manuscript, an important class of fuzzy numbers is considered. This class is denominated trapezoidal fuzzy numbers (see Definition 1). This family includes another relevant set of fuzzy numbers: the triangular fuzzy numbers (see, for instance, [25–27]). Furthermore, trapezoidal fuzzy numbers could be used to approximate an arbitrary fuzzy number (see, for instance, [14, 15]).

Definition 1 (see [22]). A fuzzy number A is called a trapezoidal fuzzy number if its membership function has the following form:

$$\mu(x) = \begin{cases} 0, & \text{if } x < a, \\ \dfrac{x-a}{b-a}, & \text{if } a \le x < b, \\ 1, & \text{if } b \le x \le c, \\ \dfrac{d-x}{d-c}, & \text{if } c < x \le d, \\ 0, & \text{if } x > d, \end{cases} \tag{1}$$

where a, b, c, and d are real numbers such that $a < b \le c < d$. In the subsequent sections, a trapezoidal fuzzy number A will be denoted by (a, b, c, d).

Figure 1 illustrates a graphical representation of the trapezoidal fuzzy number $A = (2, 6, 12, 16)$.

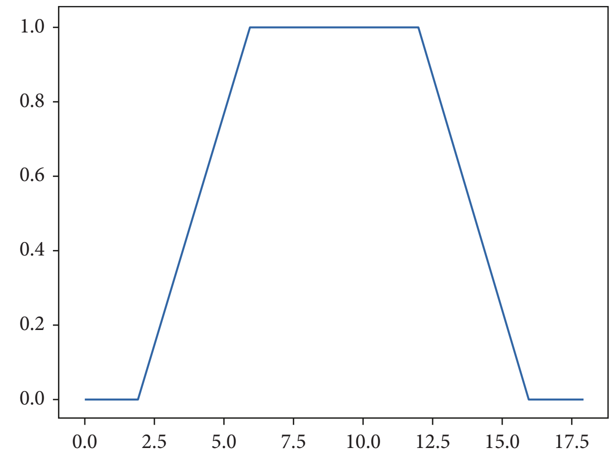

Figure 1: Trapezoidal fuzzy number $A = (2, 6, 12, 16)$.

Definition 2 (see [22]). A fuzzy number A is called a triangular fuzzy number if its membership function has the following form:

$$\mu(x) = \begin{cases} 0, & \text{if } x < a, \\ \dfrac{x-a}{b-a}, & \text{if } a \le x < b, \\ \dfrac{c-x}{c-b}, & \text{if } b \le x \le c, \\ 0, & \text{if } x > c, \end{cases} \tag{2}$$

where a, b, and c are real numbers such that $a < b < c$. A triangular fuzzy number A will be denoted by (a, b, c).

Figure 2 illustrates a graphical representation of the triangular fuzzy number $A = (5, 10, 15)$.

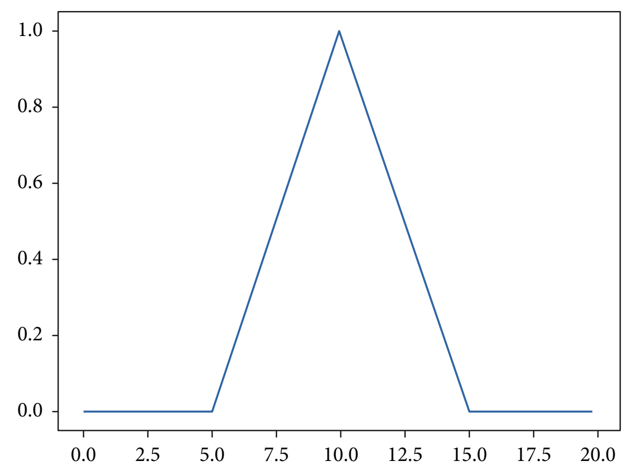

Figure 2: Triangular fuzzy number $A = (5, 10, 15)$.

Definition 3 (see [22]). Given a fuzzy set A defined on $\mathbb{R}$ with the membership function μ and any $\alpha \in [0, 1]$, the α-cut, denoted by A_α, is defined to be the set $A_\alpha := \{x \in \mathbb{R}: \mu(x) \ge \alpha\}$, for $\alpha \in [0, 1]$, and A_0 is the closure of $\{x \in \mathbb{R}: \mu(x) > 0\}$.

Remark 4. Definition 19.1 and Definition 3 imply that for each $\alpha \in [0, 1]$, $(a, b, c, d)_\alpha = [(b-a)\alpha + a, d - (d-c)\alpha]$, for the trapezoidal fuzzy numbers.

In the next definition, standard arithmetic operations on real numbers are extended to fuzzy numbers. In Definition 5, $\mu_{A \star B}$ denotes the membership function of fuzzy number $A \star B$, and equation (3) illustrates how to calculate such a membership function from the membership functions of the fuzzy numbers A and B, for details see [23].

Definition 5. Let $\star$ denote any of the four basic arithmetic operations and let $A, B \in \mathfrak{F}(\mathbb{R})$. Then, a fuzzy set on $\mathbb{R}$, $A \star B$, is defined by the following expression:

$$\mu_{A \star B}(u) = \sup_{u = x \star y} \min\{\mu_A(x), \mu_B(y)\}, \tag{3}$$

for all $u \in \mathbb{R}$.

A direct consequence of the previous definition is shown in the following result [28].

Lemma 6. *If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers, then the basic operators for the trapezoidal fuzzy numbers are as follows:*

(a) $A +^* B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$
(b) $A -^* B = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4)$

(c) $\lambda A = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4)$, for each $\lambda \geq 0$

Let D denote the set of all the closed bounded intervals on the real line. The following partial order is defined: for $A = [a_l, a_u], B = [b_l, b_u] \in D$, and $A \leq B$ if and only if $a_l \leq b_l$ and $a_u \leq b_u$. Furthermore, for $\tilde{\mu}, \tilde{\nu} \in \mathfrak{F}(\mathbb{R})$, we define the following equation:

$$\tilde{\mu} \leq^* \tilde{\nu} \text{ if and only if } \tilde{\mu}_\alpha \leq \tilde{\nu}_\alpha, \tag{4}$$

for all $\alpha \in [0, 1]$.

Remark 7. Observe that "$\leq^*$" corresponds to a partial order of $\mathfrak{F}(\mathbb{R})$. A partial order is a reflexive, transitive, and antisymmetric binary relation. In this case, $(\mathfrak{F}(\mathbb{R}), \leq^*)$ is a partially ordered set or poset. Moreover, if $\tilde{x}$ satisfies that $x \leq^* \tilde{x}$ for each $x \in \mathfrak{F}(\mathbb{R})$, then $\tilde{x}$ is an upper bound for $\mathfrak{F}(\mathbb{R})$, see [23]. If the set of upper bounds of $\mathfrak{F}(\mathbb{R})$ has the least element, then this element is called the supremum of $\mathfrak{F}(\mathbb{R})$.

For $A, B \in D$, $A = [a_l, a_u]$, and $B = [b_l, b_u]$, we define the following equation:

$$d(A; B) = \max\left\{|a_l - b_l|, |a_u - b_u|\right\}. \tag{5}$$

It is possible to check that d defines a metric on D, and (D, d) is a complete metric space.

Now, define $\hat{d}$: $\mathfrak{F}(\mathbb{R}) \times \mathfrak{F}(\mathbb{R}) \longrightarrow \mathbb{R}$ by the following equation:

$$\hat{d}(\mu, \nu) = \sup_{\alpha \in [0,1]} d(\mu_\alpha, \nu_\alpha), \tag{6}$$

with $\mu, \nu \in \mathfrak{F}(\mathbb{R})$. It is straightforward to prove that $\hat{d}$ is a metric in $\mathfrak{F}(\mathbb{R})$, see [29].

5. Fuzzy Fisheries Problem Formulation

Suppose that two countries or companies (players) extract fish from the same region; this way each country has its utilities. Moreover, each country is interested in maximizing the total sum of the discounted consumption utilities, taking into account the actions of the other country, so that it always gets its best catch. It is assumed that the model satisfies the following properties:

(a) The two countries extract the same type of fish, that is, there is no differentiation of products.
(b) The countries do not cooperate (there is no collusion).
(c) The countries have a market power, in other words, the extraction decision of each country affects the price of the good.
(d) The countries compete for the amount of extraction and choose them simultaneously. This implies that the two countries compete for the quantity of fish that is extracted from the region, and simultaneously choose how much to fish.
(e) The countries are economically rational and act strategically, seeking to maximize their profits given the decisions of their competitors.

Under the assumptions mentioned above, the model will be described below. To this end, let $X = [0, \infty)$ and consider a continuous function h: $X \longrightarrow X$. This function is referred to as the recruitment function. Suppose that at each time $t \in \{0, 1, \ldots\}$, the current stock of fish or the state of the system is $x_t = x \in X$, then Player (Country) 1 extracts $a_{1,t} = a_1 \in A(x) := [0, h(x)]$ and Player 2 extracts $a_{2,t} = a_2 \in A(x) = [0, h(x)]$, with the condition that $x > a_1 + a_2$, i.e., the two countries together cannot extract more fish than is produced. The set $A(x)$ represents the admissible actions for each player, i.e., the amount of fish that each player decides to catch when the stock of the fish is x, with $x \in X$. Then, the remaining fish in the region is $x - a_1 - a_2$, which represents the fish biomass level. Now, this quantity is affected by the reproduction function h, then the fish biomass level in the next period is $h(x_t - a_{1,t} - a_{2,t})$. Thus, the dynamic of the system is modelled according to the following difference equation:

$$x_{t+1} = h\left(x_t - a_{1,t} - a_{2,t}\right)\theta, \tag{7}$$

with $x_0 = x \in X$ known, $t = 0, 1, \ldots$ and $\theta \in [0, 1]$ represents the fish mortality rate or the proportion of stock that migrates from the region. The transition law described in equation (7) will be referred to in the subsequent sections simply as h.

Now, the admissible harvesting policies will be defined for each player. Firstly, the admissible histories are defined up to time t as follows: $\mathbb{H}_0 := X$ and $\mathbb{H}_t := \mathbb{K} \times \mathbb{H}_{t-1}$, $t \in \{1, 2 \ldots\}$, where $\mathbb{K} := \{(x, a_1, a_2)$: $x \in X$ and a_1, $a_2 \in A(x)\}$. Consider that A is the set of all possible actions, i.e., $A = \cup_{x \in X} A(x)$ and $X = A$. Then, a plan or harvesting policy is a sequence of functions $\pi = \{\pi_t\}$, such that for each $t \in \{0, 1, \ldots\}$, π_t: $\mathbb{H}_t \longrightarrow A$ satisfies the feasibility requirement that $\pi_t(h_t) \in A(x_t)$, $t = 0, 1, \ldots$. The family of all strategies for player i will be denoted by Π_i, for $i = 1, 2$. Let $\Pi := \Pi_1 \times \Pi_2$, so an element of Π will be denoted by $\hat{\pi}$ and will be called a multistrategy. Let $\mathbb{F}$ be the set of all functions f: $X \longrightarrow A$ such that $f(x) \in A(x)$ for all $x \in X$. Thus, a strategy $\pi = \{f_0, f_1, \ldots\}$ is called Markov if for each $t = 0, 1, \ldots, f_t \in \mathbb{F}$. A particular class of Markov policies is the family of stationary policies. A stationary policy is a Markov policy, $\pi = \{f_0, f_1, \ldots\}$, such that $f_t = f \in \mathbb{F}$ for all $t = 0, 1, \ldots$. The set of all stationary strategies will be denoted by $\mathbb{F}$.

The revenue of harvesting for each Country is measured by a utility function, so consider that U_1: $A \longrightarrow \mathbb{R}$ represents the utility function for Country 1 and U_2: $A \longrightarrow \mathbb{R}$ for Country 2. Now, suppose that each Player has an ambiguous revenue, then the utility function for each player is characterized by a fuzzy utility function, specifically a trapezoidal fuzzy function, see Definition 1. Then, consider for each $i = 1, 2$, the following utility function:

$$\begin{aligned}\widetilde{U}_i(a) &= \left(\frac{1}{2}, 1, 1, 2\right)U_i(a) +^* (\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon)\\ &= \left(\frac{1}{2}U_i(a) + \varepsilon, U_i(a) + 2\varepsilon, U_i(a) + 3\varepsilon, 2U_i(a) + 4\varepsilon\right),\end{aligned} \tag{8}$$

with $a \in A(x)$, $\varepsilon > 0$, and $x \in X$. The utility function described in equation (8) is used to model the ambiguity, vagueness, or approximate characteristics of the fisheries problem. This vagueness can be the result either of a lack of exactness in the measure of the elements that are necessary for the determination of the states of nature or the purely subjective interpretation of these states [30]. Moreover, recently fuzzy utility functions have been applied in algorithm data mining, see for instance [31].

Remark 8. Note that equation (8) models the fact that the utility is approximately in the interval $[U_i(a_i) + 2\varepsilon; U_i(a_i) + 3\varepsilon]$ instead of receiving U_i as it happens for a crisp utility with $\varepsilon > 0$ arbitrary.

In this way, the fuzzy game is conformed for the components: $(X, A, \{A(x)\colon x \in X\}, h, \{\widetilde{U}_i\colon i = 1, 2\})$. In rest of the manuscript, it is assumed that $\varepsilon > 0$ is fixed. The following definitions are a generalization of those corresponding to the crisp case, see, for instance, [2, 32].

Definition 9. Let β_i be a fixed number in $(0, 1)$, define the fuzzy discounted payment function for country $i = 1, 2$, as follows:

$$\begin{aligned}\widetilde{V}_i(x, \hat{\pi}) &= \sum_{t=0}^{\infty}{}^{*} \beta_i^t \widetilde{U}_i(a_t)\\ &= \left(\frac{1}{2}V_i(x, \hat{\pi}) + \frac{\varepsilon}{1-\beta_i}, V_i(x, \hat{\pi}) + \frac{2\varepsilon}{1-\beta_i}, V_i(x, \hat{\pi}) + \frac{3\varepsilon}{1-\beta_i}, 2V_i(x, \hat{\pi}) + \frac{4\varepsilon}{1-\beta_i}\right),\end{aligned} \tag{9}$$

for each multistrategy $\hat{\pi} \in \Pi$ and an initial state $x \in X$. Number β_i is called the player's discount factor for $i = 1, 2$. And $V_i(x, \hat{\pi})$ denotes the total discounted utility when Player i has an initial stock of fish $x \in X$ and the multistrategy $\hat{\pi} \in \Pi$ is applied, then V_i is defined as $V_i(x, \hat{\pi}) = \sum_{t=0}^{\infty} \beta_i^t U_i(a_t)$.

Definition 10. A multistrategy $\hat{\pi}_o = (\pi_{1,o}, \pi_{2,o}) \in \Pi$ is a fuzzy Nash equilibrium of the game if

$$\begin{aligned}\widetilde{V}_1\big(x, (\pi_1, \pi_{2,o})\big) &\leq^* \widetilde{V}_1\big(x, \hat{\pi}_o\big),\\ \widetilde{V}_2\big(x, (\pi_{1,o}, \pi_2)\big) &\leq^* \widetilde{V}_2\big(x, \hat{\pi}_o\big),\end{aligned} \tag{10}$$

for all $\pi_1 \in \Pi_1$, $\pi_2 \in \Pi_2$, and $x \in X$.

Remark 11. A fuzzy Nash equilibrium in the context of fisheries is conformed by the harvest strategies in such a way that each player obtains the greatest benefit under the condition of maintaining the biomass level of the population.

Observe that Definition 10 implies the following characterization of the total payment function:

$$\begin{aligned}\widetilde{V}_1\big(x, (\pi_{1,o}, \pi_{2,o})\big) &= \sup_{\pi_1 \in \Pi_1}{}^{*}\ \widetilde{V}_1\big(x, (\pi_1, \pi_{2,o})\big),\\ \widetilde{V}_2\big(x, (\pi_{1,o}, \pi_{2,o})\big) &= \sup_{\pi_2 \in \Pi_2}{}^{*}\ \widetilde{V}_2\big(x, (\pi_{1,o}, \pi_2)\big),\end{aligned} \tag{11}$$

for all $\pi_1 \in \Pi_1$, $\pi_2 \in \Pi_2$, and $x \in X$. In this case, the optimal value function for Players 1 and 2 is defined, respectively, as follows

$$\begin{aligned}\widetilde{V}_1(x) &= \widetilde{V}_1\big(x, (\pi_{1,o}, \pi_{2,o})\big),\\ \widetilde{V}_2(x) &= \widetilde{V}_2\big(x, (\pi_{1,o}, \pi_{2,o})\big),\end{aligned} \tag{12}$$

$x \in X$ and $(\pi_{1,o}, \pi_{2,o}) \in \Pi$ is called a fuzzy Nash equilibrium.

Now, the analogous concepts for the crisp game will be defined, which is obtained by the following components: $(X, A, \{A(x)\colon x \in X\}, h, \{U_i\colon i = 1, 2\})$. Then, a Nash equilibrium $\hat{\pi} = (\pi_{1,o}, \pi_{2,o}) \in \Pi$ for the crisp game satisfies that

$$V_1\big(x, (\pi_1, \pi_{2,o})\big) \leq V_1\big(x, \hat{\pi}_o\big), \tag{13}$$

and

$$V_2\big(x, (\pi_{1,o}, \pi_2)\big) \leq V_2\big(x, \hat{\pi}_o\big), \tag{14}$$

for all $\pi_1 \in \Pi_1$, $\pi_2 \in \Pi_2$, and $x \in X$. Next, a result is presented which allows relating the equilibrium for both games, crisp and fuzzy.

Lemma 12. *If $\hat{\pi}_o = (\pi_{1,o}, \pi_{2,o})$ is a Nash equilibrium for the crisp game, then $\hat{\pi}_o$ is a Nash equilibrium for the fuzzy game.*

Proof. Let $\hat{\pi}_o = (\pi_{1,o}, \pi_{2,o})$ be a Nash equilibrium for the crisp game, fixed. Then, the α-cut of $\widetilde{V}_1$ (see Definitions 3 and 9) is given by the following equation:

$$\widetilde{V}_{1,\alpha} := \left[\frac{V_1(x,\hat{\pi})}{2} + \frac{\varepsilon}{1-\beta} + \alpha\left(\frac{V_1(x,\hat{\pi})}{2} + \frac{\varepsilon}{1-\beta}\right), 2V_1(x,\hat{\pi}) + \frac{4\varepsilon}{1-\beta} - \alpha\left(V_1(x,\hat{\pi}) + \frac{2\varepsilon}{1-\beta}\right)\right]. \tag{15}$$

Now, applying equation (13), it follows that

$$\begin{aligned}
&\frac{1}{2}V_1(x,\hat{\pi}) + \frac{\varepsilon}{1-\beta} + \alpha\left(\frac{1}{2}V_1(x,\hat{\pi}) + \frac{\varepsilon}{1-\beta}\right)\\
&\quad\geq \frac{1}{2}V_1\left(x,\left(\pi_1,\pi_{2,o}\right)\right) + \frac{\varepsilon}{1-\beta} + \alpha\left(\frac{1}{2}V_1\left(x,\left(\pi_1,\pi_{2,o}\right)\right) + \frac{\varepsilon}{1-\beta}\right),\\
&2V_1(x,\hat{\pi}) + 4\frac{\varepsilon}{1-\beta} - \alpha\left(V_1(x,\hat{\pi}) + 2\frac{\varepsilon}{1-\beta}\right)\\
&\quad\geq 2V_1\left(x,\left(\pi_1,\pi_{2,o}\right)\right) + \frac{\varepsilon}{1-\beta} - \alpha\left(\frac{1}{2}V_1\left(x,\left(\pi_1,\pi_{2,o}\right)\right) + \frac{\varepsilon}{1-\beta}\right).
\end{aligned} \tag{16}$$

In this way, it is obtained that the following inequality is valid:

$$\hat{V}_1\left(x,\left(\pi_1,\pi_{2,o}\right)\right) \leq^* \hat{V}_1\left(x,\hat{\pi}_o\right). \tag{17}$$

Similarly, it can be shown that

$$\hat{V}_2\left(x,\left(\pi_{1,o},\pi_2\right)\right) \leq^* \hat{V}_2\left(x,\hat{\pi}_o\right). \tag{18}$$

From equations (17) and (18), it is possible to conclude that $(\pi_{1,o},\pi_{2,o})$ is a Nash equilibrium for the fuzzy game (see Definition 9).

In the next section, an example will be presented to illustrate the theory exposed. □

6. Cournot–Nash Dynamic Equilibrium

In this section, an example of the fuzzy fisheries model will be presented. For this example, the Cournot–Nash equilibrium is determined, which consists of giving extraction strategies in such a way that each player obtains his greatest benefit preserving the biomass reproduction. Then, suppose that the reproduction function is $h(u) = u^{\delta}$, $\delta \in [0,1]$, and $u \in [0,\infty]$ and the utility function is considered as $U_i(a) = \ln(a)$, for $a \in (0,\infty)$, $i = 1, 2$, with $U_i(0) = -\infty$. It is generally believed that the free and noncooperative exploitation of fish in a common lake or sea by two or more countries leads to the excessive consumption of fish, so it is important to propose an appropriate fisheries policy. In this section, a proposal in this direction is provided.

Theorem 13. *A Cournot–Nash equilibrium for the fuzzy fisheries model is given by the pair of stationary policies* (f_1, f_2), *with*

$$f_i(x) = \frac{\beta_j\delta(1-\beta_i\delta)}{1-(1-\beta_i\delta)\left(1-\beta_j\delta\right)}x, \tag{19}$$

for each $x \in X$ *and* $i, j = 1, 2$ *with* $i \neq j$. *And the optimal fuzzy payments for each player are given by the following equation:*

$$\widetilde{V}_i(x) = \left(\frac{1}{2}V_i(x) + \frac{\varepsilon}{1-\beta_i}, V_i(x) + \frac{2\varepsilon}{1-\beta_i}, V_i(x) + \frac{3\varepsilon}{1-\beta_i}, 2V_i(x) + \frac{4\varepsilon}{1-\beta_i}\right), \tag{20}$$

where $V_i(x) = (1-\beta_i\delta)^{-1}\ln(x) + C_i$ *for* $i, j = 1, 2$ *with* $i \neq j$, $x \in X$, *and*

$$C_i = \frac{1}{1-\beta_i}\left[\frac{\beta_i\delta}{1-\beta_i\delta}\ln\left(\frac{\beta_i\beta_j\delta^2}{1-(1-\beta_i\delta)\left(1-\beta_j\delta\right)}\right) + \ln\left(\frac{\beta_i\delta(1-\beta_i\delta)}{1-(1-\beta_i\delta)\left(1-\beta_j\delta\right)}\right) + \frac{\beta_i\theta}{1-\beta_i\delta}\right]. \tag{21}$$

Proof. Firstly the crisp optimal payments for each player $\{V_i\colon i=1,2\}$ are calculated. To this end, the value iteration approach [33] is applied. Let $x \in X$ be fixed and consider that $V_{1,0}(x) = V_{2,0}(x) = 0$. Then,

$$V_{i,1}(x) = \max_{a\in[0,x^{\delta}]} U_i(a) \tag{22}$$

for $i = 1, 2$. This implies that $V_{i,1}(x) = \delta \ln(x)$ and the maximizer is $f_{i,1}(x) = x^{\delta}$ for $i = 1, 2$. Following the scheme of value iteration functions, suppose that Player 2 applies the harvest strategy $f_{2,2} \in \mathbb{F}$, then the best response of Player I is determined by the following equation:

$$\begin{aligned} V_{1,2}(x) &= \max_{a_1\in[0,x^{\delta}]}\left\{\ln(a_1) + \beta_1 V_1^1\left(\left(x - a_1 - f_{2,2}(x)\right)^{\delta}\theta\right)\right\} \\ &= \max_{a_1\in[0,x^{\delta}]}\left\{\ln(a_1) + \beta_1\delta^2 \ln\left(x - a_1 - f_{2,2}(x)\right) + \beta_1\delta\ln(\theta)\right\}. \end{aligned} \tag{23}$$

On the other hand, if Player 1 applies $f_{1,2} \in \mathbb{F}$ as the harvest strategy, the corresponding optimality equation to characterize the best response of Player 2 is given by the following equation:

$$\begin{aligned} V_{2,2}(x) &= \max_{a_2\in[0,x^{\delta}]}\left\{\ln(a_2) + \beta_2 V_1^2\left(\left(x - f_{1,2}(x) - a_2\right)^{\delta}\theta\right)\right\} \\ &= \max_{a_2\in[0,x^{\delta}]}\left\{\ln(a_2) + \beta_2\delta^2 \ln\left(x - f_{1,2}(x) - a_2(x)\right) + \beta_2\delta\ln(\theta)\right\}. \end{aligned} \tag{24}$$

Then, the first-order conditions of equations (23) and (24) are as follows:

$$\begin{aligned} \frac{1}{a_1} - \frac{\beta_1\delta^2}{x - a_1 - f_{2,2}(x)} &= 0, \\ \frac{1}{a_2} - \frac{\beta_2\delta^2}{x - f_{1,2}(x) - a_2} &= 0. \end{aligned} \tag{25}$$

Solving both equations simultaneously, it yields that

$$f_{i,2}(x) = \frac{\beta_j\delta^2 x}{\left(\beta_i\delta^2+1\right)\left(\beta_j\delta^2+1\right)-1}, \tag{26}$$

for $i = 1, 2$, with $i \neq j$.

Substituting equation (26) into equations (23) and (24), it is obtained that

$$\begin{aligned} V_{i,2}(x) = &\left(1+\beta_i\delta^2\right)\ln(x) + \ln\left(\frac{\beta_j}{\left(\beta_i\delta^2+1\right)\left(\beta_j\delta^2+1\right)-1}\right) \\ &+ \beta_i\delta^2\ln\left(\frac{\beta_i\beta_j\delta^2}{\left(\beta_i\delta^2+1\right)\left(\beta_j\delta^2+1\right)-1}\right) + \beta_i\delta\ln(\theta), \end{aligned} \tag{27}$$

for $i = 1, 2$, with $i \neq j$.

Now, suppose that for $n \geq 3$ the following expressions hold:

$$V_{i,n-1}(x) = K_{i,n-1}\ln(x) + C_{i,n-1}, \tag{28}$$

where $K_{i,n-1} = \sum_{m=0}^{n-3}(\beta_i\delta)^m + \beta_i^{n-2}\delta^{n-1}$ for $i = 1, 2$, and the constant $C_{i,n-1}$, is defined by the following equation:

$$\begin{aligned} C_{i,n-1} = &\ln\left(\frac{\beta_j\delta K_{j,n-2}}{\left(\beta_i\delta K_{i,n-2}+1\right)\left(\beta_j\delta K_{j,n-2}+1\right)-1}\right) + \beta_i C_{i,n-2} \\ &+ \beta_i\delta K_{i,n-2}\ln\left(\frac{\beta_i\beta_j\delta^2 K_{i,n-2}K_{j,n-2}}{\left(\beta_i\delta K_{i,n-2}+1\right)\left(\beta_j\delta K_{j,n-2}+1\right)-1}\right) + \beta_i K_{j,n-2}\ln(\theta), \end{aligned} \tag{29}$$

for $i = 1, 2$, with $i \neq j$.

Then, for $n \geq 3$, if Player 2 chooses the stationary strategy $f_{2,n} \in \mathbb{F}$, the best response of Player 1 is implicitly defined by the following equation:

$$\begin{aligned} V_{1,n}(x) &= \max_{a_1 \in [0,x^\delta]} \left\{ \ln(a_1) + \beta_1 \left[K_{1,n-1} \ln\left(\left(x - a_1 - f_{2,n}(x)\right)^\delta \theta \right) + C_{1,n-1} \right] \right\} \\ &= \max_{a_1 \in [0,x^\delta]} \left\{ \ln(a_1) + \beta_1 \delta K_{1,n-1} \ln\left(x - a_1 - f_{2,n}(x)\right) + \beta_1 C_{1,n-1} + K_{1,n-1}\beta_1 \ln(\theta) \right\}. \end{aligned} \tag{30}$$

Analogously, if Player 1 applies the harvest strategy $f_{1,n} \in \mathbb{F}$, the best response for Player 2 is characterized by the following equation:

$$\begin{aligned} V_{2,n}(x) &= \max_{a_2 \in [0,x^\delta]} \left\{ \ln(a_2) + \beta_2 \left[K_{2,n-1} \ln\left(\left(x - f_{1,n}(x) - a_2\right)^\delta \theta \right) + C_{2,n-1} \right] \right\} \\ &= \max_{a_2 \in [0,x^\delta]} \left\{ \ln(a_2) + \beta_2 \delta K_{2,n-1} \ln\left(x - f_{1,n}(x) - a_2\right) + \beta_2 C_{2,n-1} + K_{2,n-1}\beta_2 \ln(\theta) \right\}. \end{aligned} \tag{31}$$

Then, the first-order equations are given by the following equation:

$$\begin{aligned} \frac{1}{a_1} - \frac{\beta_1 \delta K_{n-1}}{x - a_1 - f_{2,n}(x)} &= 0, \\ \frac{1}{a_2} - \frac{\beta_2 \delta \widehat{K}_{n-1}}{x - f_{1,n}(x) - a_2} &= 0, \end{aligned} \tag{32}$$

solving simultaneously, it is concluded that

$$f_{i,n}(x) = \frac{\beta_j \delta K_{j,n-1} x}{\left(\beta_i \delta K_{i,n-1} + 1\right)\left(\beta_j \delta K_{j,n-1} + 1\right) - 1}, \tag{33}$$

for $i = 1, 2$, with $i \neq j$.

This implies that

$$\begin{aligned} V_{i,n}(x) = &\left(1 + \beta_i \delta K_{i,n-1}\right) \ln(x) + \ln\left(\frac{\beta_2 \delta K_{j,n-1}}{\left(\beta_1 \delta K_{i,n-1} + 1\right)\left(\beta_2 \delta K_{j,n-1} + 1\right) - 1} \right) + \beta_1 C_{i,n-1} \\ &+ \beta_i \delta K_{i,n-1} \ln\left(\frac{\beta_i \beta_j \delta^2 K_{i,n-1} K_{j,n-1}}{\left(\beta_i \delta K_{i,n-1} + 1\right)\left(\beta_j \delta K_{j,n-1} + 1\right) - 1} \right) + \beta_i K_{i,n-1} \ln(\theta), \end{aligned} \tag{34}$$

for $i = 1, 2$, with $i \neq j$.

Therefore, it is concluded that

$$V_{i,n}(x) = K_{i,n} \ln(x) + C_{i,n}, \tag{35}$$

where

$$\begin{aligned} K_{i,n} &= \sum_{m=0}^{n-2} (\beta_i \delta)^m + \beta_i^{n-1} \delta^n, \\ C_{i,n} &= \ln\left(\frac{\beta_j \delta K_{j,n-1}}{\left(\beta_i \delta K_{i,n-1} + 1\right)\left(\beta_j \delta K_{j,n-1} + 1\right) - 1} \right) + \beta_i C_{i,n-1} \\ &+ \beta_i \delta K_{i,n-1} \ln\left(\frac{\beta_i \beta_j \delta^2 K_{i,n-1} K_{j,n-1}}{\left(\beta_i \delta K_{i,n-1} + 1\right)\left(\beta_j \delta K_{j,n-1} + 1\right) - 1} \right) + \beta_i K_{j,n-1} \ln(\theta), \end{aligned} \tag{36}$$

for $i = 1, 2$, with $i \neq j$.

From equation (35), it yields that

$$V_i(x) = \lim_{n \longrightarrow \infty} V_{i,n}(x) = \frac{1}{1-\beta_i \delta} \ln(x) + C_i, \quad (37)$$

where C_i, $i = 1, 2$ correspond to the limits of the sequences $\{C_{i,n}\}$ for $i = 1, 2$. Now, the Nash equilibrium will be determined. To this end, consider that Player 2 selects a stationary strategy $f_2 \in \mathbb{F}$, then the best response of Player 1 is determined by the following dynamic programming equation [33]:

$$V_1(x) = \max_{a_1 \in [0, x^\delta]} \left\{ \ln(a_1) + \beta_1 E\left[V_1\left(\left(x - a_1 - f_2(x) \right)^\delta \theta \right) \right] \right\}. \quad (38)$$

The first-order condition of equation (38) is given by the following equation:

$$\frac{1}{a_1} - \frac{\beta_1 \delta (1 - \beta_1 \delta)^{-1}}{x - a_1 - g(x)} = 0, \quad (39)$$

from which it is obtained that

$$f_1(x) = \frac{x - f_2(x)}{1 + \beta_1 \delta (1 - \beta_1 \delta)^{-1}}. \quad (40)$$

An analogous analysis for Player 2 leads to the following relation:

$$f_2(x) = \frac{x - f_1(x)}{1 + \beta_2 \delta (1 - \beta_2 \delta)^{-1}}. \quad (41)$$

Solving equations (40) and (41) simultaneously, it yields that

$$f_i(x) = \frac{\beta_j \delta (1 - \beta_i \delta)}{1 - (1 - \beta_i \delta)(1 - \beta_j \delta)} x, \quad (42)$$

for $i = 1, 2$, with $i \neq j$.

Thus, for $x \in X$,

$$\begin{aligned} K_i \ln(x) + C_i &= (1 + \beta_i \delta K_i) \ln(x) + \ln\left(\frac{\beta_j K_j}{\beta_i \beta_j \delta K_i K_j + \beta_i K_i + \beta_j K_j} \right) \\ &\quad + \beta_i \delta K_i \ln\left(\frac{\beta_i \beta_j \delta K_i K_j}{\beta_i \beta_j \delta K_i K_j + \beta_i K_i + \beta_j K_j} \right) + \beta_i (K_i \theta + C_i), \end{aligned} \quad (43)$$

for $i = 1, 2$, with $i \neq j$.

Consequently, $K_i = 1 + \beta_i \delta K_i$, then

$$\begin{aligned} K_i &= \frac{1}{1 - \beta_i \delta}, \\ C_i &= \frac{1}{1 - \beta_i} \left[\frac{\beta_i \delta}{1 - \beta_i \delta} \ln\left(\frac{\beta_i \beta_j \delta^2}{1 - (1 - \beta_i \delta)(1 - \beta_j \delta)} \right) + \ln\left(\frac{\beta_j \delta (1 - \beta_i \delta)}{1 - (1 - \beta_i \delta)(1 - \beta_j \delta)} \right) + \frac{\beta_i \theta}{1 - \beta_i \delta} \right], \end{aligned} \quad (44)$$

for $i = 1, 2$, with $i \neq j$.

Furthermore, (f_1, f_2) satisfies Definition 10, i.e., it is a Nash equilibrium. Then, via Lemma 12, the result follows.

In the particular case when $\beta_1 = \beta_2 = \beta$, the following corollary is obtained: □

Corollary 14. *The optimal fuzzy discounted payments for each player coincide, i.e., for each $x \in X$,*

$$\begin{aligned} \widetilde{V}_1(x) &= \left(\frac{1}{2} W(x) + \frac{\varepsilon}{1 - \beta}, W(x) + \frac{2\varepsilon}{1 - \beta}, W(x) + \frac{3\varepsilon}{1 - \beta}, 2W(x) + \frac{4\varepsilon}{1 - \beta} \right) \\ &= \widetilde{V}_2(x), \end{aligned} \quad (45)$$

with $W(x) := K\ln(x) + C$, *where*

$$K = \frac{1}{1-\beta\delta},$$
$$C = \frac{1}{1-\beta}\left[\frac{\beta\delta}{1-\beta\delta}\ln\left(\frac{\beta\delta}{2-\beta\delta}\right) + \ln\left(\frac{1-\beta\delta}{2-\beta\delta}\right) + \frac{\beta\theta}{1-\beta\delta}\right]. \tag{46}$$

Furthermore, (f_1, f_2) is a Cournot–Nash equilibrium, with $f_1(x) = f_2(x) = f_D(x) := 1 - \beta\delta/2 - \beta\delta x$, for $x \in X$.

Proof. This result is an immediate consequence of Theorem 13, it is enough to make the following substitution: $\beta = \beta_1 = \beta_2$. □

Remark 15. Note that the results in [10] represent a special case of Corollary 14 *when* $\varepsilon \longrightarrow 0$.

7. Cournot–Nash Equilibrium for the Average Utility

Now, suppose that the players (countries) follow an average reward performance criterion to choose a harvesting policy. Thus, consider that

$$j_i(\hat{\pi}, x) := \lim_{T\longrightarrow\infty}\frac{1}{T}\sum_{t=0}^{T-1} U_i(a_t), \tag{47}$$

for $x \in X$, $\hat{\pi} \in \Pi$, and $i = 1, 2$. j_i is called the long-run average utility for player i, $i = 1, 2$. Then, the average utility value function for each $i \in \{1, 2\}$ is defined as follows:

$$J_i(x) := \sup_{\hat{\pi}\in\Pi} j_i(\hat{\pi}, x), \quad x \in X. \tag{48}$$

And, in a similar way to the discounted case, a Nash equilibrium is defined (see Definition 10). Suppose that player 2 chooses $f_2 \in \mathbb{F}$ as a harvesting policy; it is well known in the literature on MDPs that the following dynamic programming equation for player 1 is valid:

$$j_1^* + w_1(x) = \sup_{a\in[0,h(x)]}\left[U_i(a) + w_1(h(x - a - f_2(x)))\right], \quad x \in X, \tag{49}$$

where j_1^* corresponds to the optimal utility of player 1 and w_1: $X \longrightarrow \mathbb{R}$ [34]. Moreover, if there exists a policy $f \in \mathbb{F}$ that maximizes the right-hand side of equation (49), the next relation holds: $j_1^* = j_1(f, x)$, for all $x \in X$. Similar arguments hold for the case of player 2.

The fuzzy control problem with respect to the long-run average objective function is defined analogously, in particular, observe that

$$\tilde{j}_i(\pi, x) = \left(\frac{j_i(\pi, x)}{2} + \varepsilon, j_i(\pi, x) + 2\varepsilon, j_i(\pi, x) + 3\varepsilon, 2j_i(\pi, x) + 4\varepsilon\right), \tag{50}$$

for $\pi \in \Pi$, $x \in X$, and $i = 1, 2$. The following result is valid for the long-run average criterion.

Lemma 16. *The Nash equilibrium for the fuzzy control problem coincides with the corresponding Nash equilibrium for the crisp problem.*

Proof. Suppose that $\hat{\pi}_o = (\pi_{1,o}, \pi_{2,o}) \in \Pi$ is a Nash equilibrium for the crisp model. Let $x \in X$ be fixed. Then, by equation (25) the following inequalities are valid:

$$\begin{aligned}
&\frac{1}{2}j_1\left(x, \left(\pi_1, \pi_{2,o}\right)\right) + \varepsilon + \alpha\left(\frac{1}{2}j_1\left(x, \left(\pi_1, \pi_{2,o}\right)\right) + \varepsilon\right) \le \frac{1}{2}j_1\left(\hat{\pi}_o, x\right) + \varepsilon + \alpha\left(\frac{1}{2}j_1\left(\hat{\pi}_o, x\right) + \varepsilon\right),\\
&2j_1\left(x, \left(\pi_1, \pi_{2,o}\right)\right) + 4\varepsilon - \alpha\left(j_1\left(x, \left(\pi_1, \pi_{2,o}\right)\right) + \varepsilon\right) \le 2j_1\left(x, \hat{\pi}_o\right) + 4\varepsilon - \alpha\left(j_1\left(x, \hat{\pi}_o\right) + \varepsilon\right),
\end{aligned} \tag{51}$$

for all $\alpha \in [0, 1]$ and $\pi_1 \in \Pi_1$. Consequently, the next inequality holds:

$$\tilde{j}_1\left(x, \left(\pi_1, \pi_{2,o}\right)\right) \le^* \tilde{j}_1\left(x, \hat{\pi}_o\right), \quad \pi_1 \in \Pi_1. \tag{52}$$

Similar arguments applied to player 2 lead to the following equation:

$$\tilde{j}_2\left(x, \left(\pi_{1,o}, \pi_2\right)\right) \le^* \tilde{j}_1\left(x, \hat{\pi}_o\right), \quad \pi_2 \in \Pi_2. \tag{53}$$

Inequalities (52) and (53) imply that $\hat{\pi}_o \in \Pi$ is a Nash equilibrium for the fuzzy control problem. Since $x \in X$ is arbitrary, the result follows.

Now, the previous result will be applied to the case when the utility function and reproduction function have the following form: $h(u) = u^{\delta}, u \in [0, \infty]$, and for $i = 1, 2$ define $U_i(a) = U(a) = \ln(a)$, for $a \in [0, 1]$ and $U_i(0) = U(0) = -\infty$, respectively. Firstly, the solution for the crisp control problem will be determined. To this end, the vanishing discount approach will be applied [34]. Thus, consider the following auxiliary functions:

$$w_{\beta}(x) \coloneqq W(x) - m_{\beta}, \rho_{\beta} \coloneqq (1-\beta)m_{\beta}, \tag{54}$$

where $m_{\beta} \coloneqq \sup_{x\in X} W(x) = C$ (see Corollary 14). Then, when $\beta\uparrow 1$ in equation (54), it is obtained that $w^*(x) \coloneqq \lim_{\beta\longrightarrow 1} w_{\beta}(x) = 1/1-\delta\ln(x)$, $x \in X$, and

$$\rho^* \coloneqq \lim_{\beta\longrightarrow 1} \rho_{\beta} = \frac{\delta}{1-\delta}\ln\left(\frac{\delta}{2-\delta}\right) + \ln\left(\frac{1-\delta}{2-\delta}\right) + \frac{\theta}{1-\delta}. \tag{55}$$

Then, if player 2 chooses the harvesting policy $f_2 \in \mathbb{F}$, the following equation holds (see (49)):

$$\rho^* + w^*(x) = \sup_{a\in[0,h(x)]} \left[U(a) + w^*\left(h\left(x - a - f_2(x)\right)\right)\right], \quad x \in X. \tag{56}$$

In a similar way, if player 1 selects the strategy $f_1 \in \mathbb{F}$, the next equation is valid:

$$\rho^* + w^*(x) = \sup_{a\in[0,h(x)]} \left[U(a) + w^*\left(h\left(x - a - f_1(x)\right)\right)\right], \quad x \in X. \tag{57}$$

Then, solving simultaneously the optimization problems described in equations (56) and (57), it is obtained that

$$f_1(x) = f_2(x) = f_A(x) \coloneqq \frac{1-\delta}{2-\delta}x, \tag{58}$$

$x \in X$. Therefore, by Lemma 16 and equation (50), the next result holds. □

Lemma 17. *The Nash equilibrium for the fuzzy control problem under the long-run average criterion is* (f_1, f_2) *and the value of the game is*

$$\tilde{j}_i = \left(\rho^*/2 + \varepsilon, \rho^* + 2\varepsilon, \rho^* + 3\varepsilon, 2\rho^* + 4\varepsilon\right), \tag{59}$$

for $i = 1, 2$.

8. Numerical Experiments

In this section, a numerical experiment is exposed. For this purpose, consider the state space normalized, i.e., $X = [0, 1]$, and take the boundary point $x = 1$ as the initial state. The state $x = 1$ represents the stable equilibrium state of the resource population when there is no extraction [13]. For numerical purposes, we consider in particular $\varepsilon = 1$; it is important to point out that the results obtained in this section can be replicated for any value of $\varepsilon > 0$. Next, two scenarios are presented to illustrate the discrepancies between the two criteria presented in the manuscript, see Table 1. The procedure for determining the values reported in the first row of Table 1 is described below. Applying the results from Corollary 14, it is obtained that $A_1 \coloneqq \tilde{V}(1) = (1.06, 2.12, 3.79, 4.24)$ and, in consequence, the corresponding membership function (see Definition 1) is given by the following equation:

$$\mu_{\tilde{V}}(y) = \begin{cases} 0, & \text{if } y < 1.06, \\ 0.94y - 1, & \text{if } 1.06 \le y < 2.12, \\ 1, & \text{if } 2.12 \le y \le 3.79, \\ -2.18y + 9.27, & \text{if } 3.79 < y \le 4.24, \\ 0, & \text{if } x > 4.24. \end{cases} \tag{60}$$

Now, applying Lemma 17, it yields that $B_1 \coloneqq \tilde{j} = (0.827, 1.655, 2.655, 3.31)$ and the membership function is as follows:

$$\mu_{\tilde{V}}(y) = \begin{cases} 0, & \text{if } y < 0.827, \\ 1.2y - 1, & \text{if } 0.827 \le y < 1.655, \\ 1, & \text{if } 1.655 \le y \le 2.655, \\ -1.526y + 5.051, & \text{if } 2.655 < y \le 3.31, \\ 0, & \text{if } x > 3.31. \end{cases} \tag{61}$$

A graphical representation of fuzzy number $\tilde{V}(1)$ and $\tilde{j}$ is illustrated in Figure 3. In a similar way, the results reported in the second row of Table 1 are determined.

For the first case shown in Table 1, the average optimal utility is less than the discounted optimal utility, under the order defined in (4), due to the following equation:

$$B_{1,\alpha} = [0.827 + 0.828\alpha, 3.31 - 0.655\alpha] \lesssim A_{1,\alpha} = [1.06(1+\alpha), 4.24 - 0.45\alpha]. \tag{62}$$

Table 1: Discounted/average optimal utilities.

x	ε	β	δ	θ	Discounted	Average
1	1	0.4	0.3	0.9	$A_1 = (1.06,\ 2.12,\ 3.79,\ 4.24)$	$B_1 = (0.827,\ 1.655,\ 2.655,\ 3.31)$
1	1	0.2	0.1	1	$A_2 = (0.87,\ 1.75,\ 3,\ 3.51)$	$B_2 = (1.01,\ 2.03,\ 3.03,\ 4.07)$

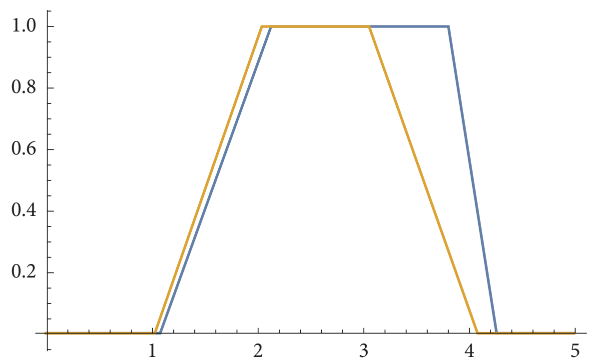

Figure 3: Blue is A_1 and orange is B_1.

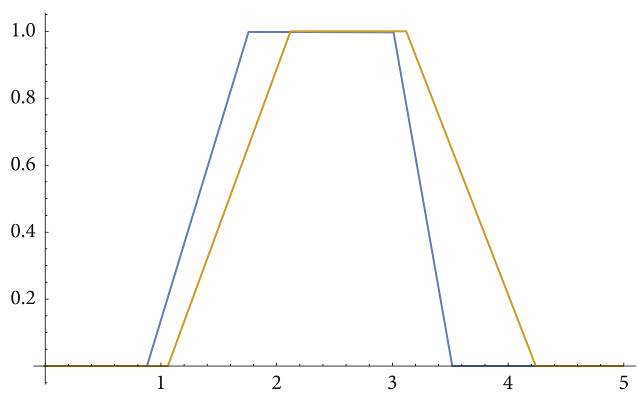

Figure 4: Blue is A_2 and orange is B_2.

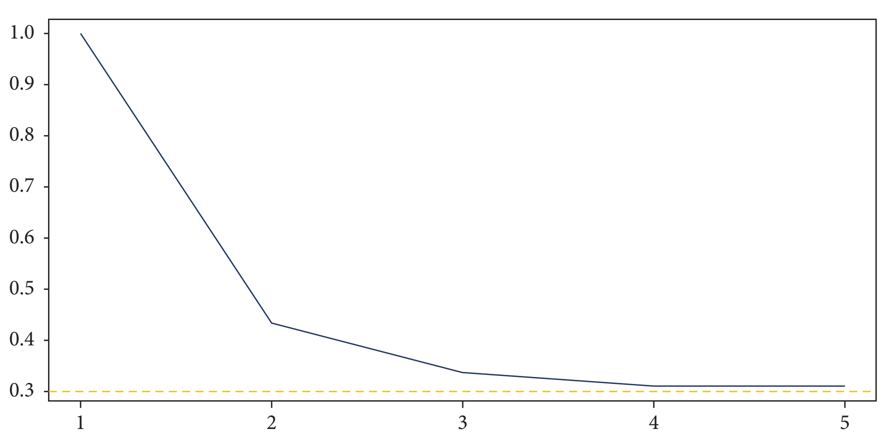

Figure 5: Biomass level for the discounted case.

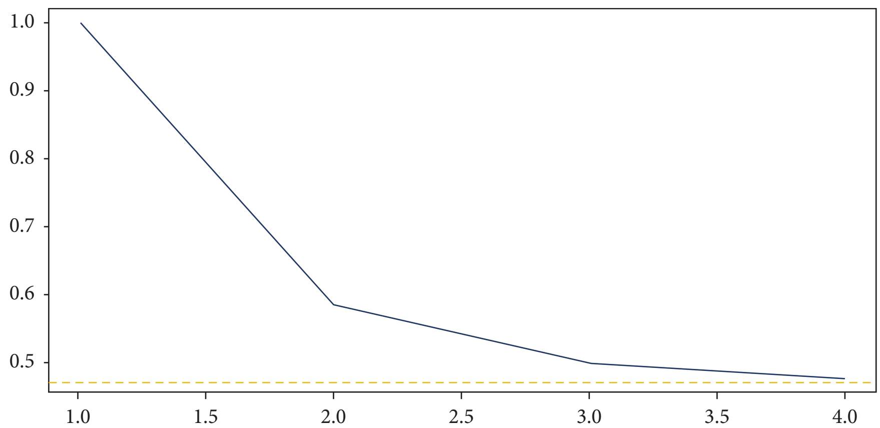

FIGURE 6: Biomass level for the average case.

This is also graphically seen in Figure 3. The second case reported in Table 1 illustrates the reverse, i.e., $A_{2,\alpha} \lesssim B_{2,\alpha}$, see Figure 4. It can be concluded that both criteria can lead to different optimal utility values. Now, consider the numerical values associated with case 1 reported in Table 1. Figures 5 and 6 show the level of biomass over time, and it can be observed that the level of biomass stabilizes in both cases. For the discounted case (see 5), the fish level converges to 0.3, and for the average case (see 6), it converges to 0.48.

Then, the biomass level for the discounted case is lower than for the average case. Similar results occur for the case of fuzzy numbers A_2 and B_2. Thus, we can conjecture that in the long run, the biomass level of the average criterion is higher than the biomass level generated by the discounted case. It is important to note that this statement is valid for the fisheries model presented in Section 5 and this fact is formally stated in Lemma 18. To establish this result, consider the following notation: let $\{x_n^D\}$ be the sequence generated by the discounted optimal policy f_D (see Corollary 14), i.e., $\{x_n^D\}$ is generated by the law motion (see (7)).

$$x_{t+1}^D = \left(x_t^D - 2 f_D\left(x_t^D\right)\right)^\delta \theta, \tag{63}$$

where $t = 0, 1, \ldots$ and $x_0^D = x \in X$. In a similar way, let $\{x_n^A\}$ be the sequence generated by the optimal policy f_A (see (58)), and in this case, $\{x_n^A\}$ satisfies that

$$x_{t+1}^A = \left(x_t^A - 2 f_A\left(x_t^A\right)\right)^\delta \theta, \tag{64}$$

with $t = 0, 1, \ldots$ and $x_0^A = x \in X$.

Lemma 18. *Under the previous notation, the following inequality holds:*

$$\overline{x}^D := \lim_{n \longrightarrow \infty} x_n^D \geq \overline{x}^A := \lim_{n \longrightarrow \infty} x_n^P. \tag{65}$$

Proof. Iterating the difference in equations (63) and (64) with the initial condition $x_0^D = x_0^A = x \in X$, it is obtained that

$$x_n^D = \theta\left(\frac{\delta\beta}{2 - \delta\beta}\right)^{\delta - \delta^{n+1}/1-\delta} x^{\delta^n}, x_n^P = \theta\left(\frac{\delta}{2 - \delta}\right)^{\delta - \delta^{n+1}/1-\delta} x^{\delta^n}. \tag{66}$$

Then, as n goes to infinity, the sequence x_n^D converges to $\overline{x}^D = \theta(\delta\beta/2 - \delta\beta)^{\delta/1-\delta}$ and x_n^P converges to $\overline{x}^P = \theta(\delta/2 - \delta)^{\delta/1-\delta}$. Now observe that $\overline{x}^D < \overline{x}^P$, due to $0 < \theta < 1$, $0 < \beta < 1$, and $0 < \delta < 1$. This confirms the statement of the lemma. □

9. Conclusions

For the fuzzy games analyzed in this paper for fisheries problems, it is important to note that the consequences presented in this paper are substantially different from the many others obtained in the studies of fisheries problems because here the Nash equilibria are presented in a fuzzy context that allows to interpret the values of the games in this context (see Theorem 13 and equation (50)). As far as the authors are aware, there is no work with this type of treatment. It has been learned about the dynamic games in the fishery models that (i) under the imposed assumptions that include a fuzzy context, the optimal strategies for both the crisp game and the fuzzy game coincide (see Theorem 13) and (ii) the numerical results indicate that the average performance is better than the discounted case, since the biomass level is larger in the average case than in the discounted one (see Section 8). These facts were not known for the fuzzy case before this work. The methodology developed here, which includes dynamics, fuzzy utilities, performance indices, fuzzy Nash equilibria, and the dynamic programming technique, is general enough and could be applied to other models not necessarily related to fisheries, such as resource management, the plastic ban problem (see [35]), the telecommunication market share problem (see [36]) and, in particular, inventory problems.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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