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Research Article

Existence Results for the System of Fractional-Order Sequential Integrodifferential Equations via Liouville–Caputo Sense

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We investigate the conditions for the existence and uniqueness of solutions in a nonlinear system of sequential fractional differential equations using the Liouville–Caputo type with varying orders. This system is enriched by nonlocal coupled integral boundary conditions. The desired outcomes are attained by employing traditional fixed-point theorems. It is essential to emphasize that the fixed-point approach proves to be an effective method for establishing the existence of solutions in boundary value problems. Furthermore, we provide constructed examples to illustrate the obtained results.

1. Introduction

Fractional calculus has become a prominent and extensively studied area of mathematical analysis during the last few decades. The significant expansion noted in this area can be attributed to the broad application of fractional calculus techniques in the development of creative mathematical models to illustrate various phenomena in the fields of science, engineering, mechanics, economics, and other fields. On this subject, references [1–8] offer comprehensive discussions and examples.

In the section that follows, we will present a survey of scholarly articles relevant to the topic at hand. On page 209 of their monograph, Miller and Ross [9] introduced the concept of sequential fractional derivative (SFD) $\mathscr{D}_{a+}^{\kappa\varrho}$, where κ is a positive integer. The papers [10, 11] explain the connection between SFDs and non-Riemann–Liouville SFDs.

The author in [12] proved that, under periodic boundary conditions, there are solutions to a nonlinear impulsive fractional differential equation (FDE) with Riemann–Liouville SFD. The monotone iterative method was employed to obtain the solutions. In reference [13], the nonexistence of solutions for an initial value problem (IVP) incorporating linear sequential FDEs with a classical first-order derivative and a Riemann–Liouville derivative is examined in the function space $\mathscr{L}^{\delta}((1,\infty),\mathscr{R}_e)$.

For a particular class of nonlinear Hadamard sequential FDEs, Klimek proved the existence and uniqueness of solutions in reference [14]. The contraction principle was used in conjunction with a set of initial conditions that included fractional derivatives to accomplish this. Within our research, "sequential" refers to the characteristic of the operator $\mathscr{D}^{\varrho}+\lambda\mathscr{D}^{\varrho-1})$, which can be expressed as a combination of the operators $\mathscr{D}^{\varsigma-1}(\mathscr{D}+\lambda)$, where $\mathscr{D}$ stands for the ordinary derivative.

The operator under discussion was first presented by Ahmad and Nieto [15] in their investigation into the existence and uniqueness of solutions for the sequential FDE with Caputo kind. Using techniques from fixed-point theory, the authors in [16] proved that there are solutions to the sequential integrodifferential problem. The authors in [17] looked into methods for solving the sequential FDE with Caputo type that included fractional Riemann–Liouville integral (RLI) boundary conditions. The study cited in [18] showcased the existence of solutions for a sequential fractional differential inclusion with Caputo-type, with boundary conditions encompassing a fractional RLI.

Referendum [19] contains several conclusions regarding the existence and uniqueness of the sequential FDE of the Caputo kind. For sequential FDEs with nonlocal boundary conditions, the authors in [20] proved the existence of solutions; for the sequential fractional differential inclusion with Hadamard-type, the authors in [21] derived existence results. The study cited in [22] discussed the mixed type of sequential FDEs. There are many practical applications for coupled systems of FDEs. The discussion that follows will cover a number of pertinent fractional systems indicated by (4) and (5). To prove that there are solutions and that they are unique for the nonlinear system of sequential FDEs with Caputo-type,

$$\begin{cases} \left({}^{c}\mathscr{D}^{\alpha} + \lambda^{c}\mathscr{D}^{\alpha-1}\right)\mathscr{X}(\tau) = \mathfrak{f}(\tau, \mathscr{X}(\tau), \mathscr{Y}(\tau)), & \tau \in [0,1], \\ \left({}^{C}\mathscr{D}^{\beta} + \lambda^{c}\mathscr{D}^{\beta-1}\right)\mathscr{Y}(\tau) = \mathfrak{g}(\tau, \mathscr{X}(\tau), \mathscr{Y}(\tau)), & \tau \in [0,1], \\ \mathscr{X}(0) = \mathscr{X}'(0) = 0, & \mathscr{X}(\zeta) = a\mathscr{I}^{\vartheta}\mathscr{X}(\eta), \\ \mathscr{Y}(0) = \mathscr{Y}'(0) = 0, & \mathscr{Y}(z) = b\mathscr{I}^{\gamma}\mathscr{Y}(\theta), \end{cases} \tag{1}$$

where $2 < \alpha, \beta \leq 3$, $\theta, z, \eta, \zeta \in (0,1)$, $\lambda > 0$, and $\vartheta, \gamma > 0$. Differential and integral operators have an impact on the nonlinearity of the function in System (4). In contrast, no differential and integral operators are used in System (1). In (5), the boundary conditions are coupled classical integral boundary conditions; in (1), the boundary conditions are coupled RLFI. There are coupled sequential fractional integrodifferential equations in System (4), while there are coupled sequential FDEs in System (1). The authors of [22] used the Leray–Schauder alternative and the Banach contraction mapping concept. An analysis proving the existence of solutions for a system of fractional order Caputo-type sequential derivatives and nonlinear coupled differential equations was presented in reference [23]. The methods utilized to attain this outcome were derived from fixed-point theory. The authors in [24] examined the stability and existence of a tripled system of sequential FDEs with multipoint boundary conditions, whereas the authors in [25] established the existence of solutions for three nonlinear sequential FDEs with nonlocal boundary conditions. The cited reference [26] contained the conclusions about the existence of solutions for a coupled system of nonlinear differential equations and inclusions incorporating SFD. The authors in [27] developed existence and uniqueness results for a system of sequential Hadamard-type FDEs, including nonlocal coupled strip conditions:

$$\begin{cases} \left({}^{H}\mathscr{D}^{\alpha} + \lambda^{H}\mathscr{D}^{\alpha-1}\right)\mathscr{X}(\tau) = \mathfrak{f}\left(\tau, \mathscr{X}(\tau), \mathscr{Y}(\tau), {}^{H}\mathscr{D}^{\omega}\mathscr{Y}(\tau)\right), & t \in (1,e), \\ \left({}^{H}\mathscr{D}^{\beta} + \lambda^{H}\mathscr{D}^{\beta-1}\right)\mathscr{Y}(\tau) = \mathfrak{g}\left(\tau, \mathscr{X}(\tau), {}^{H}\mathscr{D}^{\delta}\mathscr{X}(\tau), \mathscr{Y}(\tau)\right), & t \in (1,e), \\ \mathscr{X}(1) = 0, & \mathscr{X}(e) = {}^{H}\mathscr{I}^{\gamma}\mathscr{Y}(\eta), \\ \mathscr{Y}(1) = 0, & \mathscr{Y}(e) = {}^{H}\mathscr{I}^{\vartheta}\mathscr{X}(\zeta), \end{cases} \tag{2}$$

where $\lambda > 0$, $\alpha, \beta \in (1,2]$, $\omega, \delta \in (0,1)$, $\gamma, \vartheta > 0$, and $\eta, \zeta \in (1,e)$. Differential and integral operators impact the nonlinearity of the function in System (4), but differential operators are included in System (2). Coupled classical integral boundary conditions are used in (5), whereas coupled Hadamard integral boundary conditions are used in (2). The Liouville–Caputo sense of coupled sequential fractional integrodifferential equations is shown in System (4). Conversely, System (2) utilises Hadamard-sense differential equations that are coupled sequential FDEs. Subramanian et al. [28] analyzed the existence results for a system of coupled higher-order fractional integrodifferential equations. In [29], the authors conducted an analysis on the coupled system of sequential fractional integrodifferential equations with Caputo-type:

$$\begin{cases} \left({}^{c}\mathscr{D}^{\alpha}+\lambda_1{}^{c}\mathscr{D}^{\alpha-1}\right)\mathscr{X}(\tau)=\mathfrak{f}\left(\tau,\mathscr{X}(\tau),\mathscr{Y}(\tau),\mathscr{I}^{\mathfrak{p}_1}\mathscr{X}(\tau),\mathscr{I}^{p_2}\mathscr{Y}(\tau)\right), & \tau\in(0,1),\\ \left({}^{c}\mathscr{D}^{\beta}+\lambda_2{}^{c}\mathscr{D}^{\beta-1}\right)\mathscr{Y}(\tau)=\mathfrak{g}\left(\tau,\mathscr{X}(\tau),\mathscr{Y}(\tau),\mathscr{I}^{\mathfrak{q}_1}\mathscr{X}(\tau),\mathscr{I}^{\mathfrak{q}_2}\mathscr{Y}(\tau)\right), & \tau\in(0,1),\\ \mathscr{X}(0)=\mathscr{X}'(0)=0, & \mathscr{X}''(0)=0,\mathscr{X}(1)=\int_0^1\mathscr{X}(s)d\mathscr{H}_1(s)+\int_0^1\mathscr{Y}(s)d\mathscr{H}_2(s),\\ \mathscr{Y}(0)=\mathscr{Y}'(0)=0, & \mathscr{Y}''(0)=0,\mathscr{Y}(1)=\int_0^1\mathscr{X}(s)d\mathscr{K}_1(s)+\int_0^1\mathscr{Y}(s)d\mathscr{K}_2(s), \end{cases} \tag{3}$$

where $3<\alpha,\beta\leq 4$, $\lambda_1,\lambda_2>0$, $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{q}_1,\mathfrak{q}_2>0$, the Riemann–Stieltjes integrals (RSIs) with bounded variation functions $\mathscr{H}_1,\mathscr{K}_1,\mathscr{H}_2,\mathscr{K}_2$. The nonlinearity of the function in System (4) is impacted by differential and integral operators, whereas System (3) includes two integrated operators. The boundary conditions in (3) use coupled RSI boundary conditions, as opposed to the coupled classical integral boundary requirements in (5). The study in [30] successfully derived existence results for a coupled system of sequential fractional integrodifferential equations with nonlocal Riemann–Liouville integral boundary conditions. Motivated by the recent works, this study introduces and examines a novel nonlinear nonlocal coupled boundary value problem (BVP) involving Liouville–Caputo fractional integrodifferential equations (LCFIEs) of varying orders. The problem is defined as follows:

$$\begin{cases} \left({}^{c}\mathscr{D}^{\vartheta}+\varkappa_1{}^{c}\mathscr{D}^{\vartheta-1}\right)\varpi(\varepsilon)=\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^{c}\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right), & \varepsilon\in[0,1],\\ \left({}^{c}\mathscr{D}^{\eta}+\mu_1{}^{c}\mathscr{D}^{\eta-1}\right)\Psi(\varepsilon)=\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),{}^{c}\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon),\Psi(\varepsilon)\right), & \varepsilon\in[0,1], \end{cases} \tag{4}$$

supplemented with the coupled classical integral boundary conditions

$$\begin{cases} \varpi(0)=0, \quad \varpi'(0)=0, \quad \varpi'(1)=0, \quad \varpi(1)=\int_0^1\varpi(\zeta)d\zeta+\int_0^1\Psi(\zeta)d\zeta,\\ \Psi(0)=0, \quad \Psi'(0)=0, \quad \Psi'(1)=0, \quad \Psi(1)=\int_0^1\varpi(\zeta)d\zeta+\int_0^1\Psi(\zeta)d\zeta, \end{cases} \tag{5}$$

where ϑ, $\eta\in(3,4]$, $\varkappa_1,\mu_1>0$, $\phi_1,\phi_2,\mathfrak{q}_1,\mathfrak{q}_2>0$, ${}^{c}\mathscr{D}^{\kappa}$ represents the Liouville–Caputo fractional derivative (LCFD) of order θ (for $\theta=\vartheta,\eta,\eta-1,\vartheta-1,\phi_1,\phi_2$), $\mathfrak{F},\mathfrak{G}$: $[0,1]\times\mathscr{R}_e^4\longrightarrow\mathscr{R}_e$ are continuous functions, and $\mathscr{I}_{0^+}^{\nu}$ denotes the fractional RLI of order ν (for $\nu=\mathfrak{q}_1,\mathfrak{q}_2$). It is noteworthy that this study contributes to the literature by addressing a unique configuration of sequential LCFIEs with distinct orders and coupled integral boundary conditions. The methodology employed involves the application of the fixed-point approach to establish both existence and uniqueness results for the problems (4) and (5). The conversion of the given problem into an equivalent fixed-point problem is followed by the utilization of Leray–Schauder alternative and Banach's fixed-point theorem to prove existence and uniqueness results, respectively. The outcomes of this research are novel and enrich the existing body of literature on BVPs involving coupled systems of sequential LCFIEs.

The document is organized in the following sections: the fundamental definitions of fractional calculus relevant to this research are introduced in Section 2. An auxiliary lemma addressing the linear versions of problems (4) and (5) is provided in Section 3. The primary findings are presented in Section 4, while Section 5 provides an illustrative example that demonstrates the results of our research. Finally, Section 6 provides our paper's conclusions.

2. Preliminaries

Initially, we delineate fundamental principles of fractional calculus.

Definition 1 (see [3]). For a locally integrable, real-valued function ϖ on $\infty\leq\mathfrak{a}\leq\mathfrak{b}+\infty$, the fractional RLI of order $\vartheta\in\mathbb{R}$ $(\vartheta>0)$ is represented by $\mathscr{I}_{\mathfrak{a}}^{\vartheta}(\zeta)$ and defined as

$$\mathscr{I}_{\mathfrak{a}}^{\vartheta}\varpi(\mathfrak{z})=\int_{\mathfrak{a}}^{\mathfrak{z}}\frac{(\mathfrak{z}-\zeta)^{\vartheta-1}}{\Gamma(\vartheta)}\varpi(\zeta)\mathrm{d}\zeta. \tag{6}$$

In this context, $\Gamma(\cdot)$ represents the well-known Gamma function.

Definition 2 (see [1]). For a $(r-1)$-times absolutely continuous function ϖ: $[\mathfrak{a},\infty)\longrightarrow\mathbb{R}$, the Caputo derivative of fractional order ϑ is defined as follows:

$$^{c}\mathscr{D}^{\vartheta}\varpi(\mathfrak{z}) = \int_{\mathfrak{M}}^{\mathfrak{z}} \frac{(\mathfrak{z}-\zeta)^{r-\vartheta-1}}{\Gamma(r-\vartheta)}\varpi^{(r)}(\zeta)d\zeta, \quad r-1<\vartheta<r, r=[\vartheta]+1, \tag{7}$$

where [ϑ] represents the integral part of the real number ϑ.

3. Auxiliary Lemma

In this section, we examine a system of linear FDEs.

$$\begin{cases} \left(^{c}\mathscr{D}^{\vartheta} + \varkappa_1 {}^{c}\mathscr{D}^{\vartheta-1}\right)\varpi(\varepsilon) = \mathfrak{h}_1(\varepsilon), & \varepsilon \in (0,1), \\ \left(^{c}\mathscr{D}^{\eta} + \mu_1 {}^{c}\mathscr{D}^{\eta-1}\right)\Psi(\varepsilon) = \mathfrak{h}_2(\varepsilon), & \varepsilon \in (0,1), \end{cases} \tag{8}$$

augmented by the boundary conditions (5), where $\mathfrak{h}1, \mathfrak{h}2 \in \mathscr{C}[0,1]$. We denote by

$$\begin{aligned}
\mathscr{A}_1 &= \frac{1}{\varkappa_1}\left(1-e^{-\varkappa_1}\right), \mathscr{A}_2 = \frac{1}{\varkappa_1^2}\left(2\varkappa_1 - 2 + e^{-\varkappa_1}\right),\\
\mathscr{A}_3 &= \frac{1}{\mu_1}\left(1-e^{-\mu_1}\right), \mathscr{A}_4 = \frac{1}{\mu_1^2}\left(2\mu_1 - 2 + e^{-\mu_1}\right),\\
\mathscr{A}_5 &= \frac{1}{\varkappa_1^2}\left[\left(\varkappa_1 - 1 + e^{-\varkappa_1}\right) - \int_0^1\left(\varkappa_1\zeta - 1 + e^{-\varkappa_1\zeta}\right)d\zeta\right],\\
\mathscr{A}_6 &= \frac{1}{\varkappa_1^3}\left[\left(\varkappa_1^2 - 2\varkappa_1 + 2 - 2e^{-\varkappa_1}\right) - \int_0^1\left(\varkappa_1^2\zeta^2 - 2\varkappa_1\zeta + 2 - 2e^{\varkappa_1\zeta}\right)d\zeta\right],\\
\mathscr{A}_7 &= \frac{1}{\mu_1^2}\int_0^1\left(\mu_1\zeta - 1 + e^{-\mu_1\zeta}\right)d\zeta,\\
\mathscr{A}_8 &= \frac{1}{\mu_1^3}\int_0^1\left(\mu_1^2\zeta^2 - 2\mu_1\zeta + 2 - 2e^{-\mu_1\zeta}\right)d\zeta,\\
\mathscr{A}_9 &= \frac{1}{\varkappa_1^2}\int_0^1\left(\varkappa_1\zeta - 1 + e^{-\varkappa_1\zeta}\right)d\zeta,\\
\mathscr{A}_{10} &= \frac{1}{\varkappa_1^3}\int_0^1\left(\varkappa_1^2\zeta^2 - 2\varkappa_1\zeta + 2 - 2e^{\varkappa_1\zeta}\right)d\zeta,\\
\mathscr{A}_{11} &= \frac{1}{\mu_1^2}\left(\mu - 1 + e^{-\mu}\right) - \frac{1}{\mu_1^2}\int_0^1\left(\mu_1\zeta - 1 + e^{-\mu_1\zeta}\right)d\zeta,\\
\mathscr{A}_{12} &= \frac{1}{\mu_1^3}\left(\mu_1^2 - 2\mu_1 + 2 - 2e^{-\mu}\right) - \frac{1}{\mu_1^3}\int_0^1\left(\mu_1^2\zeta^2 - 2\mu_1\zeta + 2 - 2e^{-\mu_1\zeta}\right)d\zeta,
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\mathcal{I}_1 &= \varkappa_1 \int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta - \int_0^1 \frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\zeta)d\zeta,\\
\mathcal{I}_2 &= \mu_1 \int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta - \int_0^1 \frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\zeta)d\zeta,\\
\mathcal{I}_3 &= \int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^\mathfrak{u} \frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&\quad + \int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^\mathfrak{u} \frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&\quad - \int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta,\\
\mathcal{I}_4 &= \int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^\mathfrak{u} \frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&\quad + \int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^\mathfrak{u} \frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&\quad - \int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta,\\
\Delta_1 &= (\mathcal{A}_2\mathcal{A}_5-\mathcal{A}_1\mathcal{A}_6)(\mathcal{A}_4\mathcal{A}_{11}-\mathcal{A}_3\mathcal{A}_{12})-(\mathcal{A}_2\mathcal{A}_9-\mathcal{A}_1\mathcal{A}_{10})(\mathcal{A}_4\mathcal{A}_7-\mathcal{A}_3\mathcal{A}_8),\\
\Delta &= \mathcal{A}_2\mathcal{A}_4\Delta_1.
\end{aligned} \tag{10}$$

Lemma 3. *If $\Delta_1 \neq 0$, then the solution $(\varpi, \Psi) \in (\mathcal{C}^4[0,1])^2$ of the BVP (4) and (5),*

$$\begin{cases}
\varpi(\varepsilon) = \sum_{i=1}^{4} \mathcal{S}_i(\varepsilon)\mathcal{I}_i + \int_0^\varepsilon e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta,\\
\Psi(\varepsilon) = \sum_{i=1}^{4} \mathcal{T}_i(\varepsilon)\mathcal{I}_i + \int_0^\varepsilon e^{-\mu_1(\varepsilon-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta,
\end{cases} \tag{11}$$

where

$$\begin{cases}
\mathcal{S}_i(\varepsilon) = \frac{1}{\varkappa_1^2}(\varsigma_i)\left(\varkappa_1\varepsilon - 1 + e^{-\varkappa_1\varepsilon}\right) + \frac{1}{\varkappa_1^3}(\Theta_i)\left(\varkappa_1^2\varepsilon^2 - 2\varkappa_1\varepsilon + 2 - 2e^{-\varkappa_1\varepsilon}\right), & i = 1,2,3,4.\\
\mathcal{T}_i(\varepsilon) = \frac{1}{\mu_1^2}(\Xi_i)\left(\mu_1\varepsilon - 1 + e^{-\mu_1\varepsilon}\right) + \frac{1}{\mu_1^3}(\Upsilon_i)\left(\mu_1^2\varepsilon^2 - 2\mu_1\varepsilon + 2 - 2e^{-\mu_1\varepsilon}\right), & i = 1,2,3,4.
\end{cases} \tag{12}$$

Proof. System (8) can be expressed equivalently as follows:

$$\begin{cases} \left({}^{c}\mathscr{D}^{\vartheta}+\varkappa_1{}^{c}\mathscr{D}^{\vartheta-1}\right)\varpi(\varepsilon)=\mathfrak{h}_1(\varepsilon), & \varepsilon\in[0,1],\\ \left({}^{c}\mathscr{D}^{\eta}+\mu_1{}^{c}\mathscr{D}^{\eta-1}\right)\Psi(\varepsilon)=\mathfrak{h}_2(\varepsilon), & \varepsilon\in[0,1]. \end{cases} \tag{13}$$

The general solutions of system (5) and (13)

$$\begin{aligned} \varpi(\varepsilon) &= \mathfrak{c}_0 e^{-\varkappa_1\varepsilon}+\frac{\mathfrak{c}_1}{\varkappa_1}\left(1-e^{-\varkappa_1\varepsilon}\right)+\frac{\mathfrak{c}_2}{\varkappa_1^2}\left(\varkappa_1\varepsilon-1+e^{-\varkappa_1\varepsilon}\right)+\frac{\mathfrak{c}_3}{\varkappa_1^3}\left(\varkappa_1^2\varepsilon^2-2\varkappa_1\varepsilon+2-2e^{-\varkappa_1\varepsilon}\right)\\ &\quad+\int_0^{\varepsilon} e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta,\\ \Psi(\varepsilon) &= \mathfrak{d}_0 e^{-\mu_1\varepsilon}+\frac{\mathfrak{d}_1}{\mu_1}\left(1-e^{-\mu_1\varepsilon}\right)+\frac{\mathfrak{d}_2}{\mu_1^2}\left(\mu_1\varepsilon-1+e^{-\mu_1\varepsilon}\right)+\frac{\mathfrak{d}_3}{\mu_1^3}\left(\mu_1^2\varepsilon^2-2\mu_1\varepsilon+2-2e^{-\mu_1\varepsilon}\right)\\ &\quad+\int_0^{\varepsilon} e^{-\mu_1(\varepsilon-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta. \end{aligned} \tag{14}$$

By applying the boundary conditions $\varpi(0)=\varpi'(0)=0$ and $\Psi(0)=\Psi'(0)=0$ from (5), we infer that $\mathfrak{c}_0=\mathfrak{c}_1=0$ and $\mathfrak{d}_0=\mathfrak{d}_1=0$. Consequently, we can deduce

$$\begin{cases} \varpi(\varepsilon)=\dfrac{\mathfrak{c}_2}{\varkappa_1^2}\left(\varkappa_1\varepsilon-1+e^{-\varkappa_1\varepsilon}\right)+\dfrac{\mathfrak{c}_3}{\varkappa_1^3}\left(\varkappa_1^2\varepsilon^2-2\varkappa_1\varepsilon+2-2e^{-\varkappa_1\varepsilon}\right)\\ +\displaystyle\int_0^{\varepsilon} e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta,\\ \Psi(\varepsilon)=\dfrac{\mathfrak{d}_2}{\mu_1^2}\left(\mu_1\varepsilon-1+e^{-\mu_1\varepsilon}\right)+\dfrac{\mathfrak{d}_3}{\mu_1^3}\left(\mu_1^2\varepsilon^2-2\mu_1\varepsilon+2-2e^{-\mu_1\varepsilon}\right)\\ +\displaystyle\int_0^{\varepsilon} e^{-\mu_1(\varepsilon-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta. \end{cases} \tag{15}$$

After differentiating system (15), we get

$$\begin{cases} \varpi'(\varepsilon)=\dfrac{\mathfrak{c}_2}{\varkappa_1^2}\left(\varkappa_1-\varkappa_1 e^{-\varkappa_1\varepsilon}\right)+\dfrac{\mathfrak{c}_3}{\varkappa_1^3}\left(2\varkappa_1^2\varepsilon-2\varkappa_1+2e^{-\varkappa_1\varepsilon}\right)\\ -\varkappa_1\displaystyle\int_0^{\varepsilon} e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta+\int_0^{\varepsilon}\frac{(\varepsilon-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\zeta)d\zeta\\ \Psi'(\varepsilon)=\dfrac{\mathfrak{d}_2}{\mu_1^2}\left(\mu_1-\mu_1 e^{-\mu_1\varepsilon}\right)+\dfrac{\mathfrak{d}_3}{\mu_1^3}\left(2\mu_1^2\varepsilon-2\mu_1+2\mu_1 e^{-\mu_1\varepsilon}\right)\\ -\chi_1\displaystyle\int_0^{\varepsilon} e^{-\mu_1(\varepsilon-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta+\int_0^{\varepsilon}\frac{(\varepsilon-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\zeta)d\zeta. \end{cases} \tag{16}$$

By setting the conditions $\varpi'(1)=\Psi'(1)=0$ from (5), we deduce

$$\begin{cases} \dfrac{\mathfrak{c}_2}{\varkappa_1}\left(1-e^{-\varkappa_1}\right)+\dfrac{\mathfrak{c}_3}{\varkappa_1^2}\left(2\varkappa_1^2-2+2e^{-\varkappa_1}\right)=\varkappa_1\displaystyle\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta \\ -\displaystyle\int_0^1\frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\zeta)d\zeta \\ \dfrac{\mathfrak{d}_2}{\mu_1}\left(1-e^{-\mu_1}\right)+\dfrac{\mathfrak{d}_3}{\mu_1^2}\left(2\mu_1-2+2\mu_1e^{-\mu_1}\right)=\chi_1\displaystyle\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta \\ -\displaystyle\int_0^1\frac{(\varepsilon-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\zeta)d\zeta. \end{cases} \tag{17}$$

Now, utilizing the final boundary conditions from (5), specifically, $\varpi(1)=\int_0^1\varpi(\zeta)d\zeta+\int_0^1\Psi(\zeta)d\zeta$ and $\Psi(1)=\int_0^1\varpi(\zeta)d\zeta+\int_0^1\Psi(\zeta)d\zeta$, by (15), we deduce

$$\begin{aligned} &\mathfrak{c}_2\left[\frac{1}{\varkappa_1^2}\left[\left(\varkappa_1-1+e^{-\varkappa_1}\right)-\int_0^1\left(\varkappa_1\zeta-1+e^{-\varkappa_1\zeta}\right)d\zeta\right]\right] \\ &+\mathfrak{c}_3\left[\frac{1}{\varkappa_1^3}\left[\left(\varkappa_1^2-2\varkappa_1+2-2e^{-\varkappa_1}\right)-\int_0^1\left(\varkappa_1^2\zeta^2-2\varkappa_1\zeta+2-2e^{\varkappa_1\zeta}\right)d\zeta\right]\right] \\ &-\mathfrak{d}_2\left[\frac{1}{\mu_1^2}\int_0^1\left(\mu_1\zeta-1+e^{-\mu_1\zeta}\right)d\zeta\right]+\mathfrak{d}_3\left[\frac{1}{\mu_1^3}\int_0^1\left(\mu_1^2\zeta^2-2\mu_1\zeta+2-2e^{-\mu_1\zeta}\right)d\zeta\right] \\ =&\int_0^1\left(\int_0^{\zeta}e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta \\ &+\int_0^1\left(\int_0^{\zeta}e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta \\ &-\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\mathfrak{u})d\mathfrak{u}\right)d\zeta, \end{aligned} \tag{18}$$

and

$$- \mathfrak{c}_2\left[\frac{1}{\varkappa_1^2}\int_0^1\left(\varkappa_1\zeta - 1 + e^{-\varkappa_1\zeta}\right)d\zeta\right]$$

$$- \mathfrak{c}_3\left[\frac{1}{\varkappa_1^3}\int_0^1\left(\varkappa_1^2\zeta^2 - 2\varkappa_1\zeta + 2 - 2e^{\varkappa_1\zeta}\right)d\zeta\right]$$

$$+ \mathfrak{d}_2\left[\frac{1}{\mu_1^2}\left(\mu - 1 + e^{-\mu}\right) - \frac{1}{\mu_1^2}\int_0^1\left(\mu_1\zeta - 1 + e^{-\mu_1\zeta}\right)d\zeta\right]$$

$$= \int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{h}_1(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta$$

$$+ \int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta$$

$$- \int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{h}_2(\mathfrak{u})d\mathfrak{u}\right)d\zeta. \tag{19}$$

Therefore, by (9), (10), (17)–(19), we find the system in the unknowns $\mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{d}_2$ and $\mathfrak{d}_3$:

$$\begin{cases} \mathscr{A}_1\mathfrak{c}_2 + \mathscr{A}_2\mathfrak{c}_3 = \mathscr{I}_1, \\ \mathscr{A}_3\mathfrak{d}_2 + \mathscr{A}_4\mathfrak{d}_3 = \mathscr{I}_2, \\ \mathscr{A}_5\mathfrak{c}_2 + \mathscr{A}_6\mathfrak{c}_3 - \mathscr{A}_7\mathfrak{d}_2 - \mathscr{A}_8\mathfrak{c}_3 = \mathscr{I}_3, \\ -\mathscr{A}_9\mathfrak{c}_2 - \mathscr{A}_{10}\mathfrak{c}_3 + \mathscr{A}_{11}\mathfrak{d}_2 + \mathscr{A}_{12}\mathfrak{c}_3 = \mathscr{I}_4. \end{cases} \tag{20}$$

By the first two equations of (21), we find $\mathfrak{c}_3 = (\mathscr{I}_1 - \mathscr{A}_1\mathfrak{c}_2/\mathscr{A}_2)$ and $\mathfrak{d}_3 = (\mathscr{I}_2 - \mathscr{A}_3\mathfrak{d}_2/\mathscr{A}_2)$, $(\mathscr{A}_2, \mathscr{A}_4 > 0)$. By substituting these values of $\mathfrak{c}_3$ and $\mathfrak{d}_3$ into the remaining two equations of (21), we derive the system in the unknowns $\mathfrak{c}_2$ and $\mathfrak{d}_2$:

$$\begin{cases} \mathfrak{c}_2\mathscr{A}_4\left(\mathscr{A}_2\mathscr{A}_5 - \mathscr{A}_1\mathscr{A}_6\right) - \mathfrak{d}_2\mathscr{A}_2\left(\mathscr{A}_4\mathscr{A}_7 - \mathscr{A}_3\mathscr{A}_8\right) = \mathscr{A}_2\mathscr{A}_4\mathscr{I}_3 - \mathscr{A}_4\mathscr{A}_6\mathscr{I}_1 + \mathscr{A}_2\mathscr{A}_8\mathscr{I}_2, \\ -\mathfrak{c}_2\mathscr{A}_4\left(\mathscr{A}_4\mathscr{A}_9 - \mathscr{A}_1\mathscr{A}_{10}\right) + \mathfrak{d}_2\mathscr{A}_2\left(\mathscr{A}_4\mathscr{A}_{11} - \mathscr{A}_3\mathscr{A}_{12}\right) = \mathscr{A}_2\mathscr{A}_4\mathscr{I}_4 + \mathscr{A}_4\mathscr{A}_{10}\mathscr{I}_1 - \mathscr{A}_2\mathscr{A}_{12}\mathscr{I}_2. \end{cases} \tag{21}$$

The determinant of system (21) is $\Delta = \mathscr{A}_2\mathscr{A}_4\Delta_1$, where Δ_1 is given by (10). By assumption of this lemma, $\Delta_1 \neq 0$, then $\Delta = 0$. Therefore, the solution of system (21) is

$$\begin{aligned} \mathfrak{c}_2 &= \frac{\mathscr{A}_2}{\Delta}\{\mathscr{I}_1\left[-\mathscr{A}_4\mathscr{A}_6\left(\mathscr{A}_4\mathscr{A}_{11} - \mathscr{A}_3\mathscr{A}_{12}\right) + \mathscr{A}_4\mathscr{A}_{10}\left(\mathscr{A}_4\mathscr{A}_7 - \mathscr{A}_3\mathscr{A}_8\right)\right] \\ &\quad + \mathscr{I}_2\left[-\mathscr{A}_2\mathscr{A}_8\left(\mathscr{A}_4\mathscr{A}_{11} - \mathscr{A}_3\mathscr{A}_{12}\right) + \mathscr{A}_2\mathscr{A}_{12}\left(\mathscr{A}_4\mathscr{A}_7 - \mathscr{A}_3\mathscr{A}_8\right)\right] \\ &\quad + \mathscr{I}_3\mathscr{A}_2\mathscr{A}_4\left(\mathscr{A}_4\mathscr{A}_{11} - \mathscr{A}_3\mathscr{A}_{12}\right) + \mathscr{I}_4\mathscr{A}_2\mathscr{A}_4\left(\mathscr{A}_4\mathscr{A}_7 - \mathscr{A}_3\mathscr{A}_8\right)\} \\ &= \varkappa_1\mathscr{I}_1 + \mu_1\mathscr{I}_2 + \varsigma_3\mathscr{I}_3 + \varsigma_4\mathscr{I}_4, \\ \mathfrak{d}_2 &= \frac{\mathscr{A}_4}{\Delta}\{\mathscr{I}_1\left[-\mathscr{A}_4\mathscr{A}_{10}\left(\mathscr{A}_2\mathscr{A}_5 - \mathscr{A}_1\mathscr{A}_6\right) - \mathscr{A}_4\mathscr{A}_6\left(\mathscr{A}_2\mathscr{A}_9 - \mathscr{A}_1\mathscr{A}_{10}\right)\right] \\ &\quad + \mathscr{I}_2\left[-\mathscr{A}_2\mathscr{A}_{12}\left(\mathscr{A}_2\mathscr{A}_5 - \mathscr{A}_1\mathscr{A}_6\right) + \mathscr{A}_2\mathscr{A}_8\left(\mathscr{A}_2\mathscr{A}_9 - \mathscr{A}_1\mathscr{A}_{10}\right)\right] \\ &\quad + \mathscr{I}_3\mathscr{A}_2\mathscr{A}_4\left(\mathscr{A}_2\mathscr{A}_9 - \mathscr{A}_1\mathscr{A}_{10}\right) + \mathscr{I}_4\mathscr{A}_2\mathscr{A}_4\left(\mathscr{A}_2\mathscr{A}_5 - \mathscr{A}_1\mathscr{A}_6\right)\} \\ &= \Xi_1\mathscr{I}_1 + \Xi_2\mathscr{I}_2 + \Xi_3\mathscr{I}_3 + \Xi_4\mathscr{I}_4. \end{aligned} \tag{22}$$

Therefore, for the constants $\mathfrak{c}_3$ and $\mathfrak{d}_3$, we obtain

$$\begin{aligned}
\mathfrak{c}_3 &= \frac{1}{\mathscr{A}_2\Delta}\{\mathscr{I}_1\mathscr{A}_2\mathscr{A}_4[(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6)(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})-(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10})(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)]\\
&\quad -\mathscr{I}_1\mathscr{A}_1\mathscr{A}_2[\mathscr{A}_4\mathscr{A}_6(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})+\mathscr{A}_4\mathscr{A}_{10}(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)]\\
&\quad -\mathscr{I}_2\mathscr{A}_1\mathscr{A}_2[\mathscr{A}_2\mathscr{A}_8(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})+\mathscr{A}_2\mathscr{A}_{12}(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)]\\
&\quad -\mathscr{I}_3\mathscr{A}_1\mathscr{A}_2^2\mathscr{A}_4(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})+\mathscr{I}_4\mathscr{A}_1\mathscr{A}_2^2\mathscr{A}_{12}(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)\}\\
&= \Theta_1\mathscr{I}_1+\Theta_2\mathscr{I}_2+\Theta_3\mathscr{I}_3+\Theta_4\mathscr{I}_4,\\
\mathfrak{d}_3 &= \frac{1}{\mathscr{A}_4\Delta}\{-\mathscr{I}_1\mathscr{A}_3\mathscr{A}_4[\mathscr{A}_4\mathscr{A}_{10}(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6)-\mathscr{A}_4\mathscr{A}_6(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10})]\\
&\quad -\mathscr{I}_2\mathscr{A}_2\mathscr{A}_4[(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6)(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})-(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10})(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)]\\
&\quad -\mathscr{I}_2\mathscr{A}_3\mathscr{A}_4[-\mathscr{A}_2\mathscr{A}_{12}(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6)+\mathscr{A}_2\mathscr{A}_8(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10})]\\
&\quad -\mathscr{I}_3\mathscr{A}_2\mathscr{A}_4^2\mathscr{A}_3(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10})+\mathscr{I}_4\mathscr{A}_2\mathscr{A}_4^2\mathscr{A}_3(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6)\}\\
&= \Upsilon_1\mathscr{I}_1+\Upsilon_2\mathscr{I}_2+\Upsilon_3\mathscr{I}_3+\Upsilon_4\mathscr{I}_4,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\varsigma_1 &= \frac{1}{\Delta_1}[-\mathscr{A}_6(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})+\mathscr{A}_{10}(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)],\\
\varsigma_2 &= \frac{\mathscr{A}_2}{\Delta_1}(\mathscr{A}_8\mathscr{A}_{11}-\mathscr{A}_7\mathscr{A}_{12}),\\
\varsigma_3 &= \frac{\mathscr{A}_2}{\Delta_1}(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12}),\\
\varsigma_4 &= \frac{\mathscr{A}_2}{\Delta_1}(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8).\\
\Xi_1 &= \frac{\mathscr{A}_2}{\Delta_1}(\mathscr{A}_5\mathscr{A}_{10}-\mathscr{A}_6\mathscr{A}_9),\\
\Xi_2 &= \frac{1}{\Delta_1}[-\mathscr{A}_{12}(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6)+\mathscr{A}_8(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10})],\\
\Xi_3 &= \frac{\mathscr{A}_4}{\Delta_1}(\mathscr{A}_2\mathscr{A}_9-\mathscr{A}_1\mathscr{A}_{10}),\\
\Xi_4 &= \frac{\mathscr{A}_4}{\Delta_1}(\mathscr{A}_2\mathscr{A}_5-\mathscr{A}_1\mathscr{A}_6).\\
\Theta_1 &= \frac{1}{\Delta_1}[-\mathscr{A}_5(\mathscr{A}_4\mathscr{A}_{11}-\mathscr{A}_3\mathscr{A}_{12})+\mathscr{A}_9(\mathscr{A}_4\mathscr{A}_7-\mathscr{A}_3\mathscr{A}_8)],
\end{aligned}$$

$$\begin{aligned}
\Theta_2 &= \frac{\mathscr{A}_1}{\Delta_1}\left(-\mathscr{A}_8\mathscr{A}_{11} + \mathscr{A}_7\mathscr{A}_{12}\right),\\
\Theta_3 &= \frac{\mathscr{A}_1}{\Delta_1}\left(\mathscr{A}_3\mathscr{A}_{12} - \mathscr{A}_4\mathscr{A}_{11}\right),\\
\Theta_4 &= \frac{\mathscr{A}_1}{\Delta_1}\left(\mathscr{A}_3\mathscr{A}_8 - \mathscr{A}_4\mathscr{A}_7\right).\\
\Upsilon_1 &= \frac{\mathscr{A}_3}{\Delta_1}\left(\mathscr{A}_6\mathscr{A}_9 - \mathscr{A}_5\mathscr{A}_{10}\right),\\
\Upsilon_2 &= \frac{1}{\Delta_1}\left[\mathscr{A}_{11}\left(\mathscr{A}_2\mathscr{A}_5 - \mathscr{A}_1\mathscr{A}_6\right) + \mathscr{A}_7\left(\mathscr{A}_2\mathscr{A}_9 - \mathscr{A}_1\mathscr{A}_{10}\right)\right],\\
\Upsilon_3 &= \frac{\mathscr{A}_3}{\Delta_1}\left(\mathscr{A}_1\mathscr{A}_{10} - \mathscr{A}_2\mathscr{A}_9\right),\\
\Upsilon_4 &= \frac{\mathscr{A}_3}{\Delta_1}\left(\mathscr{A}_1\mathscr{A}_6 - \mathscr{A}_2\mathscr{A}_5\right).
\end{aligned} \tag{24}$$

where $\varsigma_i, \Xi_i, \Theta_i, \Upsilon_i, i = 1, \ldots, 4$ are given by (24).

By replacing the constants $\mathfrak{c}2, \mathfrak{d}2, \mathfrak{c}3$, and $\mathfrak{d}3$ in system (15), we can solve problems (4) and (5). It is possible to compute the reverse of this result directly.

The Leray–Schauder alternative is now presented; it will be used to demonstrate that there are solutions to problems (4) and (5). □

Theorem 4 (see [1]). *Let $\mathscr{E}$ be a Banach space and $\mathscr{T}\colon \mathscr{E} \longrightarrow \mathscr{E}$ be a completely continuous operator. Let $\mathfrak{F} = \{\varphi \in \mathscr{E}, \varphi = \nu\mathscr{T}(\varphi)$ for some $0 < \nu < 1\}$. Then, either the set $\mathfrak{F}$ is unbounded or $\mathscr{T}$ has at least one fixed point.*

4. Main Results

We consider the space $\mathscr{U} = \{\varpi \in \mathscr{C}[0,1], {}^c\mathscr{D}^{p_2}\varpi \in \mathscr{C}[0,1]\}$ and $\mathscr{V} = \{\Psi \in \mathscr{C}[0,1], {}^c\mathscr{D}^{p_1}\Psi \in \mathscr{C}[0,1]\}$ equipped, respectively, with the norms $\|\varpi\|_{\mathscr{U}} = \|\varpi\| + \|{}^c\mathscr{D}^{p_2}\varpi\|$ and $\|\Psi\|_{\mathscr{V}} = \|\Psi\| + \|{}^c\mathscr{D}^{p_1}\Psi\|$, where $\|\cdot\|$ is the supremum norm, that is $\|w\| = \sup_{\varepsilon\in[0,1]|w(\varepsilon)|}$ for $w \in \mathscr{C}[0,1]$. The spaces $(\mathscr{U}, \|\cdot\|_{\mathscr{U}})$ and $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ are Banach spaces, and the product space $\mathscr{U}\times\mathscr{V}$ endowed with the norm $\|(\varpi, \mathscr{Y})\|_{\mathscr{U}\times\Psi} = \|\varpi\|_{\mathscr{U}} + \|\Psi\|_{\mathscr{V}}$ is also a Banach space. Utilizing Lemma 3, we define the operator $\Pi\colon \mathscr{U}\times\mathscr{V} \longrightarrow \mathscr{U}\times\mathscr{V}$ as follows: $\Pi(\varpi,\Psi) = (\Pi_1(\varpi,\Psi), \Pi_2(\varpi,\Psi))$ for $(\varpi,\Psi) \in \mathscr{U}\times\mathscr{V}$, where the operators $\Pi_1\colon \mathscr{U}\times\Psi \longrightarrow \mathscr{U}$ and $\Pi_2\colon \varpi\times\Psi \longrightarrow \Psi$ are given by

$$\begin{aligned}
&\Pi_1(\varpi,\Psi)(\varepsilon)\\
&= \mathscr{S}_1(\varepsilon)\left[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\right.\\
&\quad\left.-\int_0^1 \frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\zeta)d\zeta\right]\\
&\quad+\mathscr{S}_2(\varepsilon)\left[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\right.\\
&\quad\left.-\int_0^1 \frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\zeta)d\zeta\right]\\
&\quad+\mathscr{S}_3(\varepsilon)\left[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\right.\\
&\quad+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\Bigg]\\
&+\mathscr{S}_4(\varepsilon)\Bigg[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&-\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\Bigg]\\
&+\int_0^\varepsilon e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta,
\end{aligned}
$$

$$
\begin{aligned}
\Pi_2(\varpi,\Psi)(\varepsilon)=&\,\mathscr{T}_1(\varepsilon)\Bigg[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\\
&-\int_0^1\frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\zeta)d\zeta\Bigg]\\
&+\mathscr{T}_2(\varepsilon)\Bigg[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\\
&-\int_0^1\frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\zeta)d\zeta\Bigg]\\
&+\mathscr{T}_3(\varepsilon)\Bigg[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&-\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\Bigg]\\
&+\mathscr{T}_4(\varepsilon)\Bigg[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\sigma)d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&-\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta\Bigg]\\
&+\int_0^\varepsilon e^{-\mu_1(\varepsilon-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)(\mathfrak{u})d\mathfrak{u}\right)d\zeta.
\end{aligned}
\tag{25}
$$

If and only if (ϖ,Ψ) acts as a fixed point of the operator $\mathscr{T}$, then the pair (ϖ,Ψ) is a solution to problems (4) and (5). The presumptions used in this section are now described.

(1) $[\mathscr{H}_1]$ The continuous functions $\mathfrak{F}$ and $\mathfrak{G}$ are defined on $[0,1]\times\mathbb{R}^4\longrightarrow\mathbb{R}$. Moreover, for $i=1,\ldots,4$, and $\mathfrak{M}_0>0$, $\mathfrak{b}_0>0$, there exist real constants such that

$$
\begin{aligned}
&\left|\mathfrak{F}(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)\right|\le\mathfrak{M}_0+\mathfrak{M}_1|\varphi_1|+\mathfrak{M}_2|\varphi_2|+\mathfrak{M}_3|\varphi_3|+\mathfrak{M}_3|\varphi_4|,\\
&\left|\mathfrak{G}(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)\right|\le\mathfrak{b}_0+\mathfrak{b}_1|\varphi_1|+\mathfrak{b}_2|\varphi_2|+\mathfrak{b}_3|\varphi_3|+\mathfrak{b}_3|\varphi_4|,
\end{aligned}
\tag{26}
$$

For all $\varepsilon\in[0,1]$ and $\varphi_i\in\mathbb{R}, i=1,\ldots,4$.

(2) $[\mathscr{H}_2]$ The continuous functions $\mathfrak{F}$ and $\mathfrak{G}$ are defined on $[0,1]\times\mathbb{R}^4 \longrightarrow \mathbb{R}$. Additionally, there are positive constants $\mathfrak{K}_0>0$ and $\mathfrak{W}_0>0$ such that

$$\begin{aligned}&\left|\mathfrak{F}(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)-\mathfrak{F}(\varepsilon,\mathscr{Y}_1,\mathscr{Y}_2,\mathscr{Y}_3,\mathscr{Y}_4)\right|\\&\quad\le\mathfrak{W}_0\left(|\varphi_1-\mathscr{Y}_1|+|\varphi_2-\mathscr{Y}_2|+|\varphi_3-\mathscr{Y}_3|+|\varphi_4-\mathscr{Y}_4|\right),\\&\left|\mathfrak{G}(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)-\mathfrak{G}(\varepsilon,\mathscr{Y}_1,\mathscr{Y}_2,\mathscr{Y}_3,\mathscr{Y}_4)\right|\\&\quad\le\mathfrak{K}_0\left(|\varphi_1-\mathscr{Y}_1|+|\varphi_2-\mathscr{Y}_2|+|\varphi_3-\mathscr{Y}_3|+|\varphi_4-\mathscr{Y}_4|\right),\end{aligned}\tag{27}$$

for all $\varepsilon\in[0,1]$ and $\varphi_i,\mathscr{Y}_i\in\mathbb{R}, i=1,2,3,4$.

$$\begin{cases}\mathscr{S}_i'(\varepsilon)=\dfrac{1}{\varkappa_1}(\varsigma_i)\left(1-\varkappa_1e^{-\varkappa_1}\right)+\dfrac{1}{\varkappa_1^2}(\Theta_i)\left(2\varkappa_1-2+2e^{-\varkappa_1}\right), & i=1,2,3,4.\\[2ex]\mathscr{T}_i'(\varepsilon)=\dfrac{1}{\varkappa_1}(\Xi_i)\left(1-\mu_1e^{-\mu_1}\right)+\dfrac{1}{\varkappa_1^2}(\Upsilon_i)\left(2\mu_1-2+2e^{-\mu_1}\right), & i=1,2,3,4.\end{cases}\tag{28}$$

We denote by $\widehat{\mathscr{S}}_i=\sup_{\varepsilon\in[0,1]}|\mathscr{S}_i(\varepsilon)|, \widehat{\mathscr{T}}_i=\sup_{\varepsilon\in[0,1]}|\mathscr{T}_i(\varepsilon)|$ for $i=1, 2, 3, 4$.

$$\begin{aligned}\mathscr{U}_1&=\widehat{\mathscr{S}}_1\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathscr{S}}_3\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathscr{S}}_4\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right)+\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)},\\\mathscr{U}_2&=\widehat{\mathscr{S}}_2\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathscr{S}}_3\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathscr{S}}_4\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right],\\\mathscr{V}_1&=\widehat{\mathscr{T}}\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathscr{T}}_3\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathscr{T}}_4\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right),\\\mathscr{V}_2&=\widehat{\mathscr{T}}_2\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathscr{T}}_3\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathscr{T}}_4\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right]+\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)},\\\mathscr{U}_1^*&=\widehat{\mathscr{S}}_1'\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathscr{S}}_3'\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathscr{S}}_4'\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right)+\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)},\\\mathscr{U}_2^*&=\widehat{\mathscr{S}}_2'\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathscr{S}}_3'\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathscr{S}}_4'\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right],\\\mathscr{V}_1^*&=\widehat{\mathscr{T}}_1'\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathscr{T}}_3'\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathscr{T}}_4'\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right),\end{aligned}\tag{29}$$

$$\begin{aligned}
\mathscr{V}_2^* &= \widehat{\mathscr{T}}_2'\frac{(2-e^{-\mu_1})}{\Gamma(\eta)} + \widehat{\mathscr{T}}_3'\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right) + \widehat{\mathscr{T}}_4'\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)} + \frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right] + \frac{(2-e^{-\mu_1})}{\Gamma(\eta)},\\
\mathscr{N}_1 &= \mathscr{U}_1 + \mathscr{V}_1 + \frac{\mathscr{U}_1^*}{\Gamma(2-\phi_2)} + \frac{\mathscr{V}_1^*}{\Gamma(2-\phi_1)},\\
\mathscr{N}_2 &= \mathscr{U}_2 + \mathscr{V}_2 + \frac{\mathscr{U}_2^*}{\Gamma(2-\phi_2)} + \frac{\mathscr{V}_2^*}{\Gamma(2-\phi_1)},\\
\mathscr{N}_3 &= \mathfrak{M}_0\mathscr{N}_1 + \mathfrak{b}_0\mathscr{N}_2,\\
\mathscr{N}_4 &= \mathfrak{M}_1\mathscr{N}_1 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{b}_3}{\Gamma(\mathfrak{q}_2+1)}\right)\mathscr{N}_2,\\
\mathscr{N}_5 &= \left(\mathfrak{M}_2 + \mathfrak{M}_3 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\mathscr{N}_1 + \mathfrak{b}_4\mathscr{N}_2,\\
\varrho_1 &= 1 + \frac{1}{\Gamma(1+\mathfrak{q}_1)},\\
\varrho_2 &= 1 + \frac{1}{\Gamma(1+\mathfrak{q}_2)}.
\end{aligned} \tag{30}$$

Theorem 5. *Assume that* $(\mathscr{H}_1)$ *holds. If*

$$\max\{\mathscr{N}_4, \mathscr{N}_5\} < 1. \tag{31}$$

Then, the BVP (4) and (5) has at least one solution $(\varpi(\varepsilon), \Psi(\varepsilon))$, $\varepsilon \in [0,1]$.

Proof. First, we prove the complete continuity of the operator Π: $\mathscr{U}\times\mathscr{V} \longrightarrow \mathscr{U}\times\mathscr{V}$. The operators Π_1 and Π_2 are implied to be continuous by the continuity of the functions $\mathfrak{F}$ and $\mathfrak{G}$, which make Π a continuous operator. Then, we prove that Π has a uniform boundary. Let $\Omega \subset \mathscr{U}\times\mathscr{V}$ be any arbitrary bounded set. Consequently, $\mathscr{L}1$ and $\mathscr{L}2$ are positive constants such that

$$\begin{aligned}
&\left|\mathfrak{F}\left(\varepsilon, \varpi(\varepsilon), \Psi(\varepsilon), {}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon), \mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)\right| \le \mathscr{L}_1,\\
&\left|\mathfrak{G}\left(\varepsilon, \varpi(\varepsilon), \Psi(\varepsilon), {}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon), \mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right)\right| \le \mathscr{L}_2,
\end{aligned} \tag{32}$$

$\forall\ (\varpi, \Psi) \in \Omega$ and $\varepsilon \in [0,1]$.

For any $(\varpi, \Psi) \in \Omega$ and $\varepsilon \in [0,1]$, we have

$$\begin{aligned}
\left|\Pi_1(\varpi,\Psi)(\varepsilon)\right| = \left|\mathscr{S}_1(\varepsilon)\right| &\Bigg[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u}, \varpi(\mathfrak{u}), \Psi(\mathfrak{u}), {}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\\
&+ \int_0^1 \frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta, \varpi(\zeta), \Psi(\zeta), {}^c\mathscr{D}^{\phi_1}\Psi(\zeta), \mathscr{I}^{\mathfrak{q}_1}\Psi(\zeta)\right)\right|d\zeta\Bigg]\\
&+ \left|\mathscr{S}_2(\varepsilon)\right|\Bigg[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u}, \varpi(\mathfrak{u}), \Psi(\mathfrak{u}), {}^c\mathscr{D}^{\phi_2}\varpi(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_2}\varpi\mathfrak{u}\right)\right|d\mathfrak{u}\right)d\zeta\\
&+ \int_0^1 \frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\zeta, \varpi(\zeta), \Psi(\zeta), {}^c\mathscr{D}^{\phi_2}\varpi(\zeta), \mathscr{I}^{\mathfrak{q}_2}\varpi(\zeta)\right)\right|d\zeta\Bigg]\\
&+ \left|\mathscr{S}_3(\varepsilon)\right|\Bigg[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^\mathfrak{u} \frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma, \varpi(\sigma), \Psi(\sigma), {}^c\mathscr{D}^{\phi_1}\Psi(\sigma), \mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+ \int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^\mathfrak{u} \frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma, \varpi(\sigma), \Psi(\sigma), {}^c\mathscr{D}^{\phi_2}\varpi(\sigma), \mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\Bigg]\\
&+|\mathcal{S}_4(\varepsilon)|\Bigg[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_1}\Psi(\sigma),\mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_2}\varpi(\sigma),\mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_2}\varpi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_2}\varpi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\Bigg]\Bigg|\\
&+\int_0^\zeta e^{-\varkappa_1(\zeta-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta,\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta,\\
&\le \mathscr{L}_1\left\{\widehat{\mathcal{S}}_1\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathcal{S}}_3\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathcal{S}}_4\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right)\right\}\\
&+\mathscr{L}_2\left\{\widehat{\mathcal{S}}_2\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathcal{S}}_3\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathcal{S}}_4\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right]\right\}\\
&+\mathscr{L}_1\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}=\mathscr{L}_1\mathscr{U}_1+\mathscr{L}_2\mathscr{U}_2.
\end{aligned}\tag{33}
$$

Then, $\|\Pi_1(\varpi,\Psi)\|\le\mathscr{L}_1\mathscr{U}_1+\mathscr{L}_2\mathscr{U}_2$, for all $(\varpi,\Psi)\in\Omega$.

Considering the definition of $\Pi_1(\varpi,\Psi)$, we get

$$
\begin{aligned}
\left|\Pi_1'(\varpi,\Psi)(\varepsilon)\right| &= |\mathcal{S}_1'(\varepsilon)|\Bigg|\Bigg[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta,\varpi(\zeta),\Psi(\zeta),{}^c\mathscr{D}^{\phi_1}\Psi(\zeta),\mathscr{I}^{\mathfrak{q}_1}\Psi(\zeta)\right)\right|d\zeta\Bigg]\\
&+|\mathcal{S}_2'(\varepsilon)|\Bigg[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_2}\varpi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_2}\varpi\mathfrak{u}\right)\right|d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\zeta,\varpi(\zeta),\Psi(\zeta),{}^c\mathscr{D}^{\phi_2}\varpi(\zeta),\mathscr{I}^{\mathfrak{q}_2}\varpi(\zeta)\right)\right|d\zeta\Bigg]\\
&+|\mathcal{S}_3'(\varepsilon)|\Bigg[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_1}\Psi(\sigma),\mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_2}\varpi(\sigma),\mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\
&+\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\Bigg]
\end{aligned}
$$

$$+\left|\mathcal{S}_4'(\varepsilon)\right|\left[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathcal{D}^{\phi_1}\Psi(\sigma),\mathcal{I}^{\mathfrak{q}_1}\Psi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\right.$$

$$+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathcal{D}^{\phi_2}\varpi(\sigma),\mathcal{I}^{\mathfrak{q}_2}\varpi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta$$

$$\left.+\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathcal{D}^{\phi_2}\varpi(\mathfrak{u}),\mathcal{I}^{\mathfrak{q}_2}\varpi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\right]$$

$$+\int_0^\zeta e^{-\varkappa_1(\zeta-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta,\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathcal{D}^{\phi_1}\Psi(\mathfrak{u}),\mathcal{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta,|$$

$$\leq\mathcal{L}_1\left\{\widehat{\mathcal{S}}_1'\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathcal{S}}_3'\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathcal{S}}_4'\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right)\right\}$$

$$+\mathcal{L}_2\left\{\widehat{\mathcal{S}}_2'\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathcal{S}}_3'\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathcal{S}}_4'\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right]\right\}$$

$$+\mathcal{L}_1\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}=\mathcal{L}_1\mathcal{U}_1^*+\mathcal{L}_2\mathcal{U}_2^*. \tag{34}$$

By utilizing the definition of the Caputo fractional derivative of order $\phi_2\in(0,1)$, we conclude

$$\left|{}^c\mathcal{D}^{\phi_2}\Pi_1(\varpi,\Psi)(\varepsilon)\right|\leq\int_0^\varepsilon\frac{(\varepsilon-\zeta)^{-\phi_2}}{\Gamma(1-\phi_2)}\left|\Pi_1'(\varpi,\Psi)\right|d\zeta$$
$$\leq\left(\mathcal{L}_1\mathcal{U}_1^*+\mathcal{L}_2\mathcal{U}_2^*\right)\int_0^\varepsilon\frac{(\varepsilon-\zeta)^{-\phi_2}}{\Gamma(1-\phi_2)}d\zeta$$
$$\leq\frac{\left(\mathcal{L}_1\mathcal{U}_1^*+\mathcal{L}_2\mathcal{U}_2^*\right)}{\Gamma(2-\phi_2)},\quad\forall\varepsilon\in[0,1], \tag{35}$$

from where we obtain

$$\left\|{}^c\mathcal{D}^{\phi_2}\Pi_1(\varpi,\Psi)(\varepsilon)\right\|\leq\frac{\left(\mathcal{L}_1\mathcal{U}_1^*+\mathcal{L}_2\mathcal{U}_2^*\right)}{\Gamma(2-\phi_2)},\quad\forall\varepsilon\in[0,1]. \tag{36}$$

Therefore, we conclude

$$\left\|\Pi_1(\varpi,\Psi)\right\|_{\mathcal{U}}=\left\|\Pi_1(\varpi,\Psi)\right\|+\left\|{}^c\mathcal{D}^{\phi_2}\Pi_1(\varpi,\Psi)\right\|$$
$$\leq\mathcal{L}_1\mathcal{U}_1+\mathcal{L}_2\mathcal{U}_2+\frac{\left(\mathcal{L}_1\mathcal{U}_1^*+\mathcal{L}_2\mathcal{U}_2^*\right)}{\Gamma(2-\phi_2)}. \tag{37}$$

In a similar manner, we have

$$\left|\mathcal{Y}_2(\varpi,\Psi)(\varepsilon)\right|$$
$$\leq\mathcal{V}_1\left\{\widehat{\mathcal{T}}_1\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathcal{T}}_3\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathcal{T}}_4\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right)\right\}$$
$$+\mathcal{V}_2\left\{\widehat{\mathcal{T}}_2\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathcal{T}}_3\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathcal{T}}_4\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right]\right\}$$
$$+\mathcal{V}_2\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}$$
$$=\mathcal{L}_1\mathcal{V}_1+\mathcal{L}_2\mathcal{V}_2,$$

$$\begin{aligned}
&\left|\mathscr{Y}_2'(\varpi,\Psi)(\varepsilon)\right| \\
&\leq \mathscr{V}_1\left\{\widehat{\mathscr{T}}_1'\frac{(2-e^{-\varkappa_1})}{\Gamma(\vartheta)}+\widehat{\mathscr{T}}_3'\left[\frac{(1-e^{-\varkappa_1})}{\varkappa_1\Gamma(\vartheta)}+\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right]+\widehat{\mathscr{T}}_4'\left(\frac{(\varkappa_1+e^{-\varkappa_1}-1)}{\varkappa_1^2\Gamma(\vartheta)}\right)\right\} \\
&\quad+\mathscr{V}_2\left\{\widehat{\mathscr{T}}_{2'}\frac{(2-e^{-\mu_1})}{\Gamma(\eta)}+\widehat{\mathscr{T}}_3'\left(\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right)+\widehat{\mathscr{T}}_4'\left[\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)}+\frac{(\mu_1+e^{-\mu_1}-1)}{\mu_1^2\Gamma(\eta)}\right]\right\} \\
&\quad+\mathscr{V}_2\frac{(1-e^{-\mu_1})}{\mu_1\Gamma(\eta)} \\
&=\mathscr{L}_1\mathscr{V}_1^*+\mathscr{L}_2\mathscr{V}_2^*, \\
&\left|{}^c\mathscr{D}^{\phi_1}\Pi_2(\varpi,\Psi)(\varepsilon)\right|\leq\frac{(\mathscr{L}_1\mathscr{V}_1^*+\mathscr{L}_2\mathscr{V}_2^*)}{\Gamma(2-\phi_1)},\quad \forall\varepsilon\in[0,1].
\end{aligned}\tag{38}$$

Therefore, we conclude

$$\begin{aligned}
\left\|\Pi_2(\varpi,\Psi)\right\|_{\mathscr{V}}&=\left\|\Pi_2(\varpi,\Psi)\right\|+\left\|{}^c\mathscr{D}^{\phi_1}\Pi_2(\varpi,\Psi)\right\| \\
&\leq\mathscr{L}_1\mathscr{V}_1+\mathscr{L}_2\mathscr{V}_2+\frac{(\mathscr{L}_1\mathscr{V}_1^*+\mathscr{L}_2\mathscr{V}_2^*)}{\Gamma(2-\phi_1)}.
\end{aligned}\tag{39}$$

Based on the inequalities (37) and (39), we ascertain that both Π_1 and Π_2 are uniformly bounded. This, in turn, implies that the operator Π is uniformly bounded. Next, we will demonstrate that Π is equicontinuous. Take $\varepsilon_1,\varepsilon_2\in[0,1]$, with $\varepsilon_1<\varepsilon_2$. We then have

$$\begin{aligned}
&\left|\Pi_1(\varpi,\Psi)(\varepsilon_2)-\Pi_1(\varpi,\Psi)(\varepsilon_1)\right| \\
&\leq\left|\mathscr{S}_1(\varepsilon_2)-\mathscr{S}_1(\varepsilon_1)\right| \\
&\times\left|\left[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\right.\right. \\
&\left.\left.+\int_0^1\frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta,\varpi(\zeta),\Psi(\zeta),{}^c\mathscr{D}^{\phi_1}\Psi(\zeta),\mathscr{I}^{\mathfrak{q}_1}\Psi(\zeta)\right)\right|d\zeta\right]\right| \\
&+\left|\mathscr{S}_2(\varepsilon_2)-\mathscr{S}_2(\varepsilon_1)\right| \\
&\times\left|\left[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_2}\varpi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_2}\varpi\mathfrak{u}\right)\right|d\mathfrak{u}\right)d\zeta\right.\right. \\
&\left.\left.+\int_0^1\frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\zeta,\varpi(\zeta),\Psi(\zeta),{}^c\mathscr{D}^{\phi_2}\varpi(\zeta),\mathscr{I}^{\mathfrak{q}_2}\varpi(\zeta)\right)\right|d\zeta\right]\right| \\
&+\left|\mathscr{S}_3(\varepsilon_2)-\mathscr{S}_3(\varepsilon_1)\right| \\
&\times\left[\left|\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_1}\Psi(\sigma),\mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\right|\right. \\
&+\left|\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_2}\varpi(\sigma),\mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\right| \\
&\left.+\left|\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&+\left|\mathcal{S}_4(\varepsilon_2)-\mathcal{S}_4(\varepsilon_1)\right| \\
&\times\left[\left|\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_1}\Psi(\sigma),\mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\right|\right. \\
&+\left|\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma,\varpi(\sigma),\Psi(\sigma),{}^c\mathscr{D}^{\phi_2}\varpi(\sigma),\mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\right| \\
&\left.+\left|\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u},\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_2}\varpi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_2}\varpi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\right|\right] \\
&+\left|\int_0^\varepsilon e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta,\varpi(\mathfrak{u}),\Psi(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\right|, \\
&\leq\mathscr{L}_1\frac{1}{\varkappa_1\Gamma(\vartheta)}\left(\left(1-e^{-\varkappa_1\varepsilon_1}\right)\left(1-e^{-\varkappa_1(\varepsilon_2-\varepsilon_1)}\right)\right)+\mathscr{L}_1\frac{1}{\varkappa_1}\left(1-e^{-\varkappa_1(\varepsilon_2-\varepsilon_1)}\right) \\
&\mathscr{L}_1\left\{\widehat{\mathcal{S}}_1(\varepsilon_2)-\widehat{\mathcal{S}}_1(\varepsilon_1)\frac{\left(2-e^{-\varkappa_1}\right)}{\Gamma(\vartheta)}+\widehat{\mathcal{S}}_3(\varepsilon_2)-\widehat{\mathcal{S}}_3(\varepsilon_1)\left[\frac{\left(1-e^{-\varkappa_1}\right)}{\varkappa_1\Gamma(\vartheta)}+\frac{\left(\varkappa_1+e^{-\varkappa_1}-1\right)}{\varkappa_1^2\Gamma(\vartheta)}\right]\right. \\
&\left.+\widehat{\mathcal{S}}_4(\varepsilon)_2-\widehat{\mathcal{S}}_4(\varepsilon)_1\left(\frac{\left(\varkappa_1+e^{-\varkappa_1}-1\right)}{\varkappa_1^2\Gamma(\vartheta)}\right)\right\} \\
&+\mathscr{L}_2\left\{\widehat{\mathcal{S}}_2(\varepsilon_2)-\widehat{\mathcal{S}}_2(\varepsilon_1)\frac{\left(2-e^{-\mu_1}\right)}{\Gamma(\eta)}+\widehat{\mathcal{S}}_3(\varepsilon_2)-\widehat{\mathcal{S}}_3(\varepsilon_1)\left(\frac{\left(\mu_1+e^{-\mu_1}-1\right)}{\mu_1^2\Gamma(\eta)}\right)\right. \\
&\left.+\widehat{\mathcal{S}}_4(\varepsilon_2)-\widehat{\mathcal{S}}_4(\varepsilon_1)\left[\frac{\left(1-e^{-\mu_1}\right)}{\mu_1\Gamma(\eta)}+\frac{\left(\mu_1+e^{-\mu_1}-1\right)}{\mu_1^2\Gamma(\eta)}\right]\right\}.
\end{aligned}
\tag{40}
$$

Because

$$
\begin{aligned}
&\mathcal{S}_i(\varepsilon_2)-\mathcal{S}_i(\varepsilon_2) \\
&=\left|\frac{1}{\varkappa_1^2}(\varsigma_i)\left(\varkappa_1\varepsilon_2-1+e^{-\varkappa_1\varepsilon_2}\right)+\frac{1}{\varkappa_1^3}(\Theta_i)\left(\varkappa_1^2\varepsilon_2^2-2\varkappa_1\varepsilon_2+2-2e^{-\varkappa_1\varepsilon_2}\right)\right. \\
&\left.-\frac{1}{\varkappa_1^2}(\varsigma_i)\left(\varkappa_1\varepsilon_1-1+e^{-\varkappa_1\varepsilon_1}\right)+\frac{1}{\varkappa_1^3}(\Theta_i)\left(\varkappa_1^2\varepsilon_1^2-2\varkappa_1\varepsilon_1+2-2e^{-\varkappa_1\varepsilon_1}\right)\right|\quad i=1,2,3,4. \\
&\leq\frac{1}{\varkappa_1^2}\left|\varsigma_i\right|\left|\varkappa_1(\varepsilon_2-\varepsilon_1)-e^{\varkappa_1\varepsilon_1}+e^{\varkappa_1\varepsilon_2}\right| \\
&+\frac{1}{\varkappa_1^3}\left|\Theta_i\right|\left|\varkappa_1^2\left(\varepsilon_2^2\left(\varepsilon_2^2-\varepsilon_1^2\right)-2\left(e^{-\varkappa_1\varepsilon_2}-e^{-\varkappa_1\varepsilon_1}\right)\right)-\varkappa_1(\varepsilon_2-\varepsilon_1)\right|\longrightarrow 0,\text{ as }\varepsilon_2\longrightarrow\varepsilon_1,\quad i=1,2,3,4.
\end{aligned}
\tag{41}
$$

Clearly, $\left|\Pi_1(\varpi,\Psi)(\varepsilon_2)-\Pi_1(\varpi,\Psi)(\varepsilon_1)\right|\longrightarrow 0,$ as $\varepsilon_2\longrightarrow\varepsilon_1$.

Also, we obtain

$$\begin{aligned}
&\left|{}^{c}\mathscr{D}^{\phi_2}\Pi_1(\varpi,\Psi)(\varepsilon_2)-{}^{c}\mathscr{D}^{\phi_2}\Pi_1(\varpi,\Psi)(\varepsilon_1)\right|\\
&\leq\frac{1}{\Gamma(1-\phi_2)}\int_0^{\varepsilon_1}\frac{\left|(\varepsilon_1-\zeta)^{\phi_2}-(\varepsilon_2-\zeta)^{\phi_2}\right|}{(\varepsilon_1-\zeta)^{\phi_2}(\varepsilon_2-\zeta)^{\phi_2}}\left|\Pi_1'(\varpi,\Psi)(\zeta)\right|d\zeta\\
&+\frac{1}{\Gamma(1-\phi_2)}\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\zeta)^{-\phi_2}\left|\Pi_1'(\varpi,\Psi)(\zeta)\right|d\zeta,\\
&=\frac{\mathscr{L}_1}{\Gamma(2-\phi_1)}\left(2(\varepsilon_2-\varepsilon_1)^{1-\phi_2}-\varepsilon_2^{1-\phi_2}+\varepsilon_1^{1-\phi_2}\right)\longrightarrow 0,\\
&as\,\varepsilon_2\longrightarrow\varepsilon_1.
\end{aligned}\tag{42}$$

In a similar manner, we have

$$\begin{aligned}
&\left|\Pi_2(\varpi,\Psi)(\varepsilon_2)-\Pi_2(\varpi,\Psi)(\varepsilon_1)\right|\longrightarrow 0,\\
&\left|{}^{c}\mathscr{D}^{\phi_2}\Pi_2(\varpi,\Psi)(\varepsilon_2)-{}^{c}\mathscr{D}^{\phi_2}\Pi_2(\varpi,\Psi)(\varepsilon_1)\right|\longrightarrow 0,\text{as}\,\varepsilon_2\longrightarrow\varepsilon_1.
\end{aligned}\tag{43}$$

Thus, the operators Π_1 and Π_2 are equicontinuous, and then Π is also equicontinuous.

Thus, we deduce that Π is compact based on the Ascoli–Arzela theorem. As a result, we determine that Π is completely continuous.

Now, let us establish that the set $\widehat{\mathfrak{F}}=\{(\varpi,\Psi)\in\mathscr{U}\times\mathscr{V}\,|\,(\mathscr{U},\mathscr{V})=\nu\Pi(\varpi,\Psi),0\leq\nu\leq 1\}$ is bounded. Let $\{(\varpi,\Psi)\in\widehat{\mathfrak{F}}$, that is, $\{(\varpi,\Psi)=\nu\Pi(\varpi,\Psi)$, for some $\nu\in[0,1]$. Then, for any $\varepsilon\in[0,1]$, we have $\varpi(\varepsilon)=\nu\Pi_1(\varpi,\Psi)(\varepsilon),\Psi(\varepsilon)=\nu\Pi_2(\varpi,\Psi)(\varepsilon)$. From these last relations, we deduce $|\varpi(\varepsilon)|\leq\left|\Pi_1(\varpi,\Psi)(\varepsilon)\right|$ and $|\Psi(\varepsilon)|\leq\left|\Pi_2(\varpi,\Psi)(\varepsilon)\right|\forall\,\varepsilon\in[0,1]$.

Then, by $\mathscr{H}_1$, we obtain

$$\begin{aligned}
|\varpi(\varepsilon)|\leq&\left|\mathscr{S}_1(\varepsilon)\right|\Bigg[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\\
&\times\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{M}_0\right|+\left|\mathfrak{M}_1\varpi(\mathfrak{u})\right|+\left|\mathfrak{M}_2\Psi(\mathfrak{u})\right|+\left|\mathfrak{M}_3{}^{c}\mathscr{D}^{\phi_1}\Psi(\mathfrak{u})\right|+\left|\mathfrak{M}_4\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right|d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left[\left|\mathfrak{M}_0\right|+\left|\mathfrak{M}_1\varpi(\zeta)\right|+\left|\mathfrak{M}_2\Psi(\zeta)\right|+\left|\mathfrak{M}_3{}^{c}\mathscr{D}^{\phi_1}\Psi(\zeta)\right|+\left|\mathfrak{M}_4\mathscr{I}^{\mathfrak{q}_1}\Psi(\zeta)\right|\right]d\zeta\Bigg]\\
&+\left|\mathscr{S}_2(\varepsilon)\right|\Bigg[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\\
&\times\left(\int_0^{\zeta}\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left[\left|\mathfrak{b}_0\right|+\mathfrak{b}_1\varpi(\mathfrak{u})\right|+\left|\mathfrak{b}_2\Psi(\varepsilon)(\mathfrak{u})\right|+\left|\mathfrak{b}_3{}^{c}\mathscr{D}^{\phi_2}\varpi(\mathfrak{u})\right|+\mathfrak{b}_4\mathscr{I}^{\mathfrak{q}_2}\varpi(\mathfrak{u})\left|\right]d\mathfrak{u}\right)d\zeta\\
&+\int_0^1\frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\left[\left|\mathfrak{b}_0\right|+\mathfrak{b}_1\varpi(\zeta)\right|+\left|\mathfrak{b}_2\Psi(\varepsilon)(\zeta)\right|+\left|\mathfrak{b}_3{}^{c}\mathscr{D}^{\phi_2}\varpi(\zeta)\right|+\mathfrak{b}_4\mathscr{I}^{\mathfrak{q}_2}\varpi(\zeta)\left|\right]d\zeta\Bigg]\\
&+\left|\mathscr{S}_3(\varepsilon)\right|\Bigg[\int_0^1\left(\int_0^{\zeta}e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\right.\right.\\
&\times\left[\left|\mathfrak{M}_0\right|+\left|\mathfrak{M}_1\varpi(\sigma)\right|+\left|\mathfrak{M}_2\Psi(\sigma)\right|+\left|\mathfrak{M}_3{}^{c}\mathscr{D}^{\phi_1}\Psi(\sigma)\right|+\left|\mathfrak{M}_4\mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right|\right]d\sigma\bigg)d\mathfrak{u}\bigg)d\zeta
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\right.\right.\\
&\times\left[\left|\mathfrak{b}_0\right|+\mathfrak{b}_1\varpi(\sigma)|+|\mathfrak{b}_2\Psi(\varepsilon)(\sigma)|+|\mathfrak{b}_3{}^c\mathscr{D}^{\phi_2}\varpi(\sigma)\right|+\mathfrak{b}_4\mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)|\big]d\sigma\Big)d\mathfrak{u}\Big)d\zeta\\
&+\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\right.\\
&\times\left[\left|\mathfrak{M}_0\right|+\left|\mathfrak{M}_1\varpi(\mathfrak{u})\right|+\left|\mathfrak{M}_2\Psi(\mathfrak{u})\right|+\left|\mathfrak{M}_3{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u})\right|+\left|\mathfrak{M}_4\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right|\right]d\mathfrak{u}\Big)d\zeta\Big]\\
&+\left|\mathscr{S}_4(\varepsilon)\right|\left[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\right.\right.\right.\\
&\times\left[\left|\mathfrak{M}_0\right|+\left|\mathfrak{M}_1\varpi(\sigma)\right|+\left|\mathfrak{M}_2\Psi(\sigma)\right|+\left|\mathfrak{M}_3{}^c\mathscr{D}^{\phi_1}\Psi(\sigma)\right|+\left|\mathfrak{M}_4\mathscr{I}^{\mathfrak{q}_1}\Psi(\sigma)\right|\right]d\sigma\Big)d\mathfrak{u}\Big)d\zeta\\
&+\int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\right.\right.\\
&\times\left[\left|\mathfrak{b}_0\right|+\mathfrak{b}_1\varpi(\sigma)|+|\mathfrak{b}_2\Psi(\varepsilon)(\sigma)|+|\mathfrak{b}_3{}^c\mathscr{D}^{\phi_2}\varpi(\sigma)\right|+\mathfrak{b}_4\mathscr{I}^{\mathfrak{q}_2}\varpi(\sigma)|\big]d\sigma\Big)d\mathfrak{u}\Big)d\zeta\\
&+\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\right.\\
&\times\left[\left|\mathfrak{b}_0\right|+\mathfrak{b}_1\varpi(\mathfrak{u})|+|\mathfrak{b}_2\Psi(\varepsilon)(\mathfrak{u})|+|\mathfrak{b}_3{}^c\mathscr{D}^{\phi_2}\varpi(\mathfrak{u})\right|+\mathfrak{b}_4\mathscr{I}^{\mathfrak{q}_2}\varpi(\mathfrak{u})|\big]d\mathfrak{u}\Big)d\zeta\Big]\\
&+\int_0^\varepsilon e^{-\varkappa_1(\varepsilon-\zeta)}\left(\int_0^\zeta\frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\right.\\
&\times\left[\left|\mathfrak{M}_0\right|+\left|\mathfrak{M}_1\varpi(\mathfrak{u})\right|+\left|\mathfrak{M}_2\Psi(\mathfrak{u})\right|+\left|\mathfrak{M}_3{}^c\mathscr{D}^{\phi_1}\Psi(\mathfrak{u})\right|+\left|\mathfrak{M}_4\mathscr{I}^{\mathfrak{q}_1}\Psi(\mathfrak{u})\right|\right]d\mathfrak{u}\Big)d\zeta,
\end{aligned}
\tag{44}
$$

which, on taking the norm for $\varepsilon\in[0,1]$, yields

$$
\begin{aligned}
\|\varpi\|\leq&\left[\mathfrak{M}_0+\mathfrak{M}_1\|\varpi\|_{\mathscr{U}}+\left(\mathfrak{M}_2+\mathfrak{M}_3+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{U}_1\\
&+\left[\mathfrak{b}_0+\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}}+\mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{U}_2.
\end{aligned}
\tag{45}
$$

Similarly, we obtain

$$
\begin{aligned}
\|\varpi'\|\leq&\left[\mathfrak{M}_0+\mathfrak{M}_1\|\varpi\|_{\mathscr{U}}+\left(\mathfrak{M}_2+\mathfrak{M}_3+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{U}_1^*\\
&+\left[\mathfrak{b}_0+\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}}+\mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{U}_2^*.
\end{aligned}
\tag{46}
$$

This implies that

$$\left\| {}^{c}\mathscr{D}^{\phi_2}\varpi \right\| \leq \frac{1}{\Gamma(2-\phi_2)}\left\{\left[\mathfrak{M}_0 + \mathfrak{M}_1\|\varpi\|_{\mathscr{U}} + \left(\mathfrak{M}_2 + \mathfrak{M}_3 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{U}_1^* + \left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}} + \mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{U}_2^*\right\}. \tag{47}$$

Thus, we have

$$\begin{aligned}\|\varpi\|_{\mathscr{U}} &= \|\varpi\| + \left\| {}^{c}\mathscr{D}^{\phi_2}\varpi \right\| \\ &\leq \left[\mathfrak{M}_0 + \mathfrak{M}_1\|\varpi\|_{\mathscr{U}} + \left(\mathfrak{M}_2 + \mathfrak{M}_3 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{U}_1 \\ &\quad + \left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}} + \mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{U}_2 \\ &\quad + \frac{1}{\Gamma(2-\phi_2)}\left\{\left[\mathfrak{M}_0 + \mathfrak{M}_1\|\varpi\|_{\mathscr{U}} + \left(\mathfrak{M}_2 + \mathfrak{M}_3 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{U}_1^* \right. \\ &\quad \left. + \left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}} + \mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{U}_2^*\right\}.\end{aligned} \tag{48}$$

Likewise, we can have

$$\begin{aligned}\|\Psi\|_{\mathscr{V}} &= \|\Psi\| + \left\| {}^{c}\mathscr{D}^{\phi_2}\Psi \right\| \\ &\leq \left[\mathfrak{M}_0 + \mathfrak{M}_1\|\varpi\|_{\mathscr{U}} + \left(\mathfrak{M}_2 + \mathfrak{M}_3 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{V}_1 \\ &\quad + \left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}} + \mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{V}_2 \\ &\quad + \frac{1}{\Gamma(2-\phi_1)}\left\{\left[\mathfrak{M}_0 + \mathfrak{M}_1\|\varpi\|_{\mathscr{U}} + \left(\mathfrak{M}_2 + \mathfrak{M}_3 + \frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\|\Psi\|_{\mathscr{V}}\right]\mathscr{V}_1^* \right. \\ &\quad \left. + \left[\mathfrak{b}_0 + \left(\mathfrak{b}_1 + \mathfrak{b}_2 + \frac{\mathfrak{M}_4}{\Gamma(1+\phi_1)}\right)\|\varpi\|_{\mathscr{U}} + \mathfrak{b}_3\|\varpi\|_{\mathscr{V}}\right]\mathscr{V}_2^*\right\}.\end{aligned} \tag{49}$$

From (48) and (49), we find

$$
\begin{aligned}
\|(\varpi,\Psi)\|_{\mathscr{U}\times\mathscr{V}} &= \|\varpi\|_{\mathscr{U}} + \|\Psi\|_{\mathscr{V}} \\
&\le \|\varpi\|_{\mathscr{U}}\left\{\mathfrak{M}_1\left[\mathscr{U}_1+\mathscr{V}_1+\frac{\mathscr{U}_1^*}{\Gamma(2-\phi_2)}+\frac{\mathscr{V}_1^*}{\Gamma(2-\phi_1)}\right]+\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{\mathfrak{b}_3}{\Gamma(\mathfrak{q}_1+1)}\right)\right.\\
&\quad\left.+\left[\mathscr{U}_2+\mathscr{V}_2+\frac{\mathscr{U}_2^*}{\Gamma(2-\phi_2)}+\frac{\mathscr{V}_2^*}{\Gamma(2-\phi_1)}\right]\right\}\\
&\quad+\|\Psi\|_{\mathscr{V}}\left\{\mathfrak{b}_4\left[\mathscr{U}_2+\mathscr{V}_2+\frac{\mathscr{U}_2^*}{\Gamma(2-\phi_2)}+\frac{\mathscr{V}_2^*}{\Gamma(2-\phi_1)}\right]+\left(\mathfrak{M}_2+\mathfrak{M}_3+\frac{\mathfrak{M}_4}{\Gamma(\mathfrak{q}_1+1)}\right)\right.\\
&\quad\left.+\left[\mathscr{U}_1+\mathscr{V}_1+\frac{\mathscr{U}_1^*}{\Gamma(2-\phi_2)}+\frac{\mathscr{V}_1^*}{\Gamma(2-\phi_1)}\right]\right\}\\
&\quad+\mathfrak{M}_0\left[\mathscr{U}_1+\mathscr{V}_1+\frac{\mathscr{U}_1^*}{\Gamma(2-\phi_2)}+\frac{\mathscr{V}_1^*}{\Gamma(2-\phi_1)}\right]\\
&\quad+\mathfrak{b}_0\left[\mathscr{U}_2+\mathscr{V}_2+\frac{\mathscr{U}_2^*}{\Gamma(2-\phi_2)}+\frac{\mathscr{V}_2^*}{\Gamma(2-\phi_1)}\right]\\
&\le \mathscr{N}_3+\max\{\mathscr{N}_4+\mathscr{N}_5\}\|\varpi,\Psi\|_{\mathscr{U}\times\Psi}.
\end{aligned}
\tag{50}
$$

By leveraging the assumption $\max\mathscr{N}4+\mathscr{N}5<1$, we deduce

$$
\|\varpi,\Psi\|_{\mathscr{U}\times\Psi}\le\frac{\mathscr{N}_3}{1-\max\{\mathscr{N}_4,\mathscr{N}_5\}}. \tag{51}
$$

Therefore, we infer that the set $\widehat{\mathfrak{F}}$ is bounded. Employing Theorem 5, we establish that the operator Π has at least one fixed point, serving as a solution to our problems (4) and (5). This concludes the proof.

Subsequently, we will establish existence and uniqueness results for problems (4) and (5), employing the Banach contraction mapping principle. We introduce the notations:

$$
\begin{aligned}
&r_1=\sup_{\varepsilon\in[0,1]}|\mathfrak{F}(\varepsilon,0,0,0,0)|, r_2=\sup_{\varepsilon\in[0,1]}|\mathfrak{G}(\varepsilon,0,0,0,0)|,\\
&\Lambda=\mathfrak{W}_0\rho_1\mathscr{U}_1+\mathfrak{K}_0\rho_2\mathscr{V}_1, \mathscr{W}=\mathfrak{W}_0\rho_1\mathscr{U}_1^*+\mathfrak{K}_0\rho_2 r_2\mathscr{V}_1^*,\\
&\Lambda^*=\mathfrak{W}_0\rho_1\mathscr{U}_2+\mathfrak{K}_0\rho_2\mathscr{V}_2, \mathscr{W}^*=\mathfrak{W}_0\rho_1\mathscr{U}_2^*+\mathfrak{K}_0\rho_2 r_2\mathscr{V}_2^*,\\
&\mathscr{Q}_1=r_1\mathscr{U}_1+r_2\mathscr{V}_1, \mathscr{Q}_1^*=r_1\mathscr{U}_1^*+r_2\mathscr{V}_1^*, \widehat{\mathscr{Q}_1}=r_1\mathscr{U}_2+r_2\mathscr{V}_2, \widehat{\mathscr{Q}_1^*}=r_1\mathscr{U}_2^*+r_2\mathscr{V}_2^*.
\end{aligned}
\tag{52}
$$

□

Theorem 6. *Assume that $\mathscr{H}_2$ holds. Further*

$$
\left[\Lambda+\Lambda^*+\frac{\mathscr{W}}{\Gamma(2-\phi_2)}+\frac{\mathscr{W}^*}{\Gamma(2-\phi_1)}\right]<1, \tag{53}
$$

then problems (4) and (5) has a unique solution.

Proof. We examine the positive value r provided by

$$
r\ge\frac{\left[\mathscr{Q}_1+\mathscr{Q}_1^*+\widehat{\mathscr{Q}_1}/\Gamma(2-\phi_2)+\widehat{\mathscr{Q}_1^*}\Gamma/(2-\phi_1)\right]}{(1-[\Lambda+\Lambda^*+\mathscr{W}/\Gamma(2-\phi_2)+\mathscr{W}^*/\Gamma(2-\phi_1)])}. \tag{54}
$$

We show that $\varepsilon\widehat{\mathfrak{B}}_r\subset\widehat{\mathfrak{B}}_r$, where $\widehat{\mathfrak{B}}_r=\{(\varpi,\Psi)\in\mathscr{U}\times\mathscr{V}, \|\varpi,\Psi\|_{\mathscr{U}\times\mathscr{V}}<r\}$. For $\varpi,\Psi\in\widehat{\mathfrak{B}}_r$, we obtain

$$\begin{aligned}
&\left|\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^{c}\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)\right|\\
&\leq\left|\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^{c}\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)-\mathfrak{F}(\varepsilon,0,0,0,0)\right|+|\mathfrak{F}(\varepsilon,0,0,0,0)|\\
&\leq\mathfrak{W}_0\left(|\varpi(\varepsilon)|+|\Psi(\varepsilon)|+\left|{}^{c}\mathscr{D}^{\phi_1}\Psi(\varepsilon)\right|+\left|\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right|\right)+r_1\\
&\leq\mathfrak{W}_0\left[\|\varpi\|_{\mathscr{U}}+\|\Psi\|_{\mathscr{V}}+\frac{1}{\Gamma(\mathfrak{q}_1+1)}\|\Psi\|_{\mathscr{V}}\right]+r_1,\\
&\leq\mathfrak{W}_0\left(\|\varpi\|_{\mathscr{U}}+\rho_1\|\Psi\|_{\mathscr{V}}\right)+r_1,\\
&\leq\mathfrak{W}_0\rho_1 r+r_1.
\end{aligned}\tag{55}$$

In a similar manner, we have

$$\left|\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^{c}\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right)\right|\leq\mathfrak{K}_0\rho_2 r+r_2.\tag{56}$$

Then,

$$\begin{aligned}
\left|\Pi_1(\varpi,\Psi)(\varepsilon)\right|&\leq\left(\mathfrak{W}_0\rho_1 r+r_1\right)\mathscr{U}_1+\left(\mathfrak{K}_0\rho_2 r+r_2\right)\mathscr{V}_1\\
&=\left(\mathfrak{W}_0\rho_1\mathscr{U}_1+\mathfrak{K}_0\rho_2\mathscr{V}_1\right)+r_2\mathscr{V}_1+\mathscr{U}_1 r_1\\
&=\Lambda r+\mathscr{Q}_1,\quad\forall\,\varepsilon\in[0,1],
\end{aligned}\tag{57}$$

and

$$\begin{aligned}
\left|\Pi_1'(\varpi,\Psi)(\varepsilon)\right|&\leq\left(\mathfrak{W}_0\rho_1 r+r_1\right)\mathscr{U}_1^*+\left(\mathfrak{K}_0\rho_2 r+r_2\right)\mathscr{V}_1^*\\
&=\left(\mathfrak{W}_0\rho_1\mathscr{U}_1^*+\mathfrak{K}_0\rho_2\mathscr{V}_1^*\right)+r_2\mathscr{V}_1^*+\mathscr{U}_1^* r_1\\
&=\Lambda^* r+\mathscr{Q}_1^*,\quad\forall\,\varepsilon\in[0,1],
\end{aligned}\tag{58}$$

which gives us

$$\begin{aligned}
\left|{}^{c}\mathscr{D}^{\phi_2}\Pi_1(\varpi,\Psi)(\varepsilon)\right|&\leq\int_0^{\varepsilon}\frac{(\varepsilon-\zeta)^{-\phi_2}}{\Gamma(1-\phi_2)}\left|\Pi_1'(\varpi,\Psi)(\zeta)\right|d\zeta,\\
&\leq\frac{1}{\Gamma(2-\phi_2)}\left(\mathscr{W}r+\mathscr{Q}_1^*\right),\quad\forall\varepsilon\in[0,1].
\end{aligned}\tag{59}$$

Therefore, we deduce

$$\begin{aligned}
\left\|\Pi_1(\varpi,\Psi)\right\|_{\mathscr{U}}&=\left\|\Pi_1(\varpi,\Psi)\right\|+\left\|{}^{c}\mathscr{D}^{\phi_2}\Pi_1(\varpi,\Psi)\right\|\\
&\leq\left(\Lambda+\frac{\mathscr{W}}{\Gamma(2-\phi_2)}\right)r+\mathscr{Q}_1+\frac{\mathscr{Q}_1^*}{\Gamma(2-\phi_2)}.
\end{aligned}\tag{60}$$

In a similar manner, we obtain

$$\begin{aligned}
&\left|\Pi_2(\varpi,\Psi)(\varepsilon)\right|\leq\Lambda^* r+\widehat{\mathscr{Q}_1}\\
&\left|\Pi_2'(\varpi,\Psi)(\varepsilon)\right|\leq\mathscr{W}^* r+\widehat{\mathscr{Q}_1}^*,\\
&\left|{}^{c}\mathscr{D}^{\phi_1}\Pi_2(\varpi,\Psi)(\varepsilon)\right|\leq\int_0^{\varepsilon}\frac{(\varepsilon-\zeta)^{-\phi_1}}{\Gamma(1-\phi_1)}\left|\Pi_2'(\varpi,\Psi)(\zeta)\right|d\zeta,\\
&\leq\frac{1}{\Gamma(2-\phi_1)}\left(\mathscr{W}^* r+\widehat{\mathscr{Q}_1}^*\right),\quad\forall\varepsilon\in[0,1],
\end{aligned}\tag{61}$$

then we conclude

$$\begin{aligned}
\left\|\Pi_2(\varpi,\Psi)\right\|_{\mathscr{V}}&=\left\|\Pi_2(\varpi,\Psi)\right\|+\left\|{}^{c}\mathscr{D}^{\phi_1}\Pi_2(\varpi,\Psi)\right\|\\
&\leq\left(\Lambda^*+\frac{\mathscr{W}^*}{\Gamma(2-\phi_1)}\right)r+\widehat{\mathscr{Q}_1}+\frac{\widehat{\mathscr{Q}_1^*}}{\Gamma(2-\phi_1)}.
\end{aligned}\tag{62}$$

By relations (60) and (62), we deduce

$$\begin{aligned}\|\Pi(\varpi,\Psi)\|_{\mathscr{U}\times\mathscr{V}} &= \|\Pi_1(\varpi,\Psi)\|_{\mathscr{U}} + \|\Pi_2(\varpi,\Psi)\|_{\mathscr{V}}\\ &\leq \left(\left[\Lambda+\Lambda^* + \frac{\mathscr{W}}{\Gamma(2-\phi_2)} + \frac{\mathscr{W}^*}{\Gamma(2-\phi_1)}\right]\right)r\\ &\quad + \left[\mathscr{Q}_1 + \mathscr{Q}_1^* + \frac{\widehat{\mathscr{Q}_1}}{\Gamma(2-\phi_2)} + \frac{\widehat{\mathscr{Q}_1^*}}{\Gamma(2-\phi_1)}\right] = r.\end{aligned} \tag{63}$$

This implies $\Pi\widehat{\mathfrak{B}}_r \subset \widehat{\mathfrak{B}}_r$.

We then show that Π is a contraction operator. For every $\varepsilon \in [0,1]$, taking into account $(\varpi_i,\Psi_i) \in \widehat{\mathfrak{B}}_r$ for $i = 1,2$, we obtain

$$\begin{aligned}&\left|\Pi_1(\varpi_1,\Psi_1)(\varepsilon) - \Pi_2(\varpi_2,\Psi_2)(\varepsilon)\right|\\ &= \left|\mathscr{S}_1(\varepsilon)\right|\left[\varkappa_1\int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi_1(\mathfrak{u}),\Psi_1(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi_1(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi_1(\mathfrak{u})\right) - \mathfrak{F}\left(\mathfrak{u},\varpi_2(\mathfrak{u}),\Psi_2(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi_2(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi_2(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\right.\\ &\quad - \int_0^1 \frac{(1-\zeta)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\zeta,\varpi_1(\zeta),\Psi_1(\zeta),{}^c\mathscr{D}^{\phi_1}\Psi_1(\zeta),\mathscr{I}^{\mathfrak{q}_1}\Psi_1(\zeta)\right)\right.\\ &\quad \left.\left.- \mathfrak{F}\left(\zeta,\varpi_2(\zeta),\Psi_2(\zeta),{}^c\mathscr{D}^{\phi_1}\Psi_2(\zeta),\mathscr{I}^{\mathfrak{q}_1}\Psi_2(\zeta)\right)\right|d\zeta\right]\\ &\quad + \left|\mathscr{S}_2(\varepsilon)\right|\left[\mu_1\int_0^1 e^{-\mu_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\mathfrak{u},\varpi_1(\mathfrak{u}),\Psi_1(\mathfrak{u}),{}^c\mathscr{D}^{\phi_2}\varpi_1(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_2}\varpi_1(\mathfrak{u})\right)\right.\right.\right.\\ &\quad \left.\left.- \mathfrak{G}\left(\mathfrak{u},\varpi_2(\mathfrak{u}),\Psi_2(\mathfrak{u}),{}^c\mathscr{D}^{\phi_2}\varpi_2(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_2}\varpi_2(\mathfrak{u})\right)\right|d\mathfrak{u}\right)d\zeta\\ &\quad - \int_0^1 \frac{(1-\zeta)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\zeta,\varpi_1(\zeta),\Psi_1(\zeta),{}^c\mathscr{D}^{\phi_2}\varpi_1(\zeta),\mathscr{I}^{\mathfrak{q}_2}\varpi_1(\zeta)\right)\right.\\ &\quad \left.\left.- \mathfrak{G}\left(\zeta,\varpi_2(\zeta),\Psi_2(\zeta),{}^c\mathscr{D}^{\phi_2}\varpi_2(\zeta),\mathscr{I}^{\mathfrak{q}_2}\varpi_2(\zeta)\right)\right|d\zeta\right]\\ &\quad + \left|\mathscr{S}_3(\varepsilon)\right|\left[\int_0^1\left(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\sigma,\varpi_1(\sigma),\Psi_1(\sigma),{}^c\mathscr{D}^{\phi_1}\Psi_1(\sigma),\mathscr{I}^{\mathfrak{q}_1}\Psi_1(\sigma)\right)\right.\right.\right.\right.\\ &\quad \left.\left.\left.- \mathfrak{F}\left(\sigma,\varpi_2(\sigma),\Psi_2(\sigma),{}^c\mathscr{D}^{\phi_1}\Psi_2(\sigma),\mathscr{I}^{\mathfrak{q}_1}\Psi_2(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\ &\quad + \int_0^1\left(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\left(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\left|\mathfrak{G}\left(\sigma,\varpi_1(\sigma),\Psi_1(\sigma),{}^c\mathscr{D}^{\phi_2}\varpi_1(\sigma),\mathscr{I}^{\mathfrak{q}_2}\varpi_1(\sigma)\right)\right.\right.\right.\\ &\quad \left.\left.\left.- \mathfrak{G}\left(\sigma,\varpi_2(\sigma),\Psi_2(\sigma),{}^c\mathscr{D}^{\phi_2}\varpi_2(\sigma),\mathscr{I}^{\mathfrak{q}_2}\varpi_2(\sigma)\right)\right|d\sigma\right)d\mathfrak{u}\right)d\zeta\\ &\quad - \int_0^1 e^{-\varkappa_1(1-\zeta)}\left(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\left|\mathfrak{F}\left(\mathfrak{u},\varpi_1(\mathfrak{u}),\Psi_1(\mathfrak{u}),{}^c\mathscr{D}^{\phi_1}\Psi_1(\mathfrak{u}),\mathscr{I}^{\mathfrak{q}_1}\Psi_1(\mathfrak{u})\right)\right.\right.\end{aligned}$$

$$
\begin{aligned}
&- \mathfrak{F}\left(\mathfrak{u}, \varpi_2(\mathfrak{u}), \Psi_2(\mathfrak{u}), {}^c\mathscr{D}^{\phi_1}\Psi_2(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_1}\Psi_2(\mathfrak{u})\right)\Big| d\mathfrak{u}\Big) d\zeta\Big] \\
&+ \left|\mathscr{S}_4(\varepsilon)\right|\Bigg[\int_0^1\Bigg(\int_0^\zeta e^{-\varkappa_1(\zeta-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\sigma)^{\vartheta-2}}{\Gamma(\vartheta-1)}\Big|\mathfrak{F}\left(\sigma, \varpi_1(\sigma), \Psi_1(\sigma), {}^c\mathscr{D}^{\phi_1}\Psi_1(\sigma), \mathscr{I}^{\mathfrak{q}_1}\Psi_1(\sigma)\right) \\
&- \mathfrak{F}\left(\sigma, \varpi_2(\sigma), \Psi_2(\sigma), {}^c\mathscr{D}^{\phi_1}\Psi_2(\sigma), \mathscr{I}^{\mathfrak{q}_1}\Psi_2(\sigma)\right)\Big| d\sigma\Bigg) d\mathfrak{u}\Bigg) d\zeta \\
&+ \int_0^1\Bigg(\int_0^\zeta e^{-\mu_1(\zeta-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\sigma)^{\eta-2}}{\Gamma(\eta-1)}\Big|\mathfrak{G}\left(\sigma, \varpi_1(\sigma), \Psi_1(\sigma), {}^c\mathscr{D}^{\phi_2}\varpi_1(\sigma), \mathscr{I}^{\mathfrak{q}_2}\varpi_1(\sigma)\right) \\
&- \mathfrak{G}\left(\sigma, \varpi_2(\sigma), \Psi_2(\sigma), {}^c\mathscr{D}^{\phi_2}\varpi_2(\sigma), \mathscr{I}^{\mathfrak{q}_2}\varpi_2(\sigma)\right)\Big| d\sigma\Bigg) d\mathfrak{u}\Bigg) d\zeta \\
&- \int_0^1 e^{-\mu_1(1-\zeta)}\Bigg(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\eta-2}}{\Gamma(\eta-1)}\Big|\mathfrak{G}\left(\mathfrak{u}, \varpi_1(\mathfrak{u}), \Psi_1(\mathfrak{u}), {}^c\mathscr{D}^{\phi_2}\varpi_1(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_2}\varpi_1(\mathfrak{u})\right) \\
&- \mathfrak{G}\left(\mathfrak{u}, \varpi_2(\mathfrak{u}), \Psi_2(\mathfrak{u}), {}^c\mathscr{D}^{\phi_2}\varpi_2(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_2}\varpi_2(\mathfrak{u})\right)\Big| d\mathfrak{u}\Bigg) d\zeta\Bigg] \\
&+ \int_0^\varepsilon e^{-\varkappa_1(\varepsilon-\zeta)}\Bigg(\int_0^\zeta \frac{(\zeta-\mathfrak{u})^{\vartheta-2}}{\Gamma(\vartheta-1)}\Big|\mathfrak{F}\left(\mathfrak{u}, \varpi_1(\mathfrak{u}), \Psi_1(\mathfrak{u}), {}^c\mathscr{D}^{\phi_1}\Psi_1(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_1}\Psi_1(\mathfrak{u})\right) \\
&- \mathfrak{F}\left(\mathfrak{u}, \varpi_2(\mathfrak{u}), \Psi_2(\mathfrak{u}), {}^c\mathscr{D}^{\phi_1}\Psi_2(\mathfrak{u}), \mathscr{I}^{\mathfrak{q}_1}\Psi_2(\mathfrak{u})\right)\Big| d\mathfrak{u}\Bigg) d\zeta. \\
&\leq \mathscr{U}_1\mathfrak{W}_0\left(\|\varpi_1-\varpi_2\| + \|\Psi_1-\Psi_2\| + \|{}^c\mathscr{D}^{\phi_1}\Psi_1 - {}^c\mathscr{D}^{\phi_1}\Psi_2\| + \|\mathscr{I}^{\mathfrak{q}_1}\Psi_1 - \mathscr{I}^{\mathfrak{q}_1}\Psi_2\|\right) \\
&+ \mathscr{V}_1\mathfrak{K}_0\left(\|\varpi_1-\varpi_2\| + \|{}^c\mathscr{D}^{\phi_2}\varpi_1 - {}^c\mathscr{D}^{\phi_2}\varpi_2\| + \|\mathscr{I}^{\mathfrak{q}_2}\varpi_1 - \mathscr{I}^{\mathfrak{q}_2}\varpi_2\| + \|\Psi_1-\Psi_2\|\right) \\
&\leq \mathscr{U}_1\mathfrak{W}_0\left(\|\varpi_1-\varpi_2\| + \rho_1\|\Psi_1-\Psi_2\| + \|{}^c\mathscr{D}^{\phi_1}\Psi_1 - {}^c\mathscr{D}^{\phi_1}\Psi_2\|\right) \\
&+ \mathscr{V}_1\mathfrak{K}_0\left(\rho_2\|\varpi_1-\varpi_2\| + \|{}^c\mathscr{D}^{\phi_2}\varpi_1 - {}^c\mathscr{D}^{\phi_2}\varpi_2\| + \|\Psi_1-\Psi_2\|\right) \\
&\leq \Lambda\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}} + \|\Psi_1-\Psi_2\|_{\mathscr{V}}\right).
\end{aligned} \tag{64}
$$

Then, we obtain

$$\left|\Pi_1'(\varpi_1, \Psi_1)(\varepsilon) - \Pi_2'(\varpi_2, \Psi_2)(\varepsilon)\right| \leq \mathscr{W}\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}} + \|\Psi_1-\Psi_2\|_{\mathscr{V}}\right). \tag{65}$$

This gives us

$$
\begin{aligned}
\left|{}^c\mathscr{D}^{\phi_2}\Pi_1(\varpi_1, \Psi_1)(\varepsilon) - {}^c\mathscr{D}^{\phi_2}\Pi_1(\varpi_2, \Psi_2)(\varepsilon)\right| &\leq \int_0^\varepsilon \frac{(\varepsilon-\zeta)^{-\phi_2}}{\Gamma(1-\phi_2)}\left|\Pi_1'(\varpi_1, \Psi_1)(\zeta) - \Pi_1'(\varpi_2, \Psi_2)(\zeta)\right| d\zeta \\
&\leq \frac{1}{\Gamma(2-\phi_2)}\mathscr{W}\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}} + \|\Psi_1-\Psi_2\|_{\mathscr{V}}\right).
\end{aligned} \tag{66}
$$

From the above inequalities, we conclude

$$\begin{aligned}
&\left\|\Pi_1(\varpi_1,\Psi_1)(\varepsilon)-\Pi_1(\varpi_2,\Psi_2)(\varepsilon)\right\|_{\mathscr{U}}\\
&=\left\|\Pi_1(\varpi_1,\Psi_1)(\varepsilon)-\Pi_1(\varpi_2,\Psi_2)(\varepsilon)\right\|+\left\|{}^c\mathscr{D}^{\phi_2}\Pi_1(\varpi_1,\Psi_1)(\varepsilon)-{}^c\mathscr{D}^{\phi_2}\Pi_1(\varpi_2,\Psi_2)(\varepsilon)\right\|\\
&\le\left[\Lambda+\frac{1}{\Gamma(2-\phi_2)}\mathscr{W}\right]\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}}+\|\Psi_1-\Psi_2\|_{\mathscr{V}}\right).
\end{aligned}\tag{67}$$

In the similar manner, we deduce

$$\left\|\Pi_2(\varpi_1,\Psi_1)-\Pi_2(\varpi_2,\Psi_2)\right\|_{\mathscr{V}}\le\left[\Lambda^*+\frac{1}{\Gamma(2-\phi_2)}\mathscr{W}^*\right]\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}}+\|\Psi_1-\Psi_2\|_{\mathscr{V}}\right).\tag{68}$$

Therefore, by (67) and (68), we obtain

$$\begin{aligned}
&\left\|\Pi(\varpi_1,\Psi_1)-\Pi(\varpi_2,\Psi_2)\right\|_{\mathscr{V}}\\
&=\left\|\Pi_1(\varpi_1,\Psi_1)-\Pi_1(\varpi_2,\Psi_2)\right\|_{\mathscr{U}}+\left\|\Pi_2(\varpi_1,\Psi_1)-\Pi_2(\varpi_2,\Psi_2)\right\|_{\mathscr{V}}\\
&\le\left[\Lambda+\frac{1}{\Gamma(2-\phi_2)}\mathscr{W}\right]\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}}+\|\Psi_1-\Psi_2\|_{\mathscr{V}}\right)\\
&+\left[\Lambda^*+\frac{1}{\Gamma(2-\phi_2)}\mathscr{W}^*\right]\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}}+\|\Psi_1-\Psi_2\|_{\mathscr{V}}\right)\\
&\le\left(\Lambda+\Lambda^*+\frac{1}{\Gamma(2-\phi_2)}\mathscr{W}+\frac{1}{\Gamma(2-\phi_2)}\mathscr{W}^*\right)\times\left(\|\varpi_1-\varpi_2\|_{\mathscr{U}}+\|\Psi_1-\Psi_2\|_{\mathscr{V}}\right).
\end{aligned}\tag{69}$$

By using the condition, we deduce that Π is a contraction. Hence, by Banach's fixed point theorem, the operator Π has a unique fixed point which corresponds to the unique solution of systems (4) and (5). This completes the proof. □

5. Example

Let $\lambda_1 = 2; \chi_1 = 3; \nu = 1; z = 1; \varepsilon = 1, \vartheta_1 = (9/2)$, $\eta = (7/2)$, $\phi_1 = (1/3)\phi_2 = (1/2), \mathfrak{q}_1 = (3/2), \mathfrak{q}_2 = (8/3)$, the system of FDEs that follows is examined.

$$\begin{cases}\left({}^c\mathscr{D}^{\vartheta}+\varkappa_1{}^c\mathscr{D}^{\vartheta-1}\right)\varpi(\varepsilon)=\mathfrak{F}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_1}\Psi(\varepsilon),\mathscr{I}^{\mathfrak{q}_1}\Psi(\varepsilon)\right), & \varepsilon\in(0,1),\\ \left({}^c\mathscr{D}^{\eta}+\mu_1{}^c\mathscr{D}^{\eta-1}\right)\Psi(\varepsilon)=\mathfrak{G}\left(\varepsilon,\varpi(\varepsilon),\Psi(\varepsilon),{}^c\mathscr{D}^{\phi_2}\varpi(\varepsilon),\mathscr{I}^{\mathfrak{q}_2}\varpi(\varepsilon)\right), & \varepsilon\in(0,1),\end{cases}\tag{70}$$

augmented with the coupled classical integral boundary conditions

$$\begin{cases}\varpi(0)=0,\quad \varpi'(0)=0,\quad \varpi(1)=0,\quad \varpi(1)=\int_0^1\varpi(\zeta)d\zeta+\int_0^1\Psi(\zeta)d\zeta,\\ \Psi(0)=0,\quad \Psi'(0)=0,\quad \Psi(1)=0,\quad \Psi(1)=\int_0^1\varpi(\zeta)d\zeta+\int_0^1\Psi(\zeta)d\zeta,\end{cases}\tag{71}$$

we have $\mathscr{A}_1 = 0.432332, \mathscr{A}_2 = 0.533834, \mathscr{A}_3 = 0.316738, \mathscr{A}_4 = 0.449976, \mathscr{A}_5 = 0.175751, \mathscr{A}_6 = 0.848132, \mathscr{A}_7 = 0.0907486, \mathscr{A}_8 = 0.050612, \mathscr{A}_9 = 0.108083, \mathscr{A}_{10} = 0.0585836, \mathscr{A}_1 = 0.432332, \mathscr{A}_2 = 0.533834, \mathscr{A}_3 = 0.316738, \mathscr{A}_4 = 0.449976, \mathscr{A}_5 = 0.175751, \mathscr{A}_6 = 0.848132, \mathscr{A}_7 = 0.0907486, \mathscr{A}_8 = 0.050612, \mathscr{A}_9 = 0.108083, \mathscr{A}_{10} = 0.0585836, \mathscr{A}_{11} = 0.137005, \mathscr{A}_{12} = 0.130885, \Delta = -0.00151637, \Delta_1 = -0.00631263, \Lambda_1 \approx 2.48283, \Lambda_2 \approx 0.418054, \Lambda_3 \approx -2.67921, \Lambda_4 \approx -2.74112, \Xi_1 \approx 5.80039, \Xi_2 \approx -5.91683, \Xi_3 \approx -2.30745, \Xi_4 \approx 19.4495, \Theta_1 \approx 0.137507, \Theta_2 \approx -0.338567, \Theta_3 \approx 1.38295, \Theta_4 \approx -0.783336, \Upsilon_1 \approx -4.08288, \Upsilon_2 \approx 6.38719, \Upsilon_3 \approx 1.62421, \Upsilon_4 \approx -13.6905, \mathscr{S}_1 \approx 0.734434, \mathscr{S}_2 \approx 0.0454712, \mathscr{S}_3 \approx -0.461503, \mathscr{S}_4 \approx -0.947352, \mathscr{T}_1 \approx 0.58003, \mathscr{T}_2 \approx -0.188326, \mathscr{T}_3 \approx 0.230742, \mathscr{T}_4 \approx 1.94492, \mathscr{U}_1 \approx 0.103373, \mathscr{U}_2 \approx 0.160156, \mathscr{V}_1 \approx 0.126236, \mathscr{V}_2 = 0.0648497, \mathscr{U}_1^* \approx 0.282148, \mathscr{U}_2^* \approx 0.299316, \mathscr{V}_1^* \approx 0.00163853, \mathscr{V}_2^* = 0.287505$.

We consider the functions

$$\begin{aligned}
\left|\varpi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)\right| &= \frac{\varepsilon}{\varepsilon^2+1}\left(3\cos\varepsilon + \frac{1}{8}\sin(\varphi_1+\varphi_2)\right) - \frac{1}{8(\varepsilon+1)^2}\varphi_3 + \frac{1}{10}\arctan\varphi_4,\\
\left|\Psi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)\right| &= \frac{\varepsilon}{(\varepsilon+2)^3}\left(5e^{-\varepsilon} + \frac{\varphi_1}{2} + 2\varphi_2\right) - \frac{\varepsilon}{6}\sin(\varphi_3+\varphi_4),
\end{aligned} \tag{72}$$

for all $\varepsilon \in [0,1], \varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{R}$. We obtain the inequalities.

$$\begin{aligned}
\left|\varpi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)\right| &\le \frac{3}{2} + \frac{1}{16}|\varphi_1| + \frac{1}{16}|\varphi_2| + \frac{1}{8}|\varphi_3| + \frac{1}{10}|\varphi_4|,\\
\left|\Psi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4)\right| &\le \frac{5}{8} + \frac{1}{16}|\varphi_1| + \frac{1}{4}|\varphi_2| + \frac{1}{6}|\varphi_3| + \frac{1}{6}|\varphi_4|,
\end{aligned} \tag{73}$$

for all $\varepsilon \in [0,1], \varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{R}$. So, we have $\mathfrak{M}_0 = (3/2), \mathfrak{M}_1 = (1/16), \mathfrak{M}_2 = (1/16), \mathfrak{M}_3 = (1/10), \mathfrak{M}_4 = (1/10), \mathfrak{b}_0 = (5/8), \mathfrak{b}_1 = (1/16), \mathfrak{b}_2 = (1/4), \mathfrak{b}_3 = (1/6), \mathfrak{b}_4 = (1/6)$. Given that $\mathscr{N}4 \approx 0.4561664746$ and $\mathscr{N}5 \approx 0.3510076689$, it follows that the condition $\max \mathscr{N}4, \mathscr{N}5 < 1$ is met. Consequently, by Theorem 5, we deduce that problems (4) and (5) has at least one solution for $\varepsilon \in [0,1]$.

We consider the functions

$$\begin{aligned}
\varpi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4) &= \frac{\varepsilon}{2} + \frac{1}{8(\varepsilon+1)^2}\left(\frac{\varphi_1}{1+|\varphi_1|} - \varphi_2\right) + \frac{1}{32}\sin^2\varphi_3 - \frac{\varepsilon}{9}\arctan\varphi_4,\\
\Psi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4) &= \frac{\varepsilon^2}{\varepsilon^3+1} - \frac{1}{16}\sin\varphi_1 + \frac{1}{10}\varphi_2 + \frac{1}{\sqrt{4+\varepsilon^2}}\cos\varphi_3 - \frac{|\varphi_4|}{6(1+|\varphi_4|)},
\end{aligned} \tag{74}$$

for all $\varepsilon \in [0,1], \varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{R}$.

We obtain the following inequalities:

$$\begin{aligned}
&\left|\varpi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4) - \varpi_1(\varepsilon,\mathscr{Y}_1,\mathscr{Y}_2,\mathscr{Y}_3,\mathscr{Y}_4)\right|\\
&\quad\le \frac{1}{8}\left(|\varphi_1-\mathscr{Y}_1| + |\varphi_2-\mathscr{Y}_2| + |\varphi_3-\mathscr{Y}_3| + |\varphi_4-\mathscr{Y}_4|\right),\\
&\left|\Psi_1(\varepsilon,\varphi_1,\varphi_2,\varphi_3,\varphi_4) - \Psi_1(\varepsilon,\mathscr{Y}_1,\mathscr{Y}_2,\mathscr{Y}_3,\mathscr{Y}_4)\right|\\
&\quad\le \frac{1}{6}\left(|\varphi_1-\mathscr{Y}_1| + |\varphi_2-\mathscr{Y}_2| + |\varphi_3-\mathscr{Y}_3| + |\varphi_4-\mathscr{Y}_4|\right),
\end{aligned} \tag{75}$$

for all $\varepsilon \in [0,1]$ and $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{R}$.

Here, $\mathfrak{K}_0 = (1/6)$ and $\sigma_0 = (1/8)$. Besides, we deduce $\rho_1 \approx 0.752252$ $\rho_2 \approx 0.249238, \Lambda \approx 0.189533, \Lambda^* \approx 0.014562, \mathscr{W} \approx 0.038958, \mathscr{W}^* \approx 0.012091$ and

$$\left[\Lambda + \Lambda^* + \frac{\mathscr{W}}{\Gamma(2-\phi_2)} + \frac{\mathscr{W}^*}{\Gamma(2-\phi_1)}\right] \approx 0.2614483331 < 1. \tag{76}$$

Hence, all the conditions of the theorem are fulfilled. Therefore, according to Theorem 6, we establish that problems (4 (5) possess a unique solution, $\varepsilon \in [0,1]$.

6. Discussion

We have provided criteria for the existence of solutions to a coupled system of nonlinear sequential LCFIEs with distinct orders, accompanied by nonlocal classical integral boundary conditions. We have given conditions for the existence of such solutions. Using a methodology that makes use of contemporary analytical tools, the results are obtained. It should be emphasized that the results that are provided in this particular context are novel and add to the corpus of literature already available on the topic. Furthermore, our results encompass cases where the system reduces to one with boundary conditions of the form: when classical integral modifies to RSI, then we get

$$\begin{cases} \varpi(0) = \varpi'(0) = 0, \varpi'(1) = 0, \varpi(1) = \int_0^1 \varpi(\zeta) d\mathscr{H}_1(\zeta) + \int_0^1 \Psi(\zeta) d\mathscr{H}_2(\zeta), \\ \Psi(0) = \Psi'(0) = 0, \Psi'(1) = 0, \Psi(1) = \int_0^1 \varpi(\zeta) d\mathscr{K}_1(\zeta) + \int_0^1 \Psi(\zeta) d\mathscr{K}_2(\zeta). \end{cases} \tag{77}$$

This work will be extended in the future to a tripled system of integromultipoint boundary conditions and nonlinear sequential LCFIEs of different orders. The multivalued analogue of the problem considered in this paper is another goal of ours.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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